UNICRITICAL LAMINATIONS

SOURAV BHATTACHARYA, ALEXANDER BLOKH, AND DIERK SCHLEICHER

Abstract. Thurston introduced *invariant (quadratic) laminations* in his 1984 preprint as a vehicle for understanding the connected Julia sets and the parameter space of quadratic polynomials. Important ingredients of his analysis of the angle doubling map σ_2 on the unit circle $\mathbb{S}^1$ were the Central Strip Lemma, non-existence of wandering polygons, the transitivity of the first return map on vertices of periodic polygons, and the non-crossing of minors of quadratic invariant laminations. We use Thurston's methods to prove similar results for *unicritical* laminations of arbitrary degree d and to show that the set of so-called *minors* of unicritical laminations themselves form a *Unicritical Minor Lamination* UML_d. In the end we verify the *Fatou conjecture* for the unicritical laminations and extend the *Lavaurs algorithm* onto UML_d.

Introduction

One of the most satisfactory strengths of polynomial dynamics is its successful interplay with symbolic dynamics: even though the Julia sets and the Mandelbrot set have very complicated topological structure, there are very simple combinatorial models that describe the original sets, often up to homeomorphism.

In the "early days of modern holomorphic dynamics", in the mid-1980's, Thurston developed his theory of quadratic laminations for connected Julia sets of quadratic polynomials, and for their parameter space, the *Mandelbrot set* [Thu85]: he introduced the concept of *invariant quadratic laminations* as subsets of the closed unit disk and described their properties; each of these models a Julia set in the sense that a natural quotient, called *pinched disk* by Douady [Dou93], is homeomorphic to the (filled-in) Julia set if and only if the latter is locally connected; see also [Sch09]. Thurston showed that invariant quadratic lamination is characterized by a unique leaf, called its *minor leaf* (a set of two or possibly one angles), and each angle is part of a unique minor leaf. Turning to parameter space, Thurston showed that the union of all minor leaves forms another lamination, the *quadratic minor lamination* QML, that models the Mandelbrot set via its own pinched

Date: January 18, 2021; in the revised form July 2, 2021; in the revised form January 30, 2022.
2010 *Mathematics Subject Classification.* Primary: 54F20; Secondary: 30C35.
Key words and phrases. complex dynamics, circle dynamics, laminations.

disk, and is homeomorphic to it exactly when the Mandelbrot set is locally connected, which continues to form a major conjecture in the field. Douady, Hubbard, and Lavaurs showed that QML and hence the Mandelbrot set can be constructed by a very simple algorithm, known as the *Lavaurs algorithm*, that inductively constructs all periodic minor leaves in the order of their period, and so that the closure of all periodic leaves is all of QML.

The issue of generalizing these results to higher degrees was actually raised by Thurston himself in the end of [Thu85]. For arbitrary degrees $d \geqslant 3$, this is a substantial and difficult task. The first question that Thurston viewed as fundamental, whether "wandering triangles" exist, was resolved in the affirmative in [BO08]; the non-existence of wandering triangles is one of the key lemmas in his quadratic theory (see Theorem II.5.2 from [Thu85]), so this creates a substantial difficulty. A second difficulty lies in the fact that parameter space of degree d polynomials has complex dimension $d-1$, so the one-dimensional theory of parameter spaces fails fundamentally. Even the combinatorial description of cubic polynomials is most difficult; see for instance [Mil08] and [Kaf06]. Thurston's work on general laminations of degree d was recently published in [TBGHLLT19]

In this manuscript, we focus on a particular class of degree d laminations called *unicritical*. Unicritical polynomials can always be parameterized as z^d+c, so they have a one-dimensional parameter space. Their connectedness locus, often called the *Multibrot set*, is known to resemble the Mandelbrot set (see for instance [Sch04]). Some properties of unicritical laminations are similar to those of quadratic laminations, e.g. the non-existence of wandering triangles [Lev98, Sch00]. The papers [AKLS09, KL09a, KL09b] showed deep and far reaching analogy between the parameter spaces of quadratic and unicritical polynomials.

However we are not aware of an article in which Thurston's methods were used to generalize his quadratic combinatorial results to the unicritical case serving as a rigorous reference for papers in which combinatorial structure of the Multibrot set is used. Further generalizations of [Thu85] would need such a paper as a step towards considering other parameter spaces, at the very least other one-dimensional parameter spaces. Moreover, the unicritical case provides a foundation for inroads into transcendental dynamics; here it could serve as a guiding case for particular high degree polynomial laminations that, rather than being totally general, are modeling particular transcendental families. All that shows, in the authors' view, a need for a rigorous treatment of the unicritical case (which in fact, was expected from the time of the first appearance of Thurston's preprint [Thu85]).

The present paper fills this gap by verifying that expectation and justifying the above mentioned similarities between quadratic and unicritical cases; to this end we extend results of [Thu85] onto unicritical σ_d-invariant laminations for *any* d. We show that *minors* of two distinct unicritical laminations cannot cross implying that all such minors in $\overline{\mathbb{D}}$ form a lamination UML_d, the *Unicritical Minor Lamination of degree d*. The *Fatou conjecture* about the density of hyperbolicity for UML_d is verified by proving that elements of UML_d associated with laminations with periodic Fatou gap of degree $k > 1$ are dense in UML_d. Finally, we generalize the *Lavaurs algorithm*.

The second part of the paper (dealing with the Fatou conjecture and generalization of the Lavaurs algorithm onto the unicritical case) is new, the proofs are quite different and more complicated than in the quadratic case. Indeed, for $d > 2$ the relation between *periodic minors* (on one hand) and *all σ_d-periodic points* (on the other hand) is much more intricate than in the quadratic case. This novelty serves as another justification for the extension of Thurston's quadratic results onto the unicritical case.

The first part of the paper is a buildup of tools and methods necessary for the second part of the paper. These are reminiscent of the original Thurston's methods adjusted in the unicritical case for which in many instances new arguments and approaches were needed. Let us give more detail below.

By [Thu85] and [Lav89], *every* periodic point of $\mathbb{S}^1$ is an endpoint of a non-degenerate leaf of QML. However this is false in the general unicritical case with $d > 2$. By our Theorem 3.20, for a given periodic point $x \in \mathbb{S}^1$ of period $p > 1$, there are the following two possibilities: (1) x is the endpoint of a non-degenerate leaf of UML_d (similar to the quadratic case) (2) x is disjoint from all non-degenerate leaves of UML_d (then x is called a *degenerate periodic minor*). Let $\mathcal{M}_p$ be the set of all non-degenerate periodic minors of the *least* period p. It is proven in Theorem 5.5 that the degenerate periodic minors of period p are located in the components of $\mathbb{S}^1 \setminus \mathcal{M}_p$ and each component contains precisely $d-2$ of them. In the quadratic case $d = 2$ and hence $d - 2 = 0$. This explains why *degenerate minors* were absent in the quadratic case.

As a result, the construction of the *Unicritical Main Cardioid* $\mathcal{UMC}$ generalizing the *Quadratic Main Cardioid* is different in the general unicritical case. Let $\mathcal{L}$ be a *unicritical lamination* with a critical Fatou gap U. We prove that if ψ is the map which collapses edges of $\sigma_d(U)$ to points, then UML_d contains a gap whose boundary is $\psi^{-1}(\mathcal{UMC})$. By our Theorem 4.19 *copies of* $\mathcal{UMC}$ are dense in UML_d which establishes the Fatou conjecture for in the unicritical case. All this requires new arguments and presents the most significant difference between the quadratic and the general unicritical cases.

The paper is organized as follows: Section 1 contains basic definitions and results. In Section 2 we study *unicritical σ_d-invariant laminations* and show that the minors of *unicritical laminations* form a *lamination* called *Unicritical Minor Lamination of degree d* and denote it by UML_d. In Section 3 we study the basic properties of UML_d. In Section 4 we prove that UML_d is a *q-lamination*. In Section 5, we prove additional facts that help us to devise an algorithm to construct UML_d similar to the *Lavaurs algorithm* [Lav89] for the quadratic case (see Section 6).

1. Preliminaries

1.1. **Laminational equivalence relations.** We will use some standard notation ($\mathbb{R}$ for reals, $\mathbb{C}$ for complex numbers, ∂A for the boundary of A etc). We begin with basic planar concepts.

Notation 1.1 (Basic planar concepts)**.** Let $\hat{\mathbb{C}}$ be the Riemann sphere. For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the component of $\hat{\mathbb{C}}\backslash X$ containing ∞. For X connected, let $\Psi_X : \hat{\mathbb{C}}\backslash\overline{\mathbb{D}} \to U^\infty(X)$ be a Riemann map with $\Psi_X(\infty) = \infty$, and $\Psi_X(z)$ tending to a positive real limit as $z \to \infty$.

Let us now introduce basic concepts of polynomial dynamics. Below P always denotes a complex polynomial of degree $d \geqslant 2$.

Definition 1.2 (The Julia set)**.** The *filled Julia set* K_P of a polynomial P is the set of all points z whose orbits do not diverge to infinity under iterations of P. The *Julia set* of P is $J_P = \partial K(P)$.

The polynomial Julia sets are studied in complex dynamics.

Notation 1.3. Consider the Julia set J_P and the filled Julia set K_P of P. Set θ_d be the map $z^d|_{U^\infty(\overline{\mathbb{D}})}$. If J_P is connected, $\Psi \circ \theta_d = P \circ \Psi$ [DH8485, Mil00]. If J_P is locally connected, Ψ extends to a continuous map $\overline{\Psi} : \hat{\mathbb{C}}\backslash\mathbb{D} \to \overline{\hat{\mathbb{C}}\backslash K_P}$, and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$ on $\mathbb{C} \setminus \mathbb{D}$ yielding a continuous surjection $\overline{\Psi}|_{\mathbb{S}^1} = \psi : \mathbb{S}^1 \to J_P$ is called the *Caratheodory loop*. We will write σ_d for $\theta_d|_{\mathbb{S}^1} = z^d|_{\mathbb{S}^1}$ throughout the paper.

We are ready to define less standard concepts, beginning with that of Laminational equivalence relations generated by polynomials.

Definition 1.4 (Laminational equivalence relations)**.** Assume that J_P is locally connected. Define an equivalence relation $\sim_P$ on $\mathbb{S}^1$ by $x \sim_P y$ iff $\psi(x) = \psi(y)$ and call $\sim_P$ the *laminational equivalence relation (generated by the polynomial P)*. The map ψ semi-conjugates σ_d and $P|_{J(P)}$, and $\sim_P$ is invariant; $\sim_P$-classes have pairwise disjoint convex hulls.

Define an equivalence relation $\sim$ on $\mathbb{S}^1$ similar to $\sim_P$ but with no references to polynomials.

Definition 1.5. An equivalence relation $\sim$ on the unit circle $\mathbb{S}^1$ is said to be *laminational* if :

(i) the graph of $\sim$ is a closed subset of $\mathbb{S}^1 \times \mathbb{S}^1$;
(ii) convex hulls of disjoint equivalence classes are disjoint;
(iii) each equivalence class of $\sim$ is finite.

The equivalence relation $\sim$ is (σ_d)-*invariant* if:

(i) $\sim$ is *forward invariant*: for a $\sim$-class $[g]$, the set $\sigma_d([g])$ is a $\sim$-class too;
(ii) for any $\sim$-class $[g]$, the map $\tau = \sigma_d|_{[g]}$ extends to $\mathbb{S}^1$ as an orientation preserving covering map $\hat{\tau}$ such that $[g]$ is the full preimage of $\tau([g])$ under the covering map $\hat{\tau}$.

The next definition relates to Definition 1.2.

Definition 1.6. Let $\sim$ be a laminational equivalence relation. Then the quotient space $\mathbb{S}^1/\sim = J_\sim$ is called the *topological Julia set*. The map $f_\sim : J_\sim \to J_\sim$, induced by σ_d, is called the *topological polynomial*. Using Moore's Theorem we embed $J_\sim$ into $\mathbb{C}$ and extend the quotient map $\psi_\sim : \mathbb{S}^1 \to J_\sim$ to a map $\psi_\sim : \mathbb{C} \to \mathbb{C}$ with the only non-degenerate *fibers* (point preimages) being the convex hulls of non-degenerate $\sim$-classes. A *Fatou domain* of $J_\sim$ ($f_\sim$) is a component of $\mathbb{C} \setminus J_\sim$.

Various types of Fatou domains are introduced in the next definition.

Definition 1.7. If U is a periodic Fatou domain of $f_\sim$ of period n then $f^n_\sim|_{\mathrm{Bd}(U)}$ is conjugate to an irrational rotation of $\mathbb{S}^1$ (and U is called a *Siegel domain*) or to σ_k for some $k > 1$ [BL02]. The complement $K_\sim$ of $U^\infty(J_\sim)$ (for U^∞ see Notation 1.1) is the *filled topological Julia set*, i.e. the union of $J_\sim$ and its bounded *Fatou domains*. If $\sim$ is fixed, we omit it from the notation. We consider $f_\sim$ as a self-mapping of $J_\sim$.

Definition 1.8. For a $\sim$-class A, call an edge $\overline{ab}$ of $\mathrm{CH}(A)$(CH denotes *convex hull*) a *($\sim$-)leaf*. All points of $\mathbb{S}^1$ are called *degenerate* $\sim$-leaves.

The limit of a converging sequence of $\sim$-leaves $\sim$ is a $\sim$-leaf (the family of all $\sim$-leaves is closed); the union of all $\sim$-leaves is a continuum. For $X \subset \overline{\mathbb{D}}$ with the property $X = \mathrm{CH}(X \cap \mathbb{S}^1)$, set $\sigma_d(X) = \mathrm{CH}(\sigma_d(X \cap \mathbb{S}^1))$; then for any leaf ℓ of $\sim$, the set $\sigma_d(\ell)$ is a leaf.

Definition 1.9. The set $\mathcal{L}_\sim$ of all leaves of $\sim$ is called the *lamination generated by* $\sim$. Any such lamination $\mathcal{L}_\sim$ is called a *q-lamination.*

1.2. **Laminations as defined by Thurston.** This subsection is based upon Thurston's seminal preprint [Thu85]. For a collection $\mathcal{R}$ of sets, let $\mathcal{R}^+$ be the union of all sets from $\mathcal{R}$.

Definition 1.10. Two distinct chords *cross* if they are non-disjoint inside $\mathbb{D}$. Two chords that do not cross are said to be *unlinked.* A *prelamination* is a collection $\mathcal{L}$ of pairwise unlinked chords in $\overline{\mathbb{D}}$, called *leaves.* If all points of the circle are elements of $\mathcal{L}$ (seen as *degenerate leaves*) and $\mathcal{L}^+$ is closed in $\mathbb{C}$, call $\mathcal{L}$ a *lamination.* Clearly, $\mathcal{L}^+$ is a continuum.

The closure of a prelamination is a lamination.

Definition 1.11 (Leaves and extensions)**.** If $\ell = \overline{ab}$ is a leaf, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$. If $\sigma_d(a) = \sigma_d(b)$, call ℓ a *critical leaf* and $\sigma_d(a)$ a *critical value.* If $\sigma_d^* : \mathcal{L}^+ \to \overline{\mathbb{D}}$ is the linear extension of σ_d over all the leaves in $\mathcal{L}$, then σ_d^* is continuous. Also, σ_d is locally one-to-one on $\mathbb{S}^1$, and σ_d^* is one-to-one on any given non-critical leaf.

Definition 1.12. We call the closure of a component of $\mathbb{D} \setminus \mathcal{L}^+$ a *gap.* For a gap G, call $G \cap \mathbb{S}^1$ the *basis* of G and the points of $G \cap \mathbb{S}^1$ *vertices* of G. A gap G is *infinite* if and only if $G \cap \mathbb{S}^1$ is infinite.

Definition 1.13. Let $\mathcal{L}$ be a lamination. The equivalence relation $\approx_\mathcal{L}$ on $\mathbb{S}^1$ defined by declaring that $x \approx_\mathcal{L} y$ if and only if there exists a finite concatenation of leaves of $\mathcal{L}$ joining x and y will be called the *equivalence relation generated by* $\mathcal{L}$.

Recall that q-laminations were defined above in Definition 1.9. Equivalently, a lamination $\mathcal{L}$ is a q-lamination if and only if the equivalence relation $\approx_\mathcal{L}$ is an invariant laminational equivalence relation as defined in Definition 1.5 and $\mathcal{L}$ consists exactly of the edges of the convex hulls of all $\approx_\mathcal{L}$-classes together with all points of $\mathbb{S}^1$.

Definition 1.14. A lamination $\mathcal{L}$ is *Thurston σ_d-invariant* if it is

(i) *forward σ_d-invariant*: for any leaf $\ell = \overline{ab} \in \mathcal{L}$, either $\sigma_d(a) = \sigma_d(b)$ or $\sigma_d(\ell) = \overline{\sigma_d(a)\sigma_d(b)} \in \mathcal{L}$,

(ii) *backward σ_d-invariant*: for any leaf $\ell = \overline{ab} \in \mathcal{L}$ there is a collection of d disjoint leaves in $\mathcal{L}$ each joining a preimage of a to a preimage of b,

(iii) *gap σ_d-invariant*: for any gap G in $\mathcal{L}$, the convex hull H of $\sigma_d(G \cap \mathbb{S}^1)$ is a gap of $\mathcal{L}$, a leaf of $\mathcal{L}$, or a point of $\mathbb{S}^1$. If H is a gap, the map $\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is the composition of a monotone and a covering map to $\mathrm{Bd}(H)$ with positive orientation, and the image of a point moving in the positive direction around $\mathrm{Bd}(G)$ moves (non-strictly) in the positive direction around $\mathrm{Bd}(H)$.

Definition 1.15. A gap G of a σ_d-invariant lamination is called *critical* if for each $y \in \sigma_d(G)$ the set $\sigma_d^{-1}(y) \cap G$ consists of at least 2 points.

A set of critical leaves $\mathcal{C} = \{\bar{c}_i\}_{i=1}^{d-1}$ is *full* if $\mathcal{C}^+$ contains no polygons. Theorem 1.16 shows how to generate *Thurston invariant laminations:*

Theorem 1.16 ([Thu85])**.** *Let $\mathcal{L}$ be a lamination without critical gaps which satisfies the following conditions:*

(i) *$\mathcal{L}$ is forward σ_d invariant and*

(ii) *there is a full collection of critical leaves in $\mathcal{L}$.*

Then there exists a σ_d-invariant lamination $S(\mathcal{L})$ which contains $\mathcal{L}$ and is obtained by taking pullbacks of the leaves in $\mathcal{L}$.

Definition 1.17. The lamination $S(\mathcal{L})$ is called the *saturation* of $\mathcal{L}$.

1.3. **Classification of gaps.**

Definition 1.18 (Periodic and (pre)periodic gaps)**.** Let G be a gap of a d-invariant lamination $\mathcal{L}$. G is *(pre)periodic* if $\sigma_d^{m+k}(G) = \sigma_d^m(G)$ for some $m \geq 0$, $k > 0$; if m,k are chosen to be minimal, then G is said to be *preperiodic* if $m > 0$ or periodic (of *period* k) if $m = 0$. If the period of G is 1, then G is said to be *invariant.* Define *pre-critical* and *(pre)critical* objects similarly to (pre)periodic objects defined above.

Definition 1.19 (Fatou gaps)**.** An infinite gap G is said to be a *Fatou* gap if its basis $G \cap \mathbb{S}^1$ is uncountable.

The next important theorem is due to Kiwi [Kiw02].

Theorem 1.20 ([Kiw02])**.** *Let $\mathcal{L}$ be a σ_d-invariant lamination. Then any infinite gap of $\mathcal{L}$ is (pre)periodic.*

Theorem 1.21 describes properties of infinite periodic gaps.

Theorem 1.21 ([BOPT20, Lemma 2.31])**.** *Let G be an infinite periodic gap of period n and set $K = \partial G$. Then $\sigma_d^n|_K : K \to K$ is the composition of a covering map and a monotone map of K. Then there are three cases.*

(i) $\sigma_d^n|_K : K \to K$ *is of degree $k > 1$.*
(ii) *The gap G has countably many vertices, finitely many of which are periodic of the same period, and the rest are preperiodic. All non-periodic edges of G are (pre)critical and isolated (in particular there are critical edges of G). There is a critical edge with a periodic endpoint among the edges of gaps from the orbit of G.*
(iii) *The map $\sigma_d^n|_K$ is monotonically semi-conjugate to an irrational circle rotation so that each fiber of this semiconjugacy is a finite concatenation of (pre)critical edges of G. Thus, there are critical leaves (edges of some images of G) with non-preperiodic endpoints.*

In particular, if all critical sets of a lamination are finite polygons and there are no critical leaves then the lamination has no infinite gaps.

Proof. All claims of the theorem are proven in [BOPT20, Lemma 2.31], except for the last claim of (2), and the last claim of the entire lemma. The first of these claims is about the existence of a critical edge with a periodic endpoint among edges of gaps from the orbit of G. We may assume that G is invariant. Consider $\sigma_d|_{\partial G}$. This is a degree one map of the Jordan curve of rational rotation number, and well-known properties of such maps imply that it has at least one periodic point attracting from one side. Since σ_d is expanding on $\mathbb{S}^1$, then there is a critical edge of G with a periodic endpoint as claimed.

Let us prove the last claim of the lemma. Suppose that all critical sets of $\mathcal{L}$ are non-degenerate finite polygons, and yet U is an infinite gap of $\mathcal{L}$. By Theorem 1.20 we may assume that U is n-periodic. If $\sigma_3^n|_{\partial U}$ is of degree greater than 1 then for some i we must have $\sigma_3|_{\partial \sigma_3^i(U)}$ k-to-1 with $k > 1$, a contradiction with the assumption that all critical sets of $\mathcal{L}$ are finite. Now, suppose that $\sigma_3^n|_{\partial U}$ is of degree one. Then by (1) and (2) there exists a critical leaf, a contradiction. □

Definition 1.22. If Theorem 1.21(1) holds, G is called a periodic gap *of degree k*. If Theorem 1.21(2) holds, G is called a *caterpillar* gap. If Theorem 1.21(3) holds, G is called a *Siegel* gap.

1.4. **Sibling (σ_d-)invariant laminations.** *Sibling invariant laminations* were introduced in [BMOV13]. This notion is slightly more restrictive than that of Thurston invariant lamination. An important advantage of sibling invariant laminations is that they are defined exclusively through properties of their leaves. Other useful properties of sibling invariant laminations are stated in Theorem 1.25. For our purposes it suffices to consider sibling invariant laminations only.

Definition 1.23. Let $\mathcal{L}$ be a lamination. A leaf $\ell' \neq \ell \in \mathcal{L}$ is called a *σ_d-sibling* of ℓ is $\sigma_d(\ell') = \sigma_d(\ell)$.

Definition 1.24. *A lamination $\mathcal{L}$ is sibling (σ_d-)invariant* if:

(i) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$
(ii) for each $\ell \in \mathcal{L}$ so that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d disjoint leaves ℓ_1, ℓ_2, $\dots$, ℓ_d, in $\mathcal{L}$ so that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, 2, \dots, d$. In other words, each $\ell \in \mathcal{L}$ so that $\sigma_d(\ell)$ is a non-degenerate leaf, has $d-1$ distinct *σ_d-siblings*.
(iii) for each $\ell \in \mathcal{L}$, there exists $\ell_1 \in \mathcal{L}$ so that $\sigma_d(\ell_1) = \ell$.

Theorem 1.25 ([BMOV13]). *The space of all sibling σ_d-invariant laminations is compact. All q-laminations are sibling σ_d-invariant. All q-laminations are Thurston invariant.*

Also, we talk interchangeably about leaves (gaps) of $\sim$ or of $\mathcal{L}_\sim$.

2. Unicritical invariant laminations: basic properties

Definition 2.1. A σ_d-invariant lamination is said to be *unicritical* if it has a unique critical set $\mathcal{C}(\mathcal{L})$ (which maps forward d-to-1).

From now on let $\mathcal{L}$ be a σ_d-invariant unicritical lamination. Normalize the circle so that its length equals 1. Easy properties of unicritical laminations are stated in Lemma 2.2 whose proof we leave to the reader. Recall that siblings are defined in Definition 1.23.

Lemma 2.2. *The entire lamination $\mathcal{L}$ is invariant under rotation by the angle $\frac{1}{d}$; in particular, so is $\mathcal{C}(\mathcal{L}) = \mathcal{C}$. The set $\mathcal{C}$ is either a regular d-gon whose vertices cut the circle into d arcs of length $\frac{1}{d}$ each, or $\mathcal{C}$ has more than d sides and each circle arc complementary to $\mathcal{C} \cap \mathbb{S}^1$ is shorter than $\frac{1}{d}$. Every leaf $\ell \in \mathcal{L}$ has $d-1$ distinct siblings, each of which is the rotation of ℓ by the angle $\frac{i}{d}, 1 \leqslant i \leqslant d-1$.*

Let (a, b) be the arc of $\mathbb{S}^1$ from the point a to the point b traversed anticlockwise; let $|(a, b)|$ be the length of this arc. Define the *length $|\overline{ab}|$ of a chord $\overline{ab}$* as $|\overline{ab}| = \min\{|(a, b)|, |(b, a)|\}$. The maximum length of a chord is $\frac{1}{2}$.

Definition 2.3. A leaf of a uncritical lamination of maximal length is called a *major* of $\mathcal{L}$. The image of a *major* of $\mathcal{L}$ is called a *minor* of $\mathcal{L}$.

By Lemma 2.2 a major of $\mathcal{L}$ has $d-1$ siblings of the same length (hence, $\mathcal{L}$ has exactly d majors). However a priori there might exist another group of d σ_d-*siblings* of the same length, etc. We will show below (see Lemma 2.6) that because of the properties of the length function this is not possible. In particular, this will imply that there is only one group of d *sibling majors*, and, therefore, a unique *minor* of a *unicritical lamination* $\mathcal{L}$.

Definition 2.4. Call a chord $\ell = \overline{ab}$ *critical* if $\sigma_d(a) = \sigma_d(b)$.

The length of a critical chord ℓ is $|\ell| = \frac{j}{d}$ where $j \in \{1, \dots, t | t = [\frac{d}{2}]\}$ where $[\frac{d}{2}]$ denotes the greatest integer not exceeding $\frac{d}{2}$. Evidently, $|\sigma_d(\ell)|$ as a function of $|\ell|$ can be described as a "sawtooth" d-to-1 map $\psi : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ with d segments of monotonicity (*laps*) and slope d or $-d$ on each of them; increasing and decreasing laps alternate and the first lap $[0, \frac{1}{2d}]$ is increasing.

Definition 2.5. We call the function ψ (see Figure 1)reflecting how $|\sigma_d(\ell)|$ depends on $|\ell|$ a *length function.*

The line $y = x$ intersects the graph of ψ at the points 0 and $\frac{1}{d+1}$.

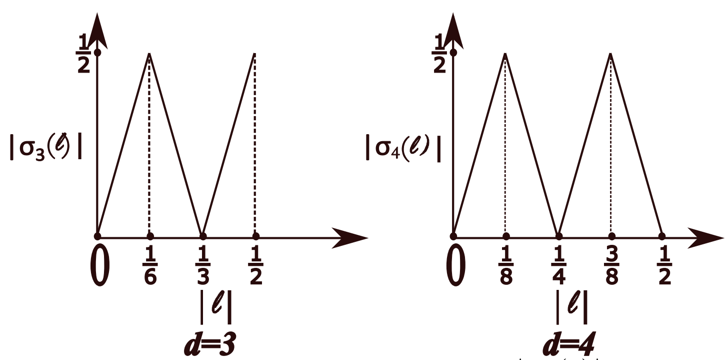

FIGURE 1. The graph of ψ reflecting how $|\sigma_d(\ell)|$ changes with $|\ell|$ when $d = 3$ and $d = 4$

Lemma 2.6. *A leaf ℓ of length $|\ell| < \frac{1}{d+1}$ will keep increasing in length under iteration of σ_d until for some iterate we have $|\sigma_d^j(\ell)| \geqslant \frac{1}{d+1}$. Further, if $\mathcal{L}$ is a σ_d-invariant unicritical lamination with a major M, then $|M| \geqslant \frac{1}{d+1}$; there are exactly d majors of $\mathcal{L}$ that are all siblings and a unique minor m of $\mathcal{L}$ such that $|m| \leqslant \frac{1}{d+1}$.*

Proof. We leave to the reader the proof of initial claims of Lemma 2.6 dealing with how the length of a leaf grows. This implies that $|M| \geqslant \frac{1}{d+1}$. If there exists another group of d majors which have the same image and are of the length $|M|$, this will imply that $1 \geqslant 2d|M| \geqslant \frac{2d}{d+1} > 1$, a contradiction (recall that $d \geqslant 2$). □

Notation 2.7. Given a σ_d-invariant unicritical lamination $\mathcal{L}$, its unique minor is denoted by $m(\mathcal{L})$.

Also, whenever we talk about circle arcs (x, y) of any length we always mean that the movement from x to y inside (x, y) is in the positive direction.

Definition 2.8. For a non-diameter chord $n = \overline{ab}$, the smaller of the two arcs into which n divides $\mathbb{S}^1$, is denoted by $H(n)$ and refer to points of $H(n)$ as located *"under"* n. Here, a is called the *initial* point of n (and $H(n)$) while b is called the *terminal* point of n (and $H(n)$). Given a minor $m = \overline{ab}$, call the arc $H(m)$ the *minor* arc.

Thus, for a non-diameter chord ℓ we can write $\ell = \overline{i(\ell)t(\ell)}$.

Definition 2.9. Call the *region* of the disk bounded by m and $H(m)$ the *region bounded by* m and denote it by $\mathcal{R}(m)$.

Definition 2.10. Given two non-diameter chords m and n, write $m \succ n$ if $n \subset \mathcal{R}(m)$, and say that n is a *successor of* m. If $m, n \in \mathrm{UML}_d, m \succ n$ and there exists no $\ell \in \mathrm{UML}_d$ satisfying $m \succ \ell \succ n$, then n is called an *immediate successor of* m.

Definition 2.11. Let $\ell = \ell_0 \in \mathcal{L}$; let $\ell_1, \dots, \ell_{d-1}$ be the $d-1$ siblings of the leaf l. There are $d+1$ complementary components to $\bigcup \ell_i$ in $\overline{\mathbb{D}}$. The component $C(\ell)$ bounded by ℓ and its $d-1$ siblings is called the *Central Strip (generated by ℓ)*.

The remaining d components are $\mathcal{R}(\ell_i), 0 \leqslant i \leqslant d-1$. Lemma 2.12 is based on Lemma 2.6; the proof is left to the reader.

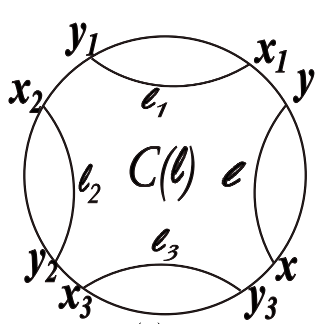

FIGURE 2. Central Strip $C(\ell)$ in σ_4 defined by a leaf ℓ and its 3 siblings: ℓ_1, ℓ_2, ℓ_3

Lemma 2.12. *Let $\mathcal{L}$ be a lamination with a major M and unique minor m. Then the length of images of a leaf $\ell \in \mathcal{L}$ increases d-fold on each step of iteration of σ_d, and $H(\sigma_d^i(\ell))$ maps onto $H(\sigma_d^{i+1}(\ell))$ homeomorphically until for some $N \in \mathbb{N}$, $\frac{|m|}{d} \leqslant |\sigma_d^N(\ell)| < |m|$. Then $H(\sigma_d^N(\ell))$ homeomorphically maps onto $H(\sigma_d^{N+1}(\ell))$, and $|m| \leqslant |\sigma_d^j(\ell)| \leqslant |M|$ for any $j > N$. Any leaf ℓ' with $|M| \geqslant \ell' \geqslant |m|$ is located in $C(m) \setminus C(M)$; in particular, if $i > N$ then $|M| \geqslant \sigma_d^i(\ell) \geqslant |m|$, the leaf $\sigma_d^i(\ell)$ is located in $C(m) \setminus C(M)$, and $\sigma_d^i(\ell)$ never separates ℓ and m.*

Definition 2.13. A *wandering triangle* is a triangle which has triangular images under σ_d so that the circular orientation on its vertices is preserved and, moreover, has images with edges that do not cross (recall that crossing chords are defined in Definition 1.10).

We want to show that a gap G of $\mathcal{L}$ is either (pre)periodic or (pre) critical(see Definition 1.18 for definition). By way of contradiction suppose G is a gap which is neither (pre)critical or (pre)periodic. Then any three vertices of G form a convex hull which is, evidently also neither (pre)periodic nor (pre)critical. Then the problem of proving that any gap G of $\mathcal{L}$ is either (pre)periodic or (pre)critical boils down to the next lemma.

Lemma 2.14. *There are no wandering triangles whose images do not intersect $\mathcal{C}(\mathcal{L})$. In particular, any gap of $\mathcal{L}$ is (pre)periodic or (pre)critical, and if $\mathcal{C}(\mathcal{L})$ is (pre)periodic then any gap of $\mathcal{L}$ is (pre)periodic.*

Proof. Let $A^0 = \text{CH}(x_0, y_0, z_0)$ be a wandering triangle whose images do not cross $\mathcal{C}(\mathcal{L})$. Set $A^i = \sigma_d(A^0)$, $\sigma_d^i(x) = x_i$, $\sigma_d^i(y) = y_i$, and $\sigma_d^i(z) = z_i$. Let L^i be the *length of the longest side* of A^i and let s^i be the *length of the shortest side* of A^i (in defining L^i and s^i we use the above introduced length function so that $L^i \geqslant \frac{1}{d+1} \geqslant s^i$ while equalities are only possible if A^i is an equilateral triangle). Evidently, there exists a continuous increasing function $\Delta : [0, \frac{1}{d+1}] \to [0, \infty)$ such that $\Delta(0) = 0$ and for every $0 \leqslant t \leqslant \frac{1}{d+1}$ the area of a triangle inscribed in the unit disk with the shortest side at least t equals $\Delta(t)$. Denote the Lebesgue measure of a set T by $\lambda(T)$.

Since $\lambda(\overline{\mathbb{D}}) < \infty$ and $\lambda(A^i) > 0$, then $\lambda(A^i) \underset{i\to\infty}{\longrightarrow} 0$. Since $\lambda(A^i) \geqslant \Delta(s^i) > 0$, then $s^i \underset{i\to\infty}{\longrightarrow} 0$. We can inductively choose a subsequence i_j such that $i_{j+1} > i_j$ is the first moment when $s^{i_{j+1}} < s^{i_j}$. Then, at the preceding moment, the triangle $A^{i_{j+1}-1}$ has a side close to $\frac{1}{d}$ (this side is the one whose image has length s^{j_i}). Hence the critical set $\mathcal{C}(\mathcal{L})$ is a regular polygon with critical edges of length $\frac{1}{d}$. Since $s^i \to 0$, the triangles $A^{i_{j+1}-1}$ have one short side and two long sides close to an edge of $\mathcal{C}(\mathcal{L})$, of whom the closer to the edge side is of length $L^{i_j} \underset{i\to\infty}{\longrightarrow} \frac{1}{d}$.

We may assume that $i_1 = 2$ and A^1 already is close to an edge of $\mathcal{C}(\mathcal{L})$ and has the above described properties. Denote the sides of A^1 by X^1, Y^1 and Z^1 where $|Z^1| \ll |Y^1| \lessapprox |X^1| \lessapprox \frac{1}{d}$. Evidently, $s^{i_1} = d(\frac{1}{d} - |X^1|)$. The triangle A^{i_2-1} must lie in the Central Strip $C(X^1)$ of X^1 (because $s^{i_2} < s^2 = s^{i_1}$), thus, A^{i_2-1} is squeezed between $\mathcal{C}(\mathcal{L})$ and the d siblings of X^1. However this implies that it has at least one side of length less than $\frac{1}{d} - |X^1| < s^1 = d(\frac{1}{d} - |X^1|)$, a contradiction with the choice of sequence i_j (by definition, the first moment the shortest side of the image of A^1 drops below s^1 must be i_2, not $i_2 - 1$). □

Now we show that if G is a non-critical periodic gap of $\mathcal{L}$, then G is finite and the first return map g of G permutes the sides of G transitively as a rational rotation. For a set T let $\text{int}(T)$ be its interior.

Lemma 2.15. *Let G be a periodic gap of $\mathcal{L}$, and there are no critical gaps in the orbit of G. Then G is finite and if g is the first return map of G, then g permutes the sides of G transitively as a rational rotation.*

Proof. By the assumption and because $\mathcal{L}$ is unicritical, the first return map $g : \text{Bd}(G) \to \text{Bd}(G)$ is a homeomorphism that acts as a bijection on the sides of G. Suppose that there are $N \geqslant 2$ cycles $\mathcal{A}_1, \dots, \mathcal{A}_N$ of sides of G. For each i, choose the leaf $\ell_i \in \mathcal{A}_i$ of shortest length s_i; by the properties of the length function, $s_i < \frac{1}{d+1}$. Let $\ell_i' \in \mathcal{A}_i$ be the leaf with $\sigma_d(\ell') = \ell_i$. Then $L_i = |\ell_i'| > \frac{1}{d+1}$.

Denote the circle arc of length s_i with the endpoints of ℓ_i by H_i. By the above $\sigma_d(\mathcal{C}(\mathcal{L})) \cap \mathbb{S}^1 \subset H_i$ for any i, and we may number these arcs so that $\{H_1 \supset \cdots \supset H_N\}$ is a nested collection. Let $N \geqslant 3$. Then ℓ_N and ℓ_1 must be edges of distinct images of G. However then some image of ℓ_1 is a leaf with endpoints contained in H_2, a contradiction with the choice of ℓ_1. Now let $N = 2$. Let ℓ_i be an edge of G_i where G_i is some image of G, $i = 1, 2$. Since G is a gap, each $\mathcal{A}_i$ has at least two leaves in G. If ℓ_1 separates $\mathcal{C}(\mathcal{L})$ from $\text{int}(G_1)$ then this implies that some image of ℓ_1 distinct from ℓ_1 will have endpoints in H_1, a contradiction with the choice of ℓ_1. Hence the opposite takes place and it is ℓ_1 that is separated from $\mathcal{C}(\mathcal{L})$ by $\text{int}(G_1)$. Now the fact that $H_1 \supset H_2$ implies that $G_2 \neq G_1$ has all its vertices in H_1 so that, again, some image of ℓ_1 distinct from ℓ_1, has endpoints in H_1, a contradiction. □

A crucial Thurston's result [Thu85] was the *Central Strip Lemma* stating that *no forward image of the minor of a quadratic lamination $\mathcal{L}$ can enter* *the Central Strip defined by the majors of the lamination $\mathcal{L}$.* A version of this result holds for unicritical laminations of arbitrary degree d. Recall that the Central Strip $C(\ell)$ of a leaf ℓ is closed.

Definition 2.16. Call components of $C(\ell)\cap\mathbb{S}^1$ *holes* of $C(\ell)$, and call their closures *closed holes* of $C(\ell)$. Extending this concept, for a compact set $A\subset\overline{\mathbb{D}}$ which equals the convex hull of the set $A\cap\mathbb{S}^1$, call a component of $\mathbb{S}^1\setminus A$ a *hole of* A; if ℓ is a chord connecting the endpoints of a hole of A, we say that the hole is located *behind* ℓ.

Observe that in the case when $A=\ell$ is a leaf, both holes of A are behind A; in this case if need be we will specify the hole using other tools.

Lemma 2.17 (Central Strip Lemma)**.** *Let $\ell\in\mathcal{L}, |\ell|<\frac{1}{d}$. If $\sigma_d^j(\ell)$ is the first time when ℓ re-enters $C(\ell)$ then there are three cases.*

(i) *If $|\ell|\leqslant\frac{1}{d(d+1)}$ then $\sigma_d(\ell)\subset C(\ell)$ and the endpoints of $\sigma_d(\ell)$ may belong to one or two closed holes of $C(\ell)$.*
(ii) *If $\frac{1}{d(d+1)}<|\ell|\leqslant\frac{1}{d+1}$ then $\sigma_d(\ell)\subset C(\ell)$ and the endpoints of $\sigma_d(\ell)$ belong to two consecutive closed holes of $C(\ell)$.*
(iii) *If $\frac{1}{d+1}<|\ell|$ then $j>1$ and the endpoints of $\sigma_d^j(\ell)$ belong to two consecutive closed holes of $C(\ell)$.*

Proof. Set $|\ell|=t$. If $x<\frac{1}{d+1}$, then $|\sigma_d(\ell)|=\psi(t)>t$(for definition of ψ see Definition 2.5) and $\sigma_d(\ell)$ must be contained in $C(\ell)$. The length of each hole of $C(\ell)$ is $\frac{1}{d}-t$. Computations show that if $t\leqslant\frac{1}{d(d+1)}$ then $\psi(t)\leqslant\frac{1}{d}-t$. Therefore in this case the endpoints of $\sigma_d(\ell)$ may belong to the same closed hole of $C(\ell)$. On the other hand, it is easy to see that if $\frac{1}{d(d+1)}<t\leqslant\frac{1}{d+1}$ then $t\leqslant\psi(t)$ and $\frac{1}{d}-t<\psi(x)$ which implies the desired.

It remains to consider the case when $\frac{1}{d+1}<t<\frac{1}{d}$ ($t\neq\frac{1}{d}$ by the assumptions of the lemma). Clearly, $\psi^i(t)<t$ for $i=1,\dots,j-1$. We claim that $\psi^i(t)>\frac{1}{d}-t$ for $i=1,\dots,j-1$. Indeed, if $i<j$ is the least with $\psi^i(t)\leqslant\frac{1}{d}-t$ then the properties of ψ imply $\psi^{i-1}(t)>t$, a contradiction with $\psi^i(t)<t$ for $i=1,\dots,j-1$. Thus, $t>\psi^{j-1}(t)>\frac{1}{d}-t$, and, hence, $\psi^j(t)>1-dt$. Since each component of $\mathbb{S}^1\cap C(\ell)$ is of length $\frac{1}{d}-t<1-dt$, then $\sigma_d^j(\ell)$ connects distinct closed holes of $C(\ell)$ and $|\sigma_d^j(\ell)|\geqslant t$. Observe now that if a leaf connects two closed holes of $C(\ell)$ that are not consecutive then its rotation by the angle $\frac{1}{d}$ will cross itself, a contradiction. □

Lemma 2.17 has important consequences.

Corollary 2.18. *Let $\mathcal{L}$ be a σ_d-invariant unicritical lamination. Let $M_1,\dots M_d$ be the majors of $\mathcal{L}$ and $m=\sigma_d(M_1)$ be its minor. Let $C(\mathcal{L})$ be the central* *strip defined by the majors of the lamination $\mathcal{L}$. Then no forward iterate of m can enter $C(\mathcal{L})$ unless it equals M_s for some s.*

Proof. If $\sigma_d^j(m) = \sigma_d^{j+1}(M_i) \subset C(\mathcal{L})$ then by Lemma 2.17 $|\sigma_d^{j+1}(M_i)| \geqslant |M_i|$ which by definition of a major implies that $\sigma_d^{j+1}(M_i) = M_s$ for any i and some (unique) s. □

Definition 2.19. A d-gon G is said to be *all-critical* if $\sigma_d(G)$ is a point.

Theorem 2.20. *Minors of distinct laminations do not cross.*

Proof. Let $\mathcal{L}_1 \neq \mathcal{L}_2$ be two laminations with *minors* m_1 and m_2. We claim that m_1 and m_2 do not cross. Indeed, suppose otherwise. Choose a non-periodic point $x \in H(m_1) \cap H(m_2)$ where $H(m_1)$ and $H(m_2)$ are the *minor arcs* determined by m_1 and m_2(see Definition 2.8). In other words, we choose x to be under both m_1 and m_2. The σ_d-preimage of the point x is an all-critical d-gon D_x. By Theorem 1.16, there exists an invariant lamination $\mathcal{L}'$ containing D_x. By Corollary 2.18 and since x is not periodic, $\mathcal{L}'$ is compatible with m_1 and m_2 so that m_1 and m_2 are diagonals of a gap G_0 of $\mathcal{L}'$. By Lemma 2.14, G_0 is a either pre-critical or pre-periodic. It cannot be pre-critical because m_1 and m_2 cannot enter their respective central strips. Thus, G_0 is pre-periodic. Choose $j \in \mathbb{N}$ such that $G = \sigma_d^j(G_0)$ is periodic and $\sigma_d^j(m_1)$ and $\sigma_d^j(m_2)$ are diagonals of G. By Lemma 2.15, the gap G is finite and the first return map permutes the sides of G transitively. Hence, some forward image of m_1 under σ_d will intersect m_1 and some forward image of m_2 under σ_d will intersect m_2, a contradiction. □

Theorem 2.21 follows from Theorem 2.20 and Theorem 1.25.

Theorem 2.21. *The space of all σ_d-invariant unicritical laminations is compact. The set of all their minors is a lamination.*

Proof. By Theorem 1.25 if a sequence of σ_d-invariant unicritical laminations converges then the limit is a σ_d-invariant laminations. Since the limit of the sequence of their critical sets then converges to a set that maps forward in the d-to-1 fashion, it follows that the limit lamination is unicritical. This implies the first claim of the theorem. Now, by Theorem 2.20 it is remains to prove that minors of σ_d-invariant unicritical laminations form a closed family of chords. Indeed, if a sequence of minors converges to a chord ℓ, then we can choose a subsequence so that the corresponding σ_d-invariant unicritical laminations converge too. Their limit lamination has ℓ has its minor and is, by the first claim of the theorem, unicritical. This proves the theorem. □

The next definition is similar to Thurston's definition of QML.

Definition 2.22. The set of all chords in the $\mathbb{D}$ which are minors of some σ_d-invariant unicritical lamination is a lamination called the *Unicritical Minor Lamination of degree d* and denoted by UML_d.

3. Basic properties of UML_d

Let us study the criterion for a non-degenerate chord m of $\overline{\mathbb{D}}$ or a point on $\mathbb{S}^1$ to be the minor of a unicritical lamination. By Lemma 2.6 we know that $|m| \leqslant \frac{1}{d+1}$; this is a necessary condition for m being a minor. However it is not sufficient (e.g., m may have forward images that cross, see Definition 1.10). Thus, to establish the desired criterion we need more conditions on iterations of m.

Definition 3.1. *Pullbacks* of m of length at most $\frac{1}{d}$ are called *'short'*.

'Short' pullbacks of a chord may be longer than the chord. There are $2d$ 'short' pullbacks of m; they are edges of a well-defined $2d$-gon. Note that if $|m| \leqslant \frac{1}{d+1}$, then d 'short' pullbacks of m are of length $\frac{|m|}{d} < |m|$ and d 'short' pullbacks of m are of length $\frac{1-|m|}{d} \geqslant |m|$ (in the latter case equality means that $|m| = \frac{1}{d+1}$). This follows from the properties of the length function (see Definition 2.5) described earlier .

Notation 3.2. Given a chord $m = \overline{ab}$ with $|m| \leqslant \frac{1}{d+1}$ denote by $D(m)$ the convex hull of $\sigma_d^{-1}(\{a, b\})$.

Lemma 3.3. *The following conditions form a criterion for a non-degenerate leaf m to be a minor of unicritical lamination:*

(i) $|m| \leqslant \frac{1}{d+1}$*;*
(ii) *m and its forward images do not cross;*
(iii) *no image of m is shorter than m;*
(iv) *If $M_1, \dots, M_d$ are d 'short' pullbacks of m of length greater than $\frac{1}{d+1}$, then forward images of m do not cross $M_1, \dots, M_d$.*

Moreover, Thurston's pullback lamination $\mathcal{L}(m)$ based on m and $M_1, \dots, M_d$, has m as its minor.

Proof. First we show that if m is the minor of a unicritical lamination $\mathcal{L}$, then it has the listed properties. Indeed, by Lemma 2.6, $|m| \leqslant \frac{1}{d+1}$. Moreover, by properties of laminations, m and its forward images do not cross. Consider d 'short' pullbacks $M_1, M_2, \dots, M_d$ of m of length greater than $\frac{1}{d+1}$. Then, again by properties of laminations, forward images of m

do not cross M_1, M_2,..., M_d. Finally, no image of m is shorter than m by Corollary 2.18.

On the other hand, we claim that if m has the listed properties then m is the minor of some unicritical lamination. Indeed, consider the $2d$-gon $D(m)$ (see Notation 3.2); it has d edges $M_1, M_2, \dots, M_d$ of length $\lambda = \frac{1-|m|}{d} \geqslant \frac{1}{d+1}$ and d edges $M'_1, M'_2, \dots, M'_d$ of length $\frac{|m|}{d} < |m|$. The leaf m cuts $\mathbb{D}$ into two pieces: a small piece T_0 and a large piece T_1. The $2d$-gon $D(m)$ cuts $\mathbb{S}^1$ in $2d$ arcs: holes $E_1, \dots, E_d$ of T behind $M_1, \dots, M_d$, and holes $E'_1, \dots, E'_d$ behind $M'_1, \dots, M'_d$. Clearly, this entire picture is symmetric with respect to the rotation by $\frac{1}{d}$. Observe that $E_1, \dots, E_d$ map in the homeomorphic fashion to T_1 by σ_d while $E'_1, \dots, E'_d$ map by σ_d to T_0 in the homeomorphic fashion.

We claim that chords $\sigma_d^j(m)$ never enter sets E'_i. Indeed, otherwise the assumption that images of m do not cross chords $M_1, \dots, M_d$ implies that for some j both endpoints of $\sigma_d^j(m)$ belong to sets $E'_1, \dots, E'_d$ and hence $\sigma_d^{j+1}(m) \subset T_0$, a contradiction with the assumption that no image of m is shorter than m. Thus, the forward orbit of m together with T forms a forward invariant lamination $\mathcal{L}_0$. Using Thurston's pullback construction and relying upon the unicriticality, we now construct a desired invariant lamination $\mathcal{L}$ (once we know that $\mathcal{L}_0$ is forward invariant the pullback construction always goes through). □

Definition 3.4. If a non-degenerate chord m satisfies the conditions from Lemma 3.3, then we will call the lamination $\mathcal{L}(m)$ constructed in Lemma 3.3 and having m as the minor the *lamination generated by* m or the *lamination corresponding to the minor* m. From now on the notation $\mathcal{L}(m)$ will be used for that lamination.

Lemma 3.5. *The following claims hold.*

(i) *If $n \in \mathrm{UML}_d$, $m \in \mathcal{L}(n)$ is a leaf, $n \succ m$, and there exists no i such that $n \succ \sigma_d^i(m) \succ m$ then $m \in \mathrm{UML}_d$. In particular, if m as above is sufficiently close to n, then $m \in \mathrm{UML}_d$.*

(ii) *If $m, n \in \mathrm{UML}_d$ with $m \succ n$ and n non-degenerate, then m is a leaf of $\mathcal{L}(n)$.*

Proof. (i) To prove that $m \in \mathrm{UML}_d$, we show that m satisfies the conditions of Lemma 3.3 one by one. Under iteration of σ_d, the length of m will continue to increase until it is at least $\frac{1}{d+1}$ and after that by Lemma 2.12 the image leaves cannot become shorter than n. Since, all leaves on the forward orbit of m are contained in $\mathcal{L}(n)$, the first three of the four hypothesis of Lemma

3.3 are satisfied. Denote by $N_1, \dots, N_d$ the d majors of $\mathcal{L}(n)$. These are sibling leaves of length at least $\frac{1}{d+1}$ that map to n under σ_d. Denote by $t_1, \dots, t_d$ the sibling chords that complement these leaves to a $2d$-gon $D(n)$, the convex hull of the majors $N_1, \dots, N_d$, which is the critical set of $\mathcal{L}(n)$.

Let $M_1, \dots, M_d$ be d chords of length at least $\frac{1}{d+1}$ that map to m under σ_d (these would be the major leaves of the lamination corresponding to m). They are longer than the major leaves of $\mathcal{L}(n)$ and, thus, not leaves of $\mathcal{L}(n)$. The d preimages of m that are leaves of $\mathcal{L}(n)$ are sibling leaves $s_1, \dots, s_d$ of length $\frac{|m|}{d}$ that complement $M_1, \dots, M_d$ to a $2d$-gon. If an iterated image $\sigma_d^{i-1}(m)$ crosses M_r for some r, it can only happen if $\sigma_d^{i-1}(m)$ is located ("squeezed") between some leaf s_j and the corresponding to it chord t_j, corresponding in the following sense: s_j is located in the hole of $D(n)$ behind t_j. However then on the next step we will have $n \succ \sigma_d^i(m) \succ m$, a contradiction. To prove the last part of claim (i) notice that m is close to n then by Lemma 2.12 the length of $\sigma_d(m)$ is greater than the length of n and the length of further images of m stays greater than that of n. Hence the assumptions of (i) hold and $m \in \mathrm{UML}_d$.

(ii) We first show that m does not intersect any leaf of $\mathcal{L}(n)$. If $\mathcal{C}(n)$ and $\mathcal{C}(m)$ are the central strips defined by the majors of the laminations $\mathcal{L}(n)$ and $\mathcal{L}(m)$ respectively then $\mathcal{C}(n) \subset \mathcal{C}(m)$. By Lemma 2.17 no image of m crosses a major of $\mathcal{L}(n)$. By construction, no image of m crosses a leaf of $\mathcal{L}(n)$. If, contrary to what is claimed, m is not a leaf of $\mathcal{L}(n)$ then m is a diagonal of a gap G of $\mathcal{L}(n)$. Let us show that this is impossible. Indeed, by the assumptions $\mathcal{L}(n)$ has no critical leaves and the critical set of $\mathcal{L}(n)$ is a finite polygon. Hence by Theorem 1.21 all gaps of $\mathcal{L}(n)$ are finite. Thus, G is finite. Since m is assumed to be a diagonal of G, then G cannot map to the critical set $D(n)$ of $\mathcal{L}(n)$ because it would contradict the Central Strip Lemma 2.17. Hence by Lemma 2.14 G eventually maps to a finite periodic gap G', and the image of m is a diagonal of G'. However then by Lemma 2.15 m will have crossing images, a contradiction. Thus, m is a leaf in $\mathcal{L}(n)$. □

Notation 3.6. Given a non-diameter chord ℓ denote by $A(\ell)$ the set (possibly empty) of all points that never exit $\overline{H(\ell)}$.

Given an arc I we denote by $|I|$ its length and that we normalize the circle length so that it equals 1.

Lemma 3.7. *Suppose that $\ell = \overline{yz}$ is a chord of length at most $\frac{1}{d}$ such that a fixed point a belongs to $[y, z]$. Then the following holds:*

(1) *The only way* $\sigma_d(y)$ *and* $\sigma_d(z)$ *can belong to* $[y, z]$ *is when one of these points equals* a *and* $|\ell| = \frac{1}{d}$.
(2) *The unique minimal invariant set contained in* $[y, z]$ *is* $\{a\}$.
(3) *If* m *is a minor with a fixed endpoint* a *then* $m = \{a\}$.

Proof. For $0 < [a, z] = \lambda < \frac{1}{d}$ we have $|[a, \sigma_d(z)]| = d\lambda < 1 - (\frac{1}{d} - \lambda) < 1 - |[y, a]|$ which implies that unless one of points y, z equals a and $|\ell| = \frac{1}{d}$, all points of $[y, z]$ except a are eventually mapped outside $[y, z]$. This shows that, except in the case when $|\ell| = \frac{1}{d}$ and either y or z equals a, the set $A(\ell)$ equals $\{a\}$ and claims (1) and (2) are immediate. In the remaining case (1) clearly holds, and $A(\ell)$ consists of a and a countable family of pullbacks of the other endpoint of ℓ towards a. Still, it follows that (2) holds in this case too. To prove (3), let $m = \overline{ab}$ be the non-degenerate minor of a lamination $\mathcal{L}$. Choose $b' \in H(m)$ with $\sigma_d(b') = b$. Then the major of $\mathcal{L}$ with an endpoint b' must cross m or be located "under" m, a contradiction (see Definition 2.8 for clarification). □

Notation 3.8. If m is the minor of a lamination $\mathcal{L}$ which is not a fixed point, then we denote by M_m the major of $\mathcal{L}$ such that $m \subset \overline{H(M_m)}$ (if m is a fixed point then M_m is not well-defined as there are two majors with the above property).

Lemma 3.9. *Let* $M = \overline{yz}$ *be a chord with* $|M| \leqslant \frac{1}{d}$ *such that* $M \succ \sigma_d(M) = m$. *Then there are two cases.*

(1) M *is a chord of length* $\frac{1}{d}$ *with a fixed endpoint.*
(2) *Otherwise* $y \leqslant \sigma_d(z) \leqslant \sigma_d(y) \leqslant z$, $\sigma_d([y, z]) = [\sigma_d(y), \sigma_d(z)]$, $\sigma_d(\ell) \in [y, z]$ *for any critical chord* $\ell \succ M$ *with* $|\ell| = \frac{1}{d}$, *and the arc* $[z - \frac{1}{d}, y + \frac{1}{d}] = I$ *contains no fixed points (so,* $\overline{H(M)}$ *and* $\overline{H(m)}$ *contain no fixed points).*

In particular, this holds if m *is a minor and* $M = M_m$.

Proof. (1) For a fixed point in $\overline{H(M)}$ the claim follows by Lemma 3.7.

(2) Suppose that no fixed point belongs to $\overline{H(M)}$. Then the first two claims follow from the fact that the image of $H(M)$ is an arc of length at most 1 but greater than $|H(M)|$. Now, since σ_d preserves orientation we see that as we rotate a critical chord ℓ from $\overline{y(y + \frac{1}{d})}$ to $\overline{(z - \frac{1}{d})z}$, the point $\sigma_d(\ell)$ moves along the circle in the negative direction from $\sigma_d(y)$ to $\sigma_d(z)$ which proves the third claim. To prove the last claim, observe that if a fixed point a belongs to $[z - \frac{1}{d}, y + \frac{1}{d}]$, then there exists a critical chord ℓ_a of length $\frac{1}{d}$ with both endpoints in I such that $\ell \succ M$. Since by the second

claim $\sigma_d(\ell_a) \in [y, z]$, then a coincides with one of the endpoints of M, a contradiction. $\square$

Notation 3.10. For a point $x \in \mathbb{S}^1$, let D_x be the convex hull of the full preimage of x under σ_d; clearly, D_x is an all-critical d-gon.

Lemma 3.11. *Every $x \in \mathbb{S}^1$ is the minor of a unicritical lamination.*

Proof. The set $D_x \cup \mathbb{S}^1$ is forward invariant, has d critical leaves, and, hence, satisfies Thurston's pullback construction conditions stated in Theorem 1.16. Thus, there exists a σ_d-invariant lamination having minor x. Such a lamination is clearly unicritical. $\square$

Notation 3.12. Denote the lamination from Lemma 3.11 by $\mathcal{L}_x$.

By our construction both $\mathcal{L}(m)$ (if m is a non-degenerate minor) and $\mathcal{L}_x$ (if $x \in \mathbb{S}^1$ is a point) are sibling invariant laminations. By Lemma 3.11, a point $x \in \mathbb{S}^1$ may be either an endpoint of a non-degenerate minor or it is a point of the circle disjoint from other minors.

Definition 3.13. A point $x \in \mathbb{S}^1$ is called a *degenerate minor* if it is disjoint from all other minors.

If x is periodic, one vertex of the d-gon D_x is periodic, the others are not. The simplest case is when x is a fixed point.

Lemma 3.14. *If x is a fixed point, $\mathcal{L}_x$ consists of isolated leaves that accumulate on points of $\mathbb{S}^1$. All gaps of $\mathcal{L}_x$ are pullbacks of D_x. If $\mathcal{L}$ is a unicritical lamination such that x belongs to the image of its critical set $\mathcal{C}(\mathcal{L})$, then $\mathcal{L} = \mathcal{L}_x$. Thus, x is a degenerate minor.*

Proof. The gap D_x has d holes. Under the map σ_d they expand to cover the circle exactly once. Moreover, for each such hole I there exists a unique fixed point $y_I \in \overline{I}$. Denote the edge of D_x connecting the endpoints of I by ℓ_I. Then the σ_d pullback of D_x with vertices in I is a $(d+1)$-gon attached to D_x at ℓ_I with vertices that include the endpoints of I and otherwise partition I into d equal arcs of length $\frac{1}{d^2}$ each. On the next step we will obtain new pullbacks of D_x with vertices that partition each of those arcs of length $\frac{1}{d^2}$ into yet d equal arcs, etc. Passing to the limit we arrive at the first claim of the lemma.

Now, let $\mathcal{L}$ be a unicritical lamination such that x belongs to the image of its critical set $\mathcal{C}(\mathcal{L}) = \mathcal{C}$. Then $D_x \subset \mathcal{C}$, and all leaves of $\mathcal{L}$ are compatible with $\mathcal{L}_x$ (i.e., do not cross leaves of $\mathcal{L}_x$). Suppose that $\mathcal{L}$ has a leaf $\ell \notin \mathcal{L}_x$.

Then ℓ is contained in one of the pullbacks of D_x which means that an eventual image of ℓ is contained in D_x but is not an edge of D_x. Evidently it is impossible because $\mathcal{L}$ is unicritical. □

Lemma 3.15 is similar except now x is periodic of period $p > 1$.

Lemma 3.15. *Let x be a non-fixed point. Then:*

(1) a unique hole H_1 of D_x contains no σ_d-fixed points while each other hole of D_x contains exactly one σ_d-fixed point;
(2) H_1 contains a non-degenerate minimal set $\mathcal{I}_x$;
(3) if x is periodic of period $p > 1$, and $\mathcal{L}_c$ is obtained from $\mathcal{L}_x$ by removing D_x and its backward orbit, then $\mathcal{L}_c$ is non-empty, and there is a Fatou gap U of $\mathcal{L}_c$ of period p that contains D_x.

Proof. (1) Clearly, $\{x\}$ is the minor of $\mathcal{L}_x$ and is not a vertex of D_x. Choose a hole H_1 of D_x such that $x \in H_1$. Then by Lemma 3.9, no fixed point belongs to H_1. Since the arcs between two consecutive fixed points are of length $\frac{1}{d-1} > \frac{1}{d}$, H_1 is a *unique* such hole of D_x.

(2) By (1) the set of points that stay forever in $\overline{H_1}$ is an invariant compact set that contains no σ_d-fixed points. Hence, there exists the desired minimal set $\mathcal{I}_x$.

(3) $\mathcal{L}_c$ is a lamination: all properties of laminations are immediate for $\mathcal{L}_c$ except for the fact that $\mathcal{L}_c$ is closed, and to see that, observe that by construction all edges of D_x are isolated in $\mathcal{L}_x$; hence, all pullbacks of edges of D_x are isolated in $\mathcal{L}_x$ and $\mathcal{L}_c$ is closed.

Since the period of x is p, the $(p-1)$-th pullback of D_x is a d-gon Δ_x emanating out of x. Taking σ_d-preimage of Δ_x we get d polygons $\Gamma^1 \subset \overline{H_1}$, $\dots$, $\Gamma^d \subset \overline{H_d}$ each of which is a $d+1$-gon called an *immediate decoration*, hanging out from an edge of D_x. Further pullbacks of D_x hang out of edges of immediate decorations (these are called simply *decorations*), etc.

Consider the convex hull G of $\mathcal{I}_x$. We claim that pullbacks of D_x can at most "touch" G (i.e., have common vertices with G), but otherwise are disjoint from G. Indeed, edges of G cannot intersect pullbacks of edges of D_x as otherwise a forward image of an edge of G will cross an edge of D_x, a contradiction. Thus, the only way the claim of the lemma fails is if G contains a pullback of an edge of D_x. Since $\mathcal{I}_x$ is invariant, then an edge of D_x contained in the boundary of H_1, is an edge of G. Since $\mathcal{I}_x$ is minimal, it follows then that $\mathcal{I}_x$ coincides with the periodic orbit of $\sigma_d(x)$. Evidently, this contradicts the assumption that an edge of G is also an edge of D_x. The claim that the intersection of a pullback of D_x and G is at most a common vertex, follows.

We claim that G is contained in a non-degenerate invariant gap of $\mathcal{L}_c$. Indeed, by the above G is contained in a non-degenerate gap G' of $\mathcal{L}_x$; edges of G' are not pullbacks of edges of D_x as at each edge of a pullback of D_x yet another pullback of D_x is attached. Hence all edges of G' are limits of pullbacks of edges of D_x, and G' remains a non-degenerate gap of $\mathcal{L}_c$, and $\mathcal{L}_c$ is not an empty lamination. Since $\mathcal{L}_c$ is obtained from $\mathcal{L}_x$ by removing D_x and its backward orbits, one can say that pullbacks of D_x do not fill up the entire unit disk.

The gap U of $\mathcal{L}_c$ that contains D_x is an infinite periodic gap which returns with degree d (it is infinite because by construction it contains immediate decorations as well as other decorations consecutively attached to them). Hence, U is a Fatou gap. Let the period of U be k. If $k < p$, then σ_d^k brings the gap U back to itself but the point $\tilde{x}$ does not return to its initial position and has is of period $\frac{p}{k} > 1$ inside U under the first return map $g = \sigma_d^k$. So, using the same technique as above applied to $\sigma_d^k|_U$, one can show that the gap U contains a non-degenerate gap of $\mathcal{L}_c$, a contradiction. Thus, $k \not< p$. On the other hand, by construction U contains D_x and all its immediate decorations which implies that $\sigma_d^p(U) = U$. Hence $k = p$ and the claim follows. □

Lemma 3.15 shows how unicritical laminations with periodic Fatou gaps can be constructed. We will now study such laminations. Throughout this section we use the same notation for the now defined objects. Let $m(\mathcal{L}) = m$ be the minor of $\mathcal{L}$. As in Lemma 3.3, there are d pullbacks of m denoted by $M_1, M_2, \dots, M_d$ that are of equal length $\lambda = \frac{1-|m|}{d} \geqslant \frac{1}{d+1}$. Since m is the minor of $\mathcal{L}$, chords $M_1, M_2, \dots, M_d$ must be leaves of $\mathcal{L}$ (one of them is, hence all of them are since $\mathcal{L}$ is unicritical). Observe that they are $1/d$-rotations of each other. By Lemma 2.17 forward images of m do not enter the central strip C of $\mathcal{L}$, i.e. the component of $\overline{\mathbb{D}} \setminus \bigcup_{i=1}^{d-1} M_i$ that has all M_i's on its boundary. In particular, no forward image of m coincides with a chord connecting two consecutive endpoints of adjacent majors of $\mathcal{L}$.

Consider now the case when the minor $m(\mathcal{L}) = m$ of $\mathcal{L}$ is periodic of period, say, n. Since $\sigma_d(\sigma_d^{n-1}(m)) = m$, then $\sigma_d^{n-1}(m)$ is one of the majors $M_1, \dots, M_d$.

Definition 3.16. We call the *major* $\sigma_d^{n-1}(m) = M$ the *primary major* of the lamination $\mathcal{L}$ and denote it by $M(\mathcal{L})$. The remaining $d-1$ *majors* of $\mathcal{L}$ are called *secondary* majors of $\mathcal{L}$.

Clearly, $M(\mathcal{L})$ and m have the same period.

Lemma 3.17. *Suppose that $\mathcal{L}$ has an n-periodic cycle of Fatou gaps with U critical. Then all d majors of $\mathcal{L}$ are edges of U and exactly one of them is periodic of period n.*

Proof. There are d majors of $\mathcal{L}$ of length at least $\frac{1}{d+1}$. By Lemma 2.17 σ_d^n maps them all to one of them which is periodic of period n. □

Definition 3.18. Given a Fatou gap U of period k, we call an edge (or vertex) of U *refixed* if its endpoints are of period k. The periodic major of $\mathcal{L}$ discovered in Lemma 3.17 is, evidently, refixed.

Lemma 3.19. *Suppose that $\mathcal{L}$ has an n-periodic cycle of Fatou gaps and U is a critical gap in that cycle. Then each edge of U is an eventual preimage of $M(\mathcal{L})$. In particular, the first return map $g|_U$ has exactly d fixed points. Two of these points are the endpoints of $M(\mathcal{L})$. Otherwise each component of $\mathrm{Bd}(U)$ from which all majors of $\mathcal{L}$ are removed contains exactly one fixed point of $g|_{\mathrm{Bd}(U)}$.*

Proof. Evidently, U has $d-1$ edges (possibly, degenerate edges, i.e. vertices) that are refixed. In particular $M(\mathcal{L})$ is a refixed edge of U. By Lemma 2.6, $|M(\mathcal{L})| \geqslant \frac{1}{d+1}$. Thus, $\mathcal{L}$ has d majors all of which are edges of U of length at least $\frac{1}{d+1}$.

Take an edge N of U. It is well-known that any edge of U is eventually mapped into either critical or periodic edge of U. Since there are no critical edges of U, we may assume that N is itself periodic. Let $\sigma_d^i(N)$ be the longest leaf in the orbit of N. By Lemma 2.6, $|\sigma_d^i(N)| \geqslant \frac{1}{d+1}$. It follows that there exists a major M' of $\mathcal{L}$ such that the endpoints of $\sigma_d^i(N)$ belong to the hole of U behind M'. However then, as we iterate $\sigma_d^i(N)$, it cannot be mapped to a chord longer than $\sigma_d^i(N)$ (by the choice of $\sigma_d^i(N)$), and neither can it enter its central strip (by Lemma 2.17), a contradiction with N being an eventual image of $\sigma_d^i(N)$. □

We will now prove the first main result of this section.

Theorem 3.20. *For a p-periodic point $x \in \mathbb{S}^1, p > 1$, there are precisely two possibilities.*

1. ***Degenerate case**: x is the degenerate minor of some unicritical lamination but there exists no unicritical lamination with a non-degenerate minor one of whose endpoints is x.*
2. ***Non-degenerate case**: x is an endpoint of a unique non-degenerate minor m of some unicritical lamination.*

Proof. Suppose that the degenerate case does not hold and there exists a minor m with endpoint x. We claim that such m is unique. Suppose otherwise. We may assume that $m = \overline{xy}$, y is period p, $H(m) = (x, y)$ (the movement from x to y inside (x, y) is clockwise) and for some periodic point $z \in (x, y)$ of period p the chord $\overline{xz}$ is also the minor of some unicritical cubic lamination. Let the corresponding *primary majors* be $M_y = \overline{x'y'}$ and $M_z = \overline{x'z'}$. Also, let U_y be the critical Fatou gap with major M_y and U_z be the critical Fatou gap with major M_z.

Denote by L and R the two siblings of M_y adjacent to M_y. Since by Lemma 2.6 $\frac{1}{d+1} \leqslant |M_z| < \frac{1}{d}$, then the point z' must belong to an arc I that separates (in the circle) M_y from one of those two siblings L and R, but does not contain the other one. Since this arc contains no p-periodic points that belong to $\mathrm{Bd}(U_y)$, then there is an edge ℓ of U_y with endpoints in I that crosses M_z. Recall that by Lemma 3.19 ℓ is an eventual preimage of M_y. Since σ_d^p acts on U_y as σ_d (after we collapse all edges of U_d to points), then after several iterations of σ_d^p the leaf ℓ will map onto a sibling M''_y of M_y distinct from M_y; it is easy to see that, as it happens, the leaf M_z will keep crossing the corresponding images of ℓ. In particular, the appropriate image of M_z will cross M''_y. However the location of M_z shows that then this image of M_z will cross one of its siblings, a contradiction. □

4. The proof of the fact that UML_d is a q-lamination

First let us classify all *gaps* of UML_d. If m is a *minor*, then by Lemma 3.9 the arc $H(m)$(for definition of $H(m)$ recall Definition 2.8) contains not *fixed points*. Thus, if we connect the center $(0, 0)$ of $\mathbb{D}$ with *fixed points* of σ_d by $d-1$ radii, then each *minor* is located in one of the thus created $d-1$ circular sectors.

Definition 4.1. Any gap G of UML_d has a side m that "faces" $(0, 0)$ (separates $(0, 0)$ from the rest of G). This edge m is called the *leading edge* of G. called

Other edges n of G are *incomparable* (with respect to the partial order $\succ$)among themselves because their sets $H(n)$ are pairwise disjoint. Thus, all edges of G are *immediate successors* of the *leading edge* m in the partial order $\succ$.

Remark 4.2. Let $\mathcal{L}$ be a lamination with *minor* $m = \overline{ab}$. $\mathcal{L}$ may have a *critical gap* $\mathcal{C}(\mathcal{L})$ which is either a *pre-periodic polygon* with dn sides ($n \geqslant 3$) or a *periodic Fatou gap* (this holds by Lemma 2.14).

Consider all three cases for $\mathcal{C}(\mathcal{L})$.

Lemma 4.3. *If $\mathcal{C}(\mathcal{L})$ is a polygon with dn sides($n \geqslant 3$), then $\sigma_d(\mathcal{C}(\mathcal{L}))$ is a gap of* UML_d.

Proof. The gap $\mathcal{C}(\mathcal{L}) = G$ can be subdivided by adding a collapsing $2d$-gon that coincides with the convex hull of d sibling edges of G with the same image which is an edge of $\sigma_d(G) = G'$. That collapsing $2d$-gon, apart from d sibling edges of it shares with G, will have d complementary edges connecting the appropriate endpoints of the sibling edges of G mentioned above. This can be done for each edge of G', thus it can be done in n ways. Each edge of G' is the minor of the corresponding lamination obtained by pulling back the appropriate $2d$-gon inside pullbacks of G. We claim that G' is a gap in UML_d. Indeed, otherwise G' contains a minor which is a diagonal of G'. However, by Lemma 2.14 and Lemma 2.15 every diagonal of G' has crossing forward images and, hence, cannot be a minor. Thus, G' is an actual gap of UML_d. □

To consider the case when $\mathcal{C}(\mathcal{L})$ is a *Fatou gap* we need to study the *unicritical version* of the *quadratic Main Cardioid*. We first prove the following Lemma:

Lemma 4.4. *Let ℓ be a critical chord of length $\frac{1}{d}$. Then there exists an invariant minimal set $\mathcal{I}(\ell) \subset \overline{H(\ell)}$, and for each such $\mathcal{I}(\ell)$ the convex hull $G(\ell)$ of $\mathcal{I}(\ell)$ has the longest edge M that separates the rest of $\mathcal{I}(\ell)$ from the center of $\mathbb{D}$; the image $m = \sigma_d(M)$ is a minor, and the unicritical lamination $\mathcal{L}(m)$ includes $G(\ell)$ as an invariant gap or leaf.*

Proof. By Lemma 3.7 we may consider a critical chord ℓ of length $\frac{1}{d}$ with no fixed point in $\overline{H(\ell)}$. The set $A(\ell) \neq \varnothing$ (since $\sigma_d(\overline{H(\ell)}) = \mathbb{S}^1$) of all points with orbits contained in $\overline{H(\ell)}$ is non-degenerate (by the "no fixed points in $\overline{H(\ell)}$" assumption), forward invariant and onto (the latter follows again because $\sigma_d(\overline{H(\ell)}) = \mathbb{S}^1$), with the circular order preserved on $A(\ell)$ except when the endpoints of ℓ belong to $A(\ell)$. Take a minimal set $\mathcal{I}(\ell) \subset \overline{H(\ell)}$. If $A(\ell)$ consists of two points, then $A(\ell) = \mathcal{I}$ is a two periodic orbit, and we associate with $A(\ell)$ the circle rotation by $\frac{1}{2}$ and the rotation number $\frac{1}{2}$. If $A(\ell)$ consists of more than two points we consider $\mathrm{CH}(A(\ell)) = U(\ell)$ with the map $\sigma_d|_{U(\ell)}$ (recall that we canonically extend σ_d onto chords of the unit circle). Since the circular orientation on $A(\ell)$ is preserved, the degree of $\sigma_d|_{\mathrm{Bd}(U(\ell))}$ is 1.

Thus, we can associate to $\sigma_d|_{\mathrm{Bd}(U(\ell))}$ (and, hence, to ℓ) a well-defined rotation number $\rho(\ell) \in (0, 1)$ (there are no fixed points in $A(\ell)$!). Consider the convex hull $G(\ell)$ of $\mathcal{I}(\ell)$. Among edges of $G(\ell)$ there is a unique edge M that "faces" the center $(0,0)$ of $\mathbb{D}$ (separates $(0,0)$ from the rest of G). This is the longest edge of $G(\ell)$. Set $m = \sigma_d(M)$. By Lemma 2.6, $|M| \geqslant \frac{1}{d+1}$, and by Lemma 3.9 $\sigma_d(\ell) \in H(m)$. Consider two cases related to the properties of any minimal invariant set $\mathcal{I} \subset \overline{H(\ell)}$ (later on we will prove that $\mathcal{I}(\ell)$ is unique).

(1) Suppose that $\mathcal{I}(\ell)$ is finite, and, hence, is a periodic orbit. This holds if and only if $\rho(\ell)$ is rational. As we map m forward, the images of m are all oriented in the same way inside $H(M)$ because, as long as the points stay in $H(\ell)$, their circular order is preserved. Their lengths stay under $\frac{1}{d}$, and on each step they are simply multiplied by d.

This allows us to evaluate lengths of all edges of $G(\ell)$ if we know the period q (not even the rotation number).

Set $\overline{ab} = m = \sigma_d(M)$ and consider the critical $2d$-gon $D(m)$. It has M and all its siblings as edges and in fact coincides with $\sigma_d^{-1}(\{a,b\})$. Since $D(m)$ and $G(\ell)$ form a forward invariant lamination, Thurston's pull back construction goes through and gives rise to the canonical pullback lamination $\mathcal{L}(m)$ that has m as its minor. Such minors are said to be *prime rational minors*.

(2) If $\mathcal{I}(\ell)$ is infinite, then it is known that $G(\ell)$ is an invariant gap such that when we collapse all its edges to points and transfer the action of σ_d onto the resulting circle we get an irrational rotation. Such gaps are called invariant *Siegel* gaps. Since edges of an invariant gap are (pre)periodic or (pre)critical, edges of $G(\ell)$ are (pre)critical. By our setup, $G(\ell)$ can only have one critical edge coinciding with ℓ. Let us show that then no two edges of $G(\ell)$ can intersect. Indeed, if they do, then after applying σ_d the appropriate number of times we would see that both ℓ and an edge ℓ' of $G(\ell)$ have a common endpoint, say, b. However ℓ' must eventually map to ℓ thus implying that $\sigma_d(\ell)$ is periodic, a contradiction. Hence no two edges of $G(\ell)$ have a common endpoint, the entire set $A(\ell)$ is minimal, and coincides with the limit set of, say, $\sigma_d(\ell)$. This case corresponds to an irrational rotation number $\rho(\ell)$. The degenerate minor $\sigma_2(\ell)$ is called a *prime irrational minor*. □

Definition 4.5. *Minors* whose existence is proven in Lemma 4.4 are called *prime minors*. If $G(\ell)$(defined in Lemma 4.4) is a *finite gap* or non-degenerate leaf then the corresponding *minor* is said to be *rational*. If $G(\ell)$ is a *Siegel gap* then the corresponding *degenerate minor* is said to be *irrational*.

Lemma 4.6. *Two distinct prime minors are not $\succ$-comparable.*

Proof. Indeed, suppose that $m \succ m'$ are two prime minors. Let $M_m = M$ and $M_{m'} = M'$ be the corresponding majors. Then by Lemma 3.9 $M' \succ M$. Choose a critical chord $\ell \succ M'$ of length $\frac{1}{d}$. Since the orbit of m is contained in $\overline{H(M)}$ and the entire orbit of m' is contained in $\overline{H(M')}$, then in fact the orbits of m and m' are contained in $\overline{H(\ell)}$ which implies that both minors are rational and have the same period. However then by the above they must have the same length and, hence, must coincide. □

Lemma 4.7. *The minimal set $\mathcal{I}$ in Lemma 4.4 is unique*

Proof. Notice that if case (2) of Lemma 4.4 holds, then the claim is immediate. So, assume that $\rho(\ell)$ is rational and show that $\mathcal{I}$ is well-defined. Suppose otherwise. Then there are two prime rational minors m and m' generated by minimal invariant sets $\mathcal{I} \subset \overline{H(\ell)}$ and $\mathcal{I}' \subset \overline{H(\ell)}$. Recall that by Theorem 2.20 m and m' do not cross, but might have a common endpoint. We claim that m and m' are $\succ$-comparable. Indeed, for geometric reasons $\sigma_d(\ell) \in \overline{H(m)} \cap \overline{H(m')}$. It follows that the only way m and m' are not $\succ$-comparable is when $\sigma_d(\ell) = y$ is the terminal point of one of them and the initial point of the other one. However then $y \in \mathcal{I} \cap \mathcal{I}'$, and since these sets are minimal, they coincide, a contradiction. Hence, m and m' are $\succ$-comparable and disjoint. We may assume that $m = \overline{xy}, m' = \overline{x'y'}$ are of period n, and $x < x' < y' < y$. The images of the arc $H(m)$ map onto each other and stay shorter than $\frac{1}{d}$ until m maps to the corresponding major M and m' maps to the corresponding major M'. However then the σ_d^n-image of $[x, x']$ stays shorter than $\frac{1}{d}$, and from that moment on the whole orbit of $[x, x']$ gets repeated. However, $[x, x']$ must expand under σ_d, a contradiction. Thus, $m = m'$. We conclude that $\mathcal{I}$, and, accordingly, m, as a function of ℓ, is unique. □

From now on we denote the minor from Lemma 4.7 by $m(\ell)$.

Lemma 4.8. *The family of prime minors forms the boundary of a closed convex set $Q = \mathrm{CH}(Q \cap \mathbb{S}^1)$. Moreover, $Q \cap \mathbb{S}^1$ is a Cantor set in which fixed points in $\mathrm{Bd}(\mathcal{UMC})$ are not isolated from either side.*

Proof. We claim that prime minors are pairwise disjoint. Indeed, the only way they intersect is when two rational prime minors m and m' have a common endpoint z which is the terminal point of, say, m and the initial endpoint of m'. Then z is a periodic point whose orbit B contains the remaining two endpoints of m and m'. Hence m and m' are edges of the same gap $\mathrm{CH}(B)$. However, by the above all edges of $\mathrm{CH}(B)$ are of distinct length, a contradiction.

As we showed above, prime minors are pairwise non-comparable in the sense of $\succ$, and pairwise disjoint. Let us show that any non-fixed point $y \in \mathbb{S}^1$ is either a prime irrational minor, or has a prime rational minor m such that $y \in H(m)$. Indeed, choose two consecutive fixed point u and v such that the positively oriented arc I from u to v contains y. Then choose an admissible critical chord ℓ with $i(\ell) \in I$ such that $\sigma_d(\ell) = y$. Set $m = m(\ell)$. It follows that $y \in H(m(\ell))$ as desired.

We claim that the union T of prime irrational minors and the endpoints of prime rational numbers is a Cantor set. Indeed, if a sequence of prime minors converges to a point $y \in \mathbb{S}^1$ then either y is an prime irrational minor, or y is an endpoint of a prime rational number because if y is "under" a prime rational minor m then the minor that converge to y will also be "under" m, a contradiction (see Definition 2.8 for clarification). Thus, T is closed. On the other hand, an isolated point $z \in T$ may exist only if two prime rational minors meet at z, a contradiction. Since T cannot contain subsegments, T is a Cantor set. Because of the symmetry, if a fixed point a had been isolated from one side, it would have been isolated from either side, a contradiction. It remains to set $Q = \mathrm{CH}(T)$. □

Definition 4.9. The set Q from Lemma 4.8 is the *unicritical analog of the Quadratic Main Cardioid*. We call it the *Unicritical Main Cardioid* and denote it by $\mathcal{UMC}$.

Theorem 4.10. *The set* $\mathcal{UMC}$ *is a gap of* UML_d.

Proof. The edges of $\mathcal{UMC}$ are rational prime minors. It also includes prime irrational minors as its "stand alone" vertices. To show that $\mathcal{UMC}$ is a gap of UML_d it suffices to show that there are no minors inside $\mathcal{UMC}$. Suppose that $m = m(\mathcal{L})$ is such a minor. By Lemma 3.9 we can choose the major $M_m = \overline{yz}$ of $\mathcal{L}$ such that $M_m \succ m$ and $\sigma_d(\ell) \in \overline{H(m)}$ for any critical chord ℓ with $\ell \succ M_m$ and $|\ell| = \frac{1}{d}$. By Lemma 3.9, $m = \sigma_d(M_m) = \overline{\sigma_d(z)\sigma_d(y)}$. Set $\ell = \overline{y(y+\frac{1}{d})}$ and consider the minor $m(\ell)$ and the associated major $M_{m(\ell)} = M(\ell)$.

By definition, $\ell \succ M(\ell)$. By construction, $\ell \succ M_m$. By Theorem 2.20 m and $m(\ell)$ do not cross. Hence, $M(\ell)$ and M_m do not cross either. If $M(\ell)$ is "under" M_m, then m is "under" $m(\ell)$, a contradiction with m being inside $\mathcal{UMC}$ (see Definition 2.8 for clarification). Thus, the only possibility left is that $M(\ell)$ has y as the initial endpoint. Then m and $m(\ell)$ share an endpoint $\sigma_d(y)$. Since m is inside $\mathcal{UMC}$, $m \succ m(\ell)$ and $M(\ell) \succ M_m$. The point $\sigma_d(y)$ belongs to the minimal set $\mathcal{I}(\ell)$ and has the orbit contained in $\overline{H(\ell)}$.

Since m is inside $\mathcal{UMC}$, the point $\sigma_d(z)$ belongs to a minor $m' \neq m(\ell)$ (otherwise $m = m(\ell)$). Hence for some $N \geqslant 2$ the point $\sigma_d^N(z)$ does not belong to $\overline{H(\ell)}$. We conclude that $\sigma_d^N(M_m)$ is a chord with one endpoint $\sigma_d^N(y) \in \overline{H(\ell)}$ and the other endpoint $\sigma_d^N(z) \notin \overline{H(\ell)}$. However this contradicts Corollary 2.18 which describes the dynamics of majors of laminations. This completes the proof. □

One can transfer Theorem 4.10 onto Fatou gaps.

Lemma 4.11. *Let $\mathcal{C}(\mathcal{L})$ is a critical Fatou gap of a unicritical lamination $\mathcal{L}$ of period n. Let $U = \sigma_d(\mathcal{C}(\mathcal{L}))$. Let $\psi : \mathrm{Bd}(U) \to \mathbb{S}^1$ be a monotone map that collapses edges of U to points. Then UML_d contains a gap $\mathcal{UMC}_U$ whose boundary is $\psi^{-1}(\mathrm{Bd}(\mathcal{UMC}))$. In particular, $\mathcal{UMC}_U \cap \mathbb{S}^1$ is a Cantor set, and the minor m of $\mathcal{L}$ has countably infinitely many immediate $\succ$-successors.*

Proof. The existence of the gap $\mathcal{UMC}_U$ is immediate from Theorem 4.10 and properties of the map ψ which semiconjugates $\sigma_d^n|_U$ and σ_d. The only thing that needs to be noticed is that when we lift $\mathrm{Bd}(\mathcal{UMC})$ to $\mathcal{UMC}_U$ using ψ, one fixed point in $\mathrm{Bd}(\mathcal{UMC})$ corresponding to the minor m of $\mathcal{L}$ is lifted to a leaf m which creates isolation on the appropriate sides of the endpoints of m in $\mathcal{UMC}_U \cap \mathbb{S}^1$. However by Lemma 4.8 fixed points in $\mathcal{UMC}$ are non-isolated on either side, hence the endpoints of m are non-isolated in $\mathcal{UMC}_U \cap \mathbb{S}^1$ as desired. □

Abusing the language, we will call a gap similar to $\mathcal{UMC}_U$ from Lemma 4.11 a *copy* of $\mathcal{UMC}$.

Lemma 4.12. *If m is a periodic minor then there are two copies of $\mathcal{UMC}$ that share m. Any non-degenerate minor with a periodic endpoint is a periodic minor.*

Proof. Our analysis implies that if m is a non-degenerate prime minor of period n, then in the associated lamination $\mathcal{L}$ there exists a finite invariant gap G such that m is one of its edges. The corresponding major $M = M_m \succ m$ is the edge of G that cuts the rest of G from the center of the circle (equivalently, from all fixed points of σ_d). At M there is a critical Fatou gap $\mathcal{C}(\mathcal{L})$ of $\mathcal{L}$ of period n. Set $U = \sigma_d(\mathcal{C}(\mathcal{L}))$. Then, by Lemma 4.11 there is a counterpart $\mathcal{UMC}_U$ of $\mathcal{UMC}$ in U such that m corresponds to the degenerate minor in $\mathcal{UMC}$ associated with angle 0, and otherwise the correspondence between $\mathcal{UMC}$ and $\mathcal{UMC}_U$ is a homeomorphism. Moreover, $\mathcal{UMC}_U$ is itself a gap of UML_d.

It follows that if we repeatedly apply the same construction, we will discover countable concatenations of copies of $\mathcal{UMC}$ whose existence is established in Lemma 4.11. This completely describes all minors with periodic endpoints. Indeed, let $m = \overline{ab} = m(\mathcal{L})$ be the minor of a unicritical lamination $\mathcal{L}$ and a be a periodic point of period k. By Lemma 3.15 there exists a critical Fatou gap W of period k such that $\sigma_d(W)$ has an edge ℓ of period k and an endpoint a. By the previous paragraph, ℓ is a minor and there are two copies of $\mathcal{UMC}$ adjacent at each other along ℓ. Moreover, the boundaries of these copies of $\mathcal{UMC}$ intersected with $\mathbb{S}^1$ are Cantor sets. It follows that $m = \ell$ as in UML_d there is no room for any other minor with endpoint a. $\square$

Definition 4.13. Let $\ell \neq \ell'$ be leaves of a lamination $\mathcal{L}$. If for some $k \in \mathbb{N}$, $\sigma_d^k(\ell)$ and ℓ' are contained in the same component of $\overline{\mathbb{D}} \setminus \ell$, then we say that ℓ maps under σ_d^k *toward* ℓ', otherwise that it maps *away from* ℓ' under σ_d^k.

The next definition is stated for the sake of completeness.

Definition 4.14. Let $\ell \neq \ell'$ be leaves of a lamination $\mathcal{L}$. A leaf $\ell_1 \in \mathcal{L}$ *separates* ℓ and ℓ' if ℓ and ℓ' lie on distinct *components* of $\overline{\mathbb{D}} \setminus \ell_1$.

We will need the next lemma which is in fact a laminational version of the fixed point theorem from continuum theory.

Lemma 4.15. *Let $\hat{\ell} \neq \ell$ be leaves of a σ_d-invariant lamination $\mathcal{L}$ moving in opposite directions by $\sigma_d^k, k > 0$. Then the following holds.*

(1) Suppose that ℓ and $\hat{\ell}$ move towards each other by the map σ_d^k and neither the leaves ℓ and $\hat{\ell}$, nor any leaf separating them, maps by σ_d^k to a leaf (including degenerate) with both endpoints in one of the boundary arcs of the strip $\mathcal{S}(\ell, \hat{\ell})$. Then there exists a σ_d^k-invariant leaf or gap that separates ℓ and $\hat{\ell}$.

(2) Suppose that ℓ and $\hat{\ell}$ move away from each other by the map σ_3^k. Then there exists a periodic leaf $y = \overline{ab}$ with $\sigma_3^k(a) = a$, $\sigma_3^k(b) = b$ that separates ℓ and ℓ'.

Proof. (1) Let T be the family of leaves of $\mathcal{L}$ that consists of ℓ and leaves u separating ℓ from $\hat{\ell}$ and either $\sigma_d^k(u) = u$ or u separates ℓ from $\sigma_d^k(u) \setminus u$. By continuity, T is closed. Also, T is nonempty as $\ell \in T$ by definition. Hence T contains a leaf t farthest from ℓ. If $\sigma_d^k(t) = t$ we are done; assume that $\sigma_d^k(t) \neq t$. By continuity and by the choice of t there must exist a gap H whose interior is separated from ℓ in $\mathbb{D}$ by t, and t is an edge of H. Let s be the edge of H defined as follows: if $\hat{\ell}$ is an edge of H, then $s = \hat{\ell}$, otherwise

s is the edge of H that separates $\hat{\ell}$ from $\sigma_d^k(s) \setminus s$. If $\sigma_d^k(s) = s$, we are done. Assume that $\sigma_d^k(s) \neq s$; then, since $s \notin T$ and by the assumptions, $\sigma_d^k(s) = t$, $\sigma_d^k(t) = s$, and H is σ_d^k-invariant.

(2) A gap G of $\mathcal{L}$ with edges $\ell, \hat{\ell}$ does not exist as otherwise the gap $\sigma_3^k(G)$ would strictly cover the gap G. Hence the family of leaves that consists of ℓ, $\hat{\ell}$, and the leaves that separate ℓ and $\hat{\ell}$ has at least one leaf that separates ℓ and $\hat{\ell}$. Clearly, this family of leaves is closed.

Let A be the set of leaves of that family that move away from $\hat{\ell}$ under σ_3^k such that for every leaf $m \in A$, if a leaf n separates ℓ and m, then n also moves away from $\hat{\ell}$ under the map σ_3^k. So, all the leaves in A move away from $\hat{\ell}$ under σ_3^k. Then $\overline{A}$ is a family of leaves, too; let $y \in \overline{A}$ be the leaf of $\overline{A}$ farthest from ℓ (i.e., every leaf $z \in A \setminus \{\ell, y\}$ separates ℓ from y). By continuity, either $y \in A$, or $\sigma_3^k(y) = y$ or $\sigma_3^k(y)$ is an endpoint of y. We claim that $y \notin A$. Indeed, suppose that y moves away from $\hat{\ell}$ under σ_3^k. There are two cases describing the dynamics of leaves and gaps close to y but separated by y from ℓ. First, it can be approached by leaves separated by y from ℓ. However by continuity this contradicts the choice of y. Second, y can be an edge of a gap G whose interior is separated by y from ℓ. Since by the assumption y maps away from $\hat{\ell}$ by σ_d^k, then the same applies to all edges of G, including the edge of G that separates y from $\hat{\ell}$. This contradicts the choice of y. Thus, $y \notin A$.

To complete the proof we need to rule out the case when σ_d^k flips y. Assume to the contrary, and consider cases. If there exists a leaf t close to y and separating ℓ and y, then t would move towards $\hat{\ell}$ under σ_3^k, a contradiction as by definition $t \in A$ moves away from $\hat{\ell}$. On the other hand, if y is an edge of a gap G such that G separates ℓ and y, then the fact that σ_d^k flips y implies that the edge of G that separates ℓ and y is mapped towards $\hat{\ell}$ and not away from $\hat{\ell}$, contradiction. Hence in either case we have a contradiction that proves the lemma. □

Let us now study properties of a minor m as a leaf of UML_d.

Lemma 4.16. *Let $m \in \mathrm{UML}_d$ be a non-degenerate non-periodic minor. If there is a sequence of leaves $\ell_i \in \mathcal{L}(m)$ with $\ell_i \succ m$ and $\ell_i \to m$, then m is the limit of periodic minors n_j with $n_j \succ m$ for any j.*

Proof. By Thurston's pullback construction the assumptions imply that we may assume that all ℓ_i's are pullbacks of majors. Hence there is a sequence of numbers $k_j \to \infty$ such that $\sigma_d^{k_j}(\ell_j) = m$. On the other hand $\sigma_d^{k_j}(m) \succ$

$m, \sigma_d^{k_j}(m) \neq m$ by properties of minors and because m is not periodic. Thus, $\sigma_d^{k_j}$ maps ℓ_j and m in opposite directions.

By Lemma 4.15 there exists a $\sigma_d^{k_j}$-invariant gap or leaf G of $\mathcal{L}(m)$ that separates m and ℓ_j. Now, consider the gap (or leaf) G. It has a leaf $\bar{g}$ that "faces" m (so that $\bar{g}$ separates m and the rest of G). We may also assume that the two arcs between the endpoints of $\bar{g}$ and the endpoints of m are shorter than $|m|$. Consider the forward orbit of $\bar{g}$. By Lemma 2.12 the length of any image of $\bar{g}$ is at least $|m|$. Hence $\bar{g} \succ \sigma_d^i(\bar{g})$ for some i then $\sigma_d^i(\bar{g})$ separates g and m (any other location of $\sigma_d^i(\bar{g})$ would imply that the length of $\sigma_d^i(\bar{g})$ is smaller than $|m|$). Since $\bar{g}$ is periodic, we may assume that some image $\bar{h}$ of $\bar{g}$ separates m and G and has no images $\sigma_d^j(\bar{h})$ such that $\bar{h} \succ \sigma_d^i(\bar{h})$.

It is easy to see that the d pullbacks $H_1, \dots, H_d$ of $\bar{h}$ that are leaves of $\mathcal{L}(m)$ are located outside of the Central Strip $C(m)$. Forward images of $\bar{h}$ cannot cross those pullbacks. By Lemma 3.3 the leaf $\bar{h}$ is a minor. Since it is periodic, it is a part of a copy of $\mathcal{UMC}_U$ of $\mathcal{UMC}$ for the appropriate Fatou gap U. □

Consider now non-degenerate and non-periodic minors.

Lemma 4.17. *Let $\mathcal{L}$ be a unicritical lamination with non-degenerate and non-periodic minor. Then there exists a lamination $\mathcal{L}' \subset \mathcal{L}$ with finite critical set $\mathcal{C}(\mathcal{L})$ such that the set $\sigma_d(\mathcal{C}(\mathcal{L}')) = H'$ is approached by leaves of $\mathcal{L}'$ from all sides and one of the following two cases holds:*

1. *$\mathcal{L}' = \mathcal{L}$;*
2. *the set $\mathcal{C}(\mathcal{L}')$ is a gap with $dn, n \geqslant 3$ edges and $\mathcal{L}$ is obtained from $\mathcal{L}'$ by inserting d sibling edges in $\mathcal{C}(\mathcal{L}')$ so that $\mathcal{L}$ has a critical $2d$-gon G inside $\mathcal{C}(\mathcal{L}')$ and then pulling G back along the backward orbit of $\mathcal{C}(\mathcal{L}')$ in $\mathcal{L}'$.*

Proof. We claim that the critical gap $\mathcal{C}(\mathcal{L})$ of $\mathcal{L}$ is finite. Indeed, if $\mathcal{C}(\mathcal{L})$ is infinite then it is periodic of period, say, N; thus, $\mathcal{C}(\mathcal{L})$ is an infinite gap that maps onto itself d-to-1 under σ_d^N. Yet then by Lemma 3.17 the associated minor must be periodic, a contradiction. Hence $\mathcal{C}(\mathcal{L})$ is a finite gap with dn edges, and $n \geqslant 2$ ($n = 1$ would imply that the minor m is degenerate, a contradiction).

Consider several cases. Suppose than $n \geqslant 3$ so that $\sigma_d(\mathcal{C}(\mathcal{L})) = H$ is a gap (not a leaf). We claim that then each edge of H is a limit edge of leaves of $\mathcal{L}$. Indeed, if ℓ is an edge of H isolated in $\mathcal{L}$, then there is a gap G that shares ℓ with H. Since $\mathcal{C}(\mathcal{L})$ is a finite gap, then G cannot be infinite because any infinite gap must have an image on which σ_d is not 1-to-1 while the only gap of $\mathcal{L}$ on which σ_d is not 1-to-1 is a finite gap $\mathcal{C}(\mathcal{L})$. Thus, G is finite.

Suppose that G at some point maps onto $\mathcal{C}(\mathcal{L})$. Then on the next step G maps to H; we may assume that $\sigma_d^N(G) = H$ for some N. it follows that G is a dn-gon while H is an n-gon. Therefore σ_d^N cannot act on G and H "swapping" them (in other words, σ_d^N cannot be "flipping" ℓ). Properties of laminations imply that σ_d^N cannot map ℓ to itself as identity either. So, σ_d^N maps ℓ to an edge $\ell' \neq \ell$ of H. Properties of laminations imply now that H maps under σ_d^N to a gap adjacent to H along ℓ', etc. Thus, iterating σ_d^N and using properties of invariant laminations we will find an infinite concatenation of images of H which is clearly impossible (e.g., because H is (pre)periodic by Lemma 2.14).

So, G is not precritical; hence, by Lemma 2.14 G is preperiodic and never maps to H. The same can be said about H, but for an even more trivial reason: H has n edges and $\mathcal{C}(\mathcal{L})$ had $dn > n$ edges, hence H is not precritical and by Lemma 2.14 H is (pre)periodic. As we apply σ_d over and over to the union $G \cup H$, both gaps G and H stay away from the critical gap $\mathcal{C}(\mathcal{L})$ and eventually map onto two distinct adjacent finite periodic gaps $\widehat{G}$ and $\widehat{H}$ that share a leaf $\hat{\ell}$. Let us show that this is impossible. Indeed, by Lemma 2.15 at every edge of G' there is an image of H' attached to G'. Then we can say the same about each such image of H', and so on. This yields infinite family of images of $\widehat{G}$ or $\widehat{H}$, a contradiction. So, if H is a gap, then all edges of H are limit edges in $\mathcal{L}$. Then we can set $\mathcal{L}' = \mathcal{L}$ and case (1) holds.

Otherwise $\mathcal{C}(\mathcal{L})$ is a $2d$-gon and $\sigma_d(\mathcal{C}(\mathcal{L}))$ is a leaf m. If m is a limit leaf from both sides, then, again, we can set $\mathcal{L}' = \mathcal{L}$ and case (1) holds. Suppose that $H = m$ is an edge of a gap G of $\mathcal{L}$. As before, G cannot be infinite because an infinite gap must have an image which maps forward no 1-to-1, and the only way it is possible under the circumstances is when an image of G equals $\mathcal{C}(\mathcal{L})$, a contradiction since $\mathcal{C}(\mathcal{L})$ is a $2d$-gon. This, G is a finite gap.

We claim that G never maps onto $\mathcal{C}(\mathcal{L})$. Indeed, if it happens then m must be periodic, a contradiction; the same reason implies that G never maps onto itself. By Lemma 2.14 G is preperiodic. There are d immediate preimages of G denoted $G_1, \dots, G_d$; these are gaps attached to the appropriate edges of $\mathcal{C}(\mathcal{L})$ denoted $\bar{g}_1, \dots, \bar{g}_d$, resp. Then $\bar{g}_1, \dots, \bar{g}_d$ are isolated sibling leaves of $\mathcal{L}$ with the same image m. Evidently, if we remove their backward obits from $\mathcal{L}$ we will obtain a new *closed* (because $\bar{g}_i$'s are isolated) invariant lamination $\mathcal{L}' \subsetneqq \mathcal{L}$ in which G remains a gap (no edge of G belongs to the union of the backward orbits of $\bar{g}_i$'s because G is not periodic). By construction, the critical gap $\mathcal{C}(\mathcal{L}')$ of the new lamination $\mathcal{L}'$ coincides with the convex hull of $\bigcup G_i$ or, equivalently, is the union of $\mathcal{C}(\mathcal{L})$ and $\bigcup G_i$ from which all leaves $\bar{g}_i$ are removed.

Now, the just described critical set $\mathcal{C}(\mathcal{L}')$ of the new lamination $\mathcal{L}'$ is a finite gap with dk edges and $k \geqslant 3$. Hence by the above proven, all edges of $G = \sigma_d(\mathcal{C}(\mathcal{L}'))$ are limit edges of $\mathcal{L}'$ as claimed. □

Finally, we can prove that UML_d is a q-lamination.

Theorem 4.18. *The lamination* UML_d *is a q-lamination.*

Proof. Consider a minor $m = m(\mathcal{L}) \in \mathrm{UML}_d$. If it is periodic then by Lemma 4.12 it is isolated in UML_d and is disjoint from all other minors. Assume that endpoints of m are non-periodic. By Lemma 4.17 there are two possibilities. First, m is the limit leaf of leaves of $\mathcal{L}(m)$ from both sides. Then by Lemma 4.16 and Lemma 3.5(1) m is the limit of minors from both sides. Finally, assume that $m = m(\mathcal{L})$ is not the limit of leaves of $\mathcal{L}$ from one side. Then by Lemma 4.17 there exists a lamination $\mathcal{L}'$ with a critical gap $\mathcal{C}(\mathcal{L}')$ with $dn, n \geqslant 3$ edges such that m is an edge of the gap $\sigma_d(\mathcal{C}(\mathcal{L}'))$ which is approached by leaves of $\mathcal{L}'$ from the outside of $\sigma_d(\mathcal{C}(\mathcal{L}'))$. By Lemma 3.5(1) m is a minor, approached by other minors from the outside of $\sigma_d(\mathcal{C}(\mathcal{L}'))$. Thus, any non-periodic minor is either itself approximated from all sides by other minors, or is an edge of a finite gap whose all edges are approached ny minors from the outside of $\sigma_d(\mathcal{C}(\mathcal{L}'))$. Evidently, all this implies that UML_d is a *q-lamination.* □

Lemma 4.19 is a UML_d-version of the density of hyperbolicity.

Lemma 4.19 (Fatou conjecture)**.** *Copies of* $\mathcal{UMC}$ *are dense in* UML_d*.*

Recall that the concept of "under" is introduced in Definition 2.8.

Proof. It suffices to prove that periodic minors are dense in UML_d. Let m be a non-periodic non-degenerate minor. By Lemma 4.16 if m is the limit of leaves $\ell_i \succ m$ then m is the limit of periodic minors ℓ_i' such that $\ell_i' \succ m$. Now, if m is not the limit of leaves $\ell_i \succ m$ with $\ell_i \in \mathcal{L}(m)$ then m is an edge of a finite gap G in UML_d and no edge of this gap is "under" m. Then m is the limit of minors all of which are "under" m. Moreover, these minors can be chosen to be limits of leaves of their respective laminations located so that the minors are "under" those leaves; hence, again by Lemma 4.16, we can find periodic minors in a small neighborhood of m located "under" m. □

5. Constructing UML_d

Let $\mathcal{L}$ be a *unicritical σ_d-invariant lamination* with a unique *critical set* $\mathcal{C}(\mathcal{L})$. If the σ_d-orbit of $\mathcal{C}(\mathcal{L})$ is finite, then $\mathcal{C}(\mathcal{L})$ is either *periodic or strictly preperiodic.*

Definition 5.1. A lamination $\mathcal{L}$ is called *Misiurewicz lamination* if $\mathcal{C}(\mathcal{L})$ is strictly *pre-periodic.* In such a case $\mathcal{C}(\mathcal{L})$ is called a *Misiurewicz critical set.*

By definition, a Misiurewicz lamination cannot have a critical leaf with a periodic endpoint or with non-preperiodic endpoints.

Lemma 5.2. *Let $\mathcal{L}$ be a Misiurewicz lamination with critical set $\mathcal{C}(\mathcal{L})$. Then all gaps of $\mathcal{L}$ are finite. Moreover, there exists a q-lamination $\mathcal{L}' \subset \mathcal{L}$ with critical set $\mathcal{C}(\mathcal{L}')$ such that all gaps of $\mathcal{L}'$ are finite, $\mathcal{C}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L}')$, and one of the following holds:*

(i) *$\mathcal{C}(\mathcal{L})$ is an all-critical d-gon and $\mathcal{C}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L}')$;*
(ii) *$\mathcal{C}(\mathcal{L})$ is a critical $2d$-gon that map onto an edge of $\sigma_d(\mathcal{C}(\mathcal{L}'))$ and $\mathcal{C}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L}')$;*
(iii) *$\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}')$.*

In particular, all leaves of $\mathcal{L}'$ are non-isolated from at least one side.

Proof. Clearly, $\mathcal{C} = \mathcal{C}(\mathcal{L})$ must be finite (if $\mathcal{C}$ is infinite then $\mathcal{C}$ is periodic but we have assumed that $\mathcal{C}$ is strictly pre-periodic). Also, all gaps of $\mathcal{L}$ are finite. Indeed, if G is an infinite gap of $\mathcal{L}$, then by Theorem 1.20 an eventual image G' of G is periodic. Consider several cases. First, the first return map of G' to itself may be of degree greater than 1. Then we may assume that G' itself coincides with the critical set $\mathcal{C}(\mathcal{L})$ of lamination $\mathcal{L}$, contradicting the assumption that $\mathcal{C}(\mathcal{L})$ is strictly preperiodic. Otherwise the first return map of G' to itself may be of degree 1. However in that case by Theorem 1.21 there must exist a critical leaf of $\mathcal{L}$ with a periodic endpoint or with both endpoints not preperiodic, and both possibilities contradict the assumption that $\mathcal{L}$ is a Misiurewicz lamination. Thus, all gaps of $\mathcal{L}$ are finite.

If G and H are gaps that never map to $\mathcal{C}$, then G and H cannot share a common leaf. Indeed, if G and H share a common leaf ℓ, we can apply σ_d over and over to the union $G \cup H$ and see that G and H stay away from $\mathcal{C}$ and eventually map onto two distinct adjacent finite periodic gaps G' and H' that share a leaf ℓ'. By Lemma 2.15 at every edge of G' there is an image of H' attached to G', at every edge of these images of H' there is an image of G', and so on. This yields infinite family of concatenated images of G' and H', a contradiction.

Finally, if G is an image of $\mathcal{C}$, then by Lemma 4.17 only the three cases from Lemma 5.2 are possible. □

Now we can extend Lemma 3.5.

Lemma 5.3. *If $n \in \mathbb{S}^1$ is a strictly preperiodic point of preperiod greater than 2 and $m \in \mathrm{UML}_d$ is a minor such that $m \succ n$ and n is not an endpoint of m, then $m \in \mathcal{L}(n)$.*

Proof. Clearly, $\mathcal{L}(n)$ is a Misiurewicz lamination. By Lemma 5.2 all gaps of $\mathcal{L}(n)$ are finite. We claim that m does not intersect any leaf of $\mathcal{L}(n)$. Indeed, by Lemma 2.17 no image of m crosses a (critical) major of $\mathcal{L}(n)$. By construction, no image of m crosses a leaf of $\mathcal{L}(n)$. If, contrary to what is claimed, m is not a leaf of $\mathcal{L}(n)$ then m is a diagonal of a gap G of $\mathcal{L}(n)$. By the above G is finite. By the Central Strip Lemma 2.17 no image of G equals $\mathcal{C}(\mathcal{L})$. Hence by Lemma 2.14 G eventually maps to a finite periodic gap G', and the image of m is a diagonal of G'. However then by Lemma 2.15 m will have crossing images, a contradiction. Thus, m is a leaf in $\mathcal{L}(n)$. □

The quadratic version of the next theorem is due to Lavaurs [Lav89].

Theorem 5.4. *If two leaves in UML_d have the same period and are $\succ$-comparable, then these two leaves are separated in $\mathbb{D}$ by a leaf in UML_d of lower period.*

In the proof of this theorem we will use the previously introduced notation.

Proof. Let $m \succ n$ be minors of period p; then their endpoints are of period p. Choose a preperiodic point $z \in H(n)$ of preperiod greater than 2 such that the period of a periodic eventual image of z is larger than p. Consider the pullback lamination $\mathcal{L}(z)$ generated by z viewed as a minor. Since the critical set $D(z)$ of $\mathcal{L}(z)$ (i.e., the convex hull of all preimages of z under σ_d) is precritical, then $\mathcal{L}(z)$ is a Misiurewicz lamination. By Lemma 5.2 there exists a Misiurewicz q-lamination $\mathcal{L}$ with the critical set $\mathcal{C}(\mathcal{L})$ such that $z \in \sigma_d(\mathcal{C}(\mathcal{L})) = \delta$. We claim that all edges of δ are minors. Indeed, if ℓ is an edge of δ then it is easy to verify that it satisfies all the conditions of Lemma 3.3 and, therefore, is a minor. Thus, δ is either a gap of UML_d, or a leaf of UML_d, or just a point. Since by Theorem 4.18 UML_d is a q-lamination, $\delta \cap \mathbb{S}^1 \subset H(n)$.

By Lemma 5.3 m and n are leaves of $\mathcal{L}(z)$. We claim that they are actually leaves of $\mathcal{L}$. Indeed, by Lemma 5.2 $\mathcal{L} \subset \mathcal{L}(z)$ and all gaps of $\mathcal{L}$ are finite. If, say, m is not a leaf of $\mathcal{L}$, then m is a diagonal of a gap G of $\mathcal{L}$. If G is (pre)periodic, then a forward image of m becomes a diagonal of a periodic gap of $\mathcal{L}$. However, by Lemma 2.15 this implies that forward images of m cross, a contradiction. Hence G is not (pre)periodic. Then by Lemma 2.14 G must be precritical and $\sigma_d^s(m) \subset \sigma_d^s(G) = \mathcal{C}(\mathcal{L})$ for some s. However then $\sigma_d^{s+1}(m) \subset \delta$, and since $\delta \cap \mathbb{S}^1 \subset H(n)$ we see that $\sigma_d^{s+1}(m) \cap \mathbb{S}^1 \subset H(n) \subset H(m)$. Since this contradicts the property of minors listed as property (3) in Lemma 3.3, we see that this is impossible. Similarly, n is a leaf of $\mathcal{L}$.

Set $\mathcal{C}(\mathcal{L}) = \mathcal{C}$, $\psi(\mathcal{C}) = \mathcal{C}'$, $\psi(m) = m'$, $\psi(n) = n'$ and $\psi(\delta) = \delta'$; these are points of $\mathcal{H}(\mathcal{L})$. Then $\mathcal{H}(\mathcal{L}) \setminus \mathcal{C}'$ has ld components for some $l \geqslant 1$.

We claim that m and n are not edges of the same gap of $\mathcal{L}$. Indeed, suppose that m and n are edges of a gap G of $\mathcal{L}$. Then G is a periodic gap of $\mathcal{L}$. By Lemma 5.2 G is finite, and by Lemma 2.15 the first return map on G acts transitively on edges of G. Hence m maps to n under some power of σ_d, a contradiction with property (3) of minors listed in Lemma 3.3. Thus, m and n are not edges of the same gap of $\mathcal{L}$. Now, the critical set of $\mathcal{L}(z)$ is a d-gon $D(z)$ with d critical edges. By construction $D(z) \subset \mathcal{C}(\mathcal{L})$. On the other hand, pullbacks of $D(z)$ are dense in $\mathcal{L}(z)$ by the construction of pullback laminations. Since $\mathcal{L} \subset \mathcal{L}(z)$, it follows that pullbacks of $\mathcal{C}(\mathcal{L})$ are dense in $\mathcal{L}$. Since m and n are not edges of the same gap of $\mathcal{L}$, it follows that for a minimal k, a σ_d^k-pullback $\mathcal{C}_{-k}$ of $\mathcal{C}(\mathcal{L})$ separates m and n and is such that $\sigma_d^k(\mathcal{C}_{-k}) = \mathcal{C}$. By our choice, k is the least number for which such gap exists; since m and n are p-periodic, $k \leqslant p-1$. Consider two cases.

(1) Let $k < p-1$. Then $\sigma_d^{k+1}(\mathcal{C}_{-k}) = \delta$ and so $\mathcal{C}_{-k}$ maps away from m by σ_d^{k+1}; also, by the property (3) of minors listed in Lemma 3.3, m maps away from $\mathcal{C}_{-k}$. Hence by Lemma 4.15 there is a leaf t of $\mathcal{L}$ that separates m and $\mathcal{C}_{-k}$ and maps to itself under σ_d^{k+1}. Moreover, the fact that t maps to itself implies that t cannot be precritical (even if $\mathcal{L}$ does have critical leaves, they must be preperiodic by the choice of z, and, hence, t, that conjecturally maps to one of them, will not be able to then map to its endpoint). We conclude that t maps onto itself under σ_d^{k+1} so that its endpoints are σ_d^{k+1}-fixed. Let us now choose the least number j such that there exists a leaf t' of period j that separates m and n. It follows that $j \leqslant k+1 < p$. Moreover, we may assume that t' is chosen among all such leaves as the closest to m. We claim that t' is a minor. Indeed, by Lemma 3.3 the only reason why t'

may not be a minor is that $\sigma_d^i(t')$ can be "under" t' for some $i < j$ (see Definition 2.8 for clarification). However then by Lemma 4.15 there exists a leaf t'' that separates m and t' and is such that $\sigma_d^i(t'') = t''$, a contradiction with the choice of t'. So, t' is a minor as desired.

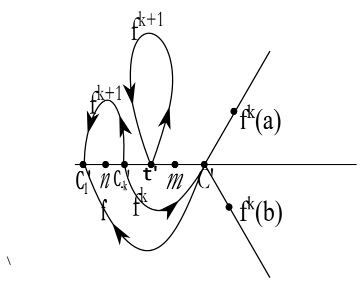

FIGURE 3. Case1: $k + 1 < p$

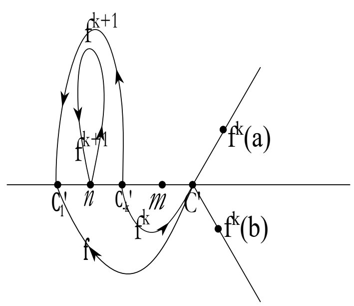

FIGURE 4. Case2: $k + 1 = p$

(2) Let $k = p - 1$. Then σ_d^p maps n to itself while mapping $\mathcal{C}_{-k}$ "under" itself. On the other hand, the endpoints a, b of n are σ_d^p-fixed which implies that small one-sides neighborhoods of a, b expand and cover themselves. Choose the edge $\ell = \overline{xy}$ of $\mathcal{C}_{-k}$ that separates n and the rest of $\mathcal{C}_{-k}$ (if $\mathcal{C}_{-k}$ is a leaf, set $\overline{xy} = \mathcal{C}_{-k}$). We may assume that the two circle arcs on the boundary of the strip between n and ℓ are (x, a) and (b, y). Then it follows from the choice of k as the least number such that the corresponding pullback of $\mathcal{C}(\mathcal{L})$ separates m and n implies that the entire strip between n and ℓ maps under σ_d, σ_d^2, $\dots$, σ_d^k so that its image is located in one of the components of $\overline{\mathbb{D}} \setminus \mathcal{C}(\mathcal{L})$. This implies that on the last step, i.e. under σ_d^p, the circular order among point x, a, b, y remains the same. Therefore, the σ_d^p-image of $\mathcal{C}_{-k}$ cannot be δ which is located *"under"* n while in the beginning n as located "under" $\ell = \overline{xy}$. This contradiction shows that $k < p - 1$ and, hence, case (1) always applies. $\square$

Recall that by Lemma 3.19 a critical n-periodic Fatou gap U of a unicritical lamination $\mathcal{L}$ has one n-periodic major and otherwise $d-2$ periodic points on its boundary that are not endpoints of any edge of U. Therefore, the minor $m(\mathcal{L})$ of $\mathcal{L}$ is a unique n-periodic non-degenerate edge of $\sigma_d(U)$ while all other n-periodic points from $\mathrm{Bd}(\sigma_d(U))$ are disjoint from all edges of $\sigma_d(U)$. In the forthcoming arguments we will use the semiconjugacy $\psi : \mathrm{Bd}(\sigma_d(U)) \to \mathbb{S}^1$ that collapses all edges of $\sigma_d(U)$ to points and seminconjugates $\sigma_d^n|_{\sigma_d(U)}$ and σ_d.

Lemma 5.5. *Let $\mathcal{L}$ be a unicritical lamination with critical Fatou gap U of period n. Then all non-degenerate edges of $\sigma_d(U)$ are non-degenerate minors. In particular, periodic minors are either disjoint from $\sigma_d(U)$¡ or contained in it and then their periods are multiples of n. Moreover, all points of $\mathrm{Bd}(\sigma_d(U))$ that are of period n and are not the endpoints of the minor $m(\mathcal{L})$ are degenerate minors.*

Proof. Let ℓ be a non-degenerate edge of $\sigma_d(U) = V$. Consider the convex hull C of $\sigma_d^{-1}(\ell)$. Clearly, C and its forward orbit (which begins with ℓ) is a forward invariant lamination. Then Thurston's pullback construction yields a lamination with minor ℓ as desired. It follows that if a minor m' is non-disjoint from the interior of V then in fact $m' \subset V$. One may say that the gap V with all the minors contained in it forms a part of UML_d. Since all edges of V are pullbacks of $m(\mathcal{L})$, the claim about periodic minors from the lemma easily follows.

Let $x \in \mathrm{Bd}(V)$ be a point of period n but not an endpoint of $m(\mathcal{L}) = m$ (m is a non-degenerate edge of V such that $\sigma_d^n(m) = m$). Assume, by way of contradiction, that there exists a non-degenerate minor $\overline{y} = m(\mathcal{L}')$ with an endpoint x (here $\mathcal{L}'$ is some unicritical lamination). By the previous paragraph $\overline{y} \subset V$. Construct the lamination $\mathcal{L}(\overline{y})$ as usual: first construct the critical set which is the convex hull $\mathcal{C}(\overline{y})$ of $\sigma_d^{-1}(\overline{y})$ and then pull it back in the compatible with itself way.

Evidently, all pullbacks of $\mathcal{C}(\overline{y})$ as well as images of $\overline{y}$ are contained in the corresponding images and pullbacks of V that are gaps of $\mathcal{L}$. The same applies to the leaves from the closure of these pullbacks of $\mathcal{C}(\overline{y})$ and images of $\overline{y}$, i.e. to all leaves of $\mathcal{L}(\overline{y})$. Thus, the restriction $\mathcal{L}(\overline{y})|_V$ of $\mathcal{L}(\overline{y})$ onto V satisfies all properties of invariant laminations except that instead of σ_d we need to consider σ_d^n. Applying ψ to it, we will obtain a σ_d-invariant unicritical lamination with $\psi(x)$ belonging to the image of its critical set. However then by Lemma 3.14 no leaf of this lamination may have an endpoint $\psi(x)$, a contradiction with the existence of the leaf $\psi(\overline{y})$. □

Now, suppose that for some k all periodic minors of periods at most k, non-degenerate and degenerate, are already constructed. Denote their union by A_k. Take a component B of $\overline{\mathbb{D}} \setminus A_k$. Let z be a periodic point of period $k+1$ that belongs to B. By Lemma 3.15 there is a lamination $\mathcal{L}$ with a critical Fatou gap U of period $k+1$ such that $z \in \sigma_d(U) = V$. By Lemma 5.5 the entire V is contained in B. Either z is an endpoint of the minor $m(\mathcal{L})$, or z is a degenerate minor located "under" $m(\mathcal{L})$. All of this can happen more than once inside B and will exhaust all periodic points of period $k+1$ that belong to B.

We need to make another observation based upon our study of the Unicritical Main Cardioid $\mathcal{UMC}$. By construction, leaves from $\mathcal{UMC}$ of periods at most k are included in A_k, and no minor intersects the interior of $\mathcal{UMC}$. Hence each minor of period $k+1$ contained in B is located between two consecutive σ_d-fixed points, and the same applies to any gap similar to the gap V considered above. Together with the previous paragraph, this yields an algorithm for constructing UML_d. The quadratic version of this algorithm was proved by Lavaurs in [Lav89].

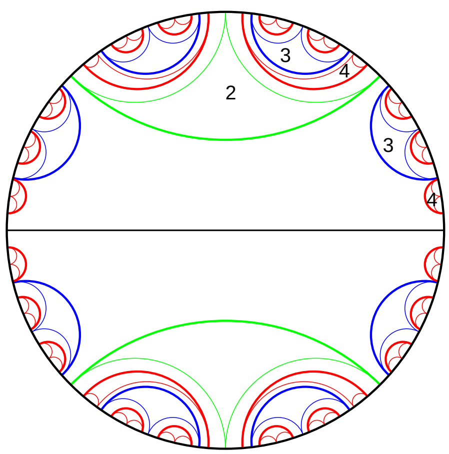

FIGURE 5. *Construction of the periodic minors of the Unicritical Minor Lamination* UML_3 *up to period* 4*; the gaps for different periods are coded by colors, and some periods are labeled.*

6. The Lavaurs Algorithm in the Unicritical Case

Let us construct leaves ofUML_d that have periodic endpoints; the closure of their collection is UML_d. The map σ_d has $d-1$ fixed points on the circle: $0, \frac{1}{d-1}, \frac{2}{d-1}, \dots, \frac{d-2}{d-1}$. By Lemma 3.14 they are *degenerate minors*. We place a *dot* on the circle at each of these points.

Now, take points of *period* 2 under σ_d. The map σ_d has d^2-1 points of period 2 on the circle given by $\{\frac{k}{d^2-1} | k = 0, 1, 2 \dots, d^2-1\}$. Some of these will correspond to *degenerate minors* and some of them will be the endpoints of *non-degenerate minors*. It follows that for any *non-degenerate* minor γ of period 2, there exist $d-2$ *degenerate* minors δ_i such that $\gamma \succ \delta_i$ for every i. Moreover, any non-degenerate minor of period 2 is located between two consecutive fixed points. Take the smallest point of period 2: $\frac{1}{d^2-1}$, connect it to the point $\frac{d}{d^2-1}$, and place a *dot* at the intermediate points $\frac{2}{d^2-1}, \frac{3}{d^2-1}, \dots, \frac{d-1}{d^2-1}$ (by Lemma 5.5 they are *degenerate minors*). Continue this way around the circle.

In general, suppose the periodic points on the circle of period less than k have been *connected* or *dotted*. Connect those of period k starting with the smallest point p of period k. Let $q_1, q_2, \dots, q_{d-1}$ be the next $d-1$ points *ordered anticlockwise* and belonging to the same component of $\overline{\mathbb{D}}$ with minors of periods less than k removed as p. Connect p to q_{d-1} and place a *dot* at the intermediate points: $q_1, \dots, q_{d-2}$. Choose the next available (not yet *connected* or *dotted*) point of period k and proceed in the same way; continue similarly around the circle. We do this for periodic points on the circle in the order of increasing periods.

Acknowledgements. The authors are grateful to the referee for useful remarks and comments.

References

[AF00] L. Alseda, N. Fagella, *Dynamics on Hubbard trees*, Fund. Math. **164** (2000), 115–141.

[AKLS09] A.Avila, J. Kahn, M. Lyubich, W.Shen, *Combinatorial rigidity for unicritical polynomials*, Ann. of Math (2) **170** (2009),783–797.

[BL02] A. Blokh, G. Levin, *Growing trees, laminations and the dynamics on the Julia set*, Ergodic Th. and Dynam. Sys. **22** (2002), 63–97.

[BMOV13] A. Blokh, D. Mimbs, L. Oversteegen, K. Valkenburg, *Laminations in the language of leaves*, Trans. Amer. Math. Soc. **365** (2013), 5367–5391.

[B008] A. Blokh, L. Oversteegen *Wandering Gaps for Weakly Hyperbolic Polynomials*, in: Complex Dynamics: Families and Friends, edited by D. Schleicher, AK Peters (2008), 139–168.

[B0PT17] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Combinatorial models for spaces of cubic polynomials*, C.R. Math. Acad. Sci (Paris) **355** (2017), 590–595.

[BOPT20] A. Blokh, L. Oversteegen, R. Ptacek, V. Timorin, *Laminational models for spaces of polynomials of any degree*, Memoirs of the AMS **265** (2020), no. 1288.

[Bra86] B. Branner, *The parameter space for cubic polynomials*, in: Chaotic Dynamics and Fractals, edited by Barnsley and Demko, Academic Press **9** (1986), 169–179.

[BrHu88] B. Branner, J. Hubbard, *The iteration of cubic polynomials, Part I: The global topology of parameter space*, Acta Math. **160** (1988), 143–206.

[Bro60] M. Brown, *A proof of the generalised Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

[Bro61] M. Brown, *The monotone union of open n-cells in an open n-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814.

[Chi07] D. Childers, *Wandering polygons and recurrent critical leaves*, Ergodic Th. and Dynam. Sys. **27** (2007), 87–107.

[Dou93] A. Douady, *Descriptions of compact sets in* $\mathbb{C}$, in: Topological methods in modern mathematics (Stony Brook, NY, 1999) edited by L. Goldberg and A. Philips, Publish or Perish, Houston, TX (1993), 429–465.

[DH8485] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes I,II*, Publications Mathématiques d'Orsay **84-02** (1984), **85-04** (1985).

[KL09a] J.Kahn, M.Lyubich *The quasi-additivity law in confromal geometry*, Ann. of Math.(2)**169**(2009), 561-593.

[KL09b] J.Kahn, M.Lyubich *Local connectivity of Julia sets for unicritical polynomials*, Ann. of Math.(2)**170**(2009), 413–426.

[Kaf06] A. Kaffl, *Hubbard Trees and Kneading Sequences for Unicritical and Cubic Polynomials*. PhD thesis, International University Bremen, 2006.

[Kiw02] J. Kiwi, *Wandering orbit portraits*, Trans. Amer. Math. Soc. **354** (2002), 1473–1485.

[Lav89] P. Lavaurs, *Systèmes dynamiques holomorphes*, Explosion de points périodiques paraboliques, Thèse de doctorat, Université Paris-Sud, Orsay (1989).

[Lev98] G. Levin, *On backward stability of holomorphic dynamical systems*, Fundamenta Mathematicae, **158** (1998), 97–107.

[Mil00] J. Milnor, *Dynamics in One Complex Variable*, Third Edition, Annals of Mathematical Studies **160**, Princeton University Press (2006).

[Mil08] J. Milnor, *Cubic polynomial maps with periodic critical orbit I*, in: Complex Dynamics: Families and Friends, edited by D. Schleicher, AK Peters, Wellesley, MA (2009), 333–411.

[Sch00] D. Schleicher, *Rational parameter rays of the Mandelbrot set*, Astérisque **261** (2000),405–443.

[Sch04] D. Schleicher, *On fibers and local connectivity of Mandelbrot and Multibrot sets*, in: Fractal geometry and applications I, edited by M. Lapidus and M. van Frankenhuysen, Proc. Sympos. Pure Math. **72**, Amer. Math Soc., Providence, RI (2004), 477–517.

[Sch09] D. Schleicher, *Appendix: Laminations, Julia sets and the Mandelbrot set*, in: Complex Dynamics: Families and Friends, edited by D. Schleicher, AK Peters, Wellesley, MA(2009), 111–137.

[Thu85] W. Thurston, *On the Geometry and Dynamics of Iterated Rational Maps*, edited by D. Schleicher and N. Selinger, in: Complex Dynamics : Families and Friends, edited by D. Schleicher, AK Peters, Wellesley, MA (2009), 3–109.

[TBGHLLT19] W. Thurston, H. Baik, Y. Gao, J. Hubbard, T. Lei, K. Lindsey, D. Thurston, *Degree-d-invariant laminations*. In: What's next? The Mathematical Legacy of William P. Thurston, edited by Dylan Thurston. Annals of Mathematics Studies **205**, Princeton University Press, Princeton, NJ (2020), 259–325.

(Sourav Bhattacharya and Alexander Blokh) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

Email address, Sourav Bhattacharya: `sourav@uab.edu`

Email address, Alexander Blokh: `ablokh@uab.edu`

(Dierk Schleicher) Aix-Marseille Université, Institut de Mathématiques de Marseille, 163 Avenue de Luminy Case 901, 13009 Marseille, France

Email address, Dierk Schleicher: `dierk.schleicher@univ-amu.fr`