OVER-ROTATION NUMBERS FOR UNIMODAL MAPS

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Abstract. We introduce *twist unimodal maps* of the interval and describe their structure. Sufficient conditions for the growth of over-rotation interval in families of maps are given.

0. Introduction

The present paper is related to a few classical results in one-dimensional dynamics, namely to a Sharkovskiĭ's theorem on periods of interval maps and Misiurewicz's results concerning periodic points of degree one circle maps.

0.1 Cycles on the interval and on the circle. To state the Sharkovskiĭ theorem let us first introduce the *Sharkovskiĭ ordering* for positive integers:

$$(*)\qquad 3 \succ_S 5 \succ_S 7 \succ_S \dots \succ_S 2\cdot 3 \succ_S 2\cdot 5 \succ_S 2\cdot 7 \succ_S \dots \succ_S 8 \succ_S 4 \succ_S 2 \succ_S 1$$

If $m \succ_S n$, say that m is *sharper* than n. Let $Sh(k)$ be the set of all integers m with $k \succeq_S m$, let $Sh(2^\infty)$ be the set $\{1, 2, 4, \dots\}$, and let $P(\varphi)$ be the set of (minimal) periods of cycles of a map φ.

Theorem S[S]. *If $g : [0, 1] \to [0, 1]$ is continuous, $m \succ_S n$ and $m \in P(g)$ then $n \in P(g)$ and so there exists $k \in \mathbb{N} \cup 2^\infty$ with $P(g) = Sh(k)$.*

Similar results for circle maps of degree one are due to Misiurewicz [M1] who used the notion of the *rotation number*. This notion was first introduced by Poincarè [P] for circle homeomorphisms, then extended to circle maps of degree one by Newhouse, Palis and Takens [NPT], and then studied in [BGMY], [I], [CGT], [M1], [M2], [ALMC], [ALMM] (see [ALM2] with an extensive list of references).

In fact, one can define rotation numbers in a variety of cases using the following approach ([MZ], [Z]). Let X be a compact metric space, $\varphi : X \to \mathbb{R}$ be a bounded

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measurable function, $f : X \to X$ be a continuous map. Then for any x the set $I_{f,\varphi}(x)$ of all limits of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x)$ is called the *φ-rotation set* of x. Clearly $I_{f,\varphi}(x)$ is a closed interval. If $I_{f,\varphi} = \{\rho_\varphi(x)\}$ then the number $\rho_\varphi(x)$ is called the *φ-rotation number* of x; if x is a periodic point then its rotation number $\rho_\varphi(x)$ is well-defined. Properties of these and related sets in case when X is an interval are studied in [B2] for a variety of functions φ and contain much information about the dynamics even for an arbitrary function φ.

For functions φ related to the dynamics of the map f one might get additional results about φ-rotation sets; e.g., this happens for rotation numbers in the circle degree one case [M1]. Let $f : S^1 \to S^1$ be a map of degree 1, $\pi : \mathbb{R} \to S^1$ be the natural projection which maps the interval $[0, 1)$ onto the whole circle. Fix a lifting F of f. Define $\varphi_f : S^1 \to \mathbb{R}$ so that $\varphi_f(x) = F(X) - X$ for any point $X \in \pi^{-1}x$; then φ_f is well-defined, the classical rotation set of a point z is $I_{f,\varphi_f}(z) = I_f(z)$ and the classical rotation number of z is $\rho_{f,\varphi_f}(z) = \rho(z)$ whenever it exists.

The *rotation set* of the map f is $I_f = \cup I_f(x)$; it follows from [NPT],[I] that I_f is a closed interval (cf. [B2]). The sum $\sum_{i=0}^{n-1}\varphi_f(f^i x) = m$ taken along the orbit of an n-periodic point x is an integer which defines a pair $(m, n) \equiv rp(x)$ called the *rotation pair* of x; denote the set of all rotation pairs of periodic points of f by $RP(f)$. For real $a \le b$ let $N(a, b) = \{(p, q) \in \mathbb{Z}^2_+ : p/q \in (a, b)\}$ (in particular $N(a, a) = \emptyset$). For $a \in \mathbb{R}$ and $l \in \mathbb{Z}_+ \cup \{2^\infty\}$ let $Q(a, l)$ be empty if a is irrational; otherwise let it be $\{(ks, ns) : s \in Sh(l)\}$ where $a = k/n$ with k, n coprime.

Theorem M1 [M1]. *For a continuous circle map f of degree 1 such that $I_f = [a, b]$ there exist $l, r \in \mathbb{Z}_+ \cup \{2^\infty\}$ such that $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$.*

The choice of φ_f is crucial for Theorem M1. By [B1, BM0] an appropriate choice of $\varphi = \varphi_f$ leads to results for interval maps similar to Theorem M1; one can even derive Theorem S from them. More precisely, let $f : [0, 1] \to [0, 1]$ be continuous, $Per(f)$ be its set of periodic points, and $Fix(f)$ be its set of fixed points. It is easy to see that if $Per(f) = Fix(f)$ then $\omega(y)$ is a fixed point for any y. Assume from now on that $Per(f) \ne Fix(f)$ and define a function $\chi_f = \chi$ as follows:

$$\chi(x) = \begin{cases} 1/2 & \text{if } (f(x) - x)(f^2(x) - f(x)) \le 0, \\ 0 & \text{if } (f(x) - x)(f^2(x) - f(x)) > 0. \end{cases}$$

For any non-fixed periodic point y of period $p(y)$ the integer $l(y) = \sum_{i=0}^{n-1}\chi(f^i y)$ is at most $p(y)/2$ and is the same for all points from the orbit of y. The pair $orp(y) = (l(y), p(y))$ is called the *over-rotation pair* of y, and *coprime over-rotation pair* if p, q are coprime. The set of all over-rotation pairs of periodic non-fixed points of f is denoted by $ORP(f)$ and the χ-rotation number $\rho_\chi(y) = \rho(y) = l(y)/p(y)$ is called the *over-rotation number* of y. Observe that by Theorem S and by the assumption that $Per(f) \ne Fix(f)$ it follows that f has a point of period 2 and that the over-rotation number of this point is 1/2; in other words, the set of all over-rotation numbers of periodic points of f includes 1/2 and, therefore, 1/2 belongs to the union of all χ-rotation sets $I_{f,\chi}(x)$ defined earlier.

We introduce the partial ordering $\Vdash$ among all pairs of integers (s,t), $0 < s \le t/2$: $(p,q) \Vdash (k,l)$ if $k/l \in (p/q, 1/2]$.

Proposition BM1 [BM0]. *If $(p,q) \Vdash (k,l)$ and $(p,q) \in ORP(f)$ then $(k,l) \in ORP(f)$.*

This proposition implies Theorem S. Indeed, let f be an interval map and consider odd periods. For any $2n+1$ the closest to $1/2$ over-rotation number of a periodic point of period $2n+1$ is $\frac{n}{2n+1}$. Clearly $\frac{n}{2n+1} < \frac{n+1}{2n+3} < \frac{1}{2}$. Hence for any periodic point x of period $2n+1$ its over-rotation pair $orp(x)$ is $\Vdash$-stronger than the pair $(n+1, 2n+3)$, and by the proposition the map f has a point of period $2n+3$. Also, for any m we have $(n, 2n+1) \Vdash (m, 2m)$, so by the same proposition the map f has a point of period $2m$. Applying this result to the maps $f, f^2, f^4, \dots$ one can prove the Sharkovskiĭ's theorem for all periods but the powers of 2; additional arguments covering the case of powers of 2 are quite easy. Below we extend the definition of the $\Vdash$-ordering; namely, if $0 < k \le n/2$ then:

1. if $(p,q) \Vdash (k,l)$ then $(p,q) \gtrdot (k,l)$,
2. if $p/q = k/l = m/n$, m, n coprime, then $(p,q) \gtrdot (k,l)$ if and only if (p/m) is sharper than (k/m) (both (p/m) and (k/m) are integers).

Theorem BM2 [BM0]. *If $(p,q) \gtrdot (k,l)$ and $(p,q) \in ORP(f)$ then $(k,l) \in ORP(f)$.*

Theorem BM2 implies a full description of sets $ORP(f)$ for interval maps, close to that from Theorem M1. It implies that the closure of the set of over-rotation numbers of periodic points of f is an interval $I_f = [\rho_f, 1/2], 0 \le \rho_f \le 1/2$, called the *over-rotation interval* of f. Observe that over-rotation numbers of periodic points of f are simply χ-rotation numbers of those points. It is natural to consider the connection between I_f, defined by the over-rotation numbers of *periodic points* of f, and the union of *all* χ-rotation sets $I_{f,\chi}(x)$ defined earlier. Before we describe this connection in detail, we would like to discuss certain difficulties related to such description; hopefully, this will help the reader understand assumptions and exclusions which are necessary here.

For over-rotation numbers, the dynamics in small neighborhoods of fixed points can play a misleading role. First of all, a point x which maps into a fixed point yields a sequence $\chi(f^i(x))$ which eventually consists of zeros and hence yields the set $I_{f,\chi}(x) = \{0\}$. However, this clearly has no implications for the dynamics of periodic points of the map. Hence *pre-fixed* points x should not be considered as we compare I_f and the union of *all* χ-rotation sets $I_{f,\chi}(x)$.

In general, points which contain some fixed points in their limit sets should not be considered here because then the seemingly rich dynamics may take place over a shrinking to zero part of the space and therefore should be ignored rather than taken into account. To explain this, let us draw analogy with the case of the entropy. It is known that for continuous interval maps it can happen so that the entropy of such maps is large (even infinite) while it is assumed on smaller and smaller invariant sets converging to fixed points of the map. Similarly, it can happen that the dynamics in a small neighborhood of, say, an attracting fixed point a is chaotic. That may lead to a rich set of sequences $\chi(f^i(x))$ and large χ-rotational sets of such points while having absolutely no bearing on the set of periodic points of the map at all (e.g., maps like that can be such that *all* points are attracted to a unique fixed point). To avoid this "artificial" richness we consider only *admissible* points.

More precisely, by a *limit measure* of a point x we mean a limit of ergodic averages of the δ-measure concentrated at x; clearly, any limit measure is invariant. The results of [B2] apply to a wide variety of functions, but we only state them as they apply to the function $\chi_f = \chi$ and over-rotation numbers. Call a point x *admissible* if any limit measure μ of x is such that $\mu(Fix(f)) = 0$; since μ is invariant, this implies that in fact the set of all points x which are eventual preimages of fixed points of f is of zero μ-measure. Since the set of discontinuities of χ is contained in the union of the set of fixed points $Fix(f)$ of f and their preimages, we see that for an admissible point x the set of discontinuities of χ is of zero limit measure for any limit measure of x.

The connection between I_f and the union of χ-rotation sets $I_{f,\chi}(x)$ for all admissible points x is established in papers [B2, B3] and illustrated in Theorem B1; the part of Theorem B1 concerning rational rotation numbers and periodic points follows from the definitions and Theorem BM2. To state the last part of Theorem B1 we define *piecewise-monotone* interval maps. Say that $f : I = [a, b] \to \mathbb{R}$ is a *piecewise-monotone* map if there are (perhaps degenerate) closed intervals $a \le C_0 = C < C_1 < \cdots < C_l \le b$ with f on each component of $[a, b] \setminus \bigcup C_i$ being monotone and $C_i, 0 \le i \le l$ being a (perhaps, degenerate) flat spot for each i. A flat spot is an interval I such that f—I is a constant. Sets C_i are said to be *critical* (sets of f). A degenerate set C_i is called a *critical point* of f. This determines $l = l(f)$; components of $[a, b] \setminus \bigcup C_i$, denoted by $I_1, \dots, I_l$, are called *laps* of f. Thus, laps and sets $C_0 \cup I_1, C_2, I_2, \dots, C_{l-1}, I_l \cup C_l$ form a partition of $[a, b]$ called a *basic* partition of $[a, b]$. For simplicity, we always assume that f is a piecewise-monotone map with only finitely many fixed points of f as well as finitely many their first preimages.

Theorem B1 [B2, B3]. *The following statements are true.*

1. *If f is continuous and $\rho_f < 1/2$ then for any $a \in (\rho_f, 1/2]$ there is an admissible point x, generic for a measure μ, such that $I_f(x) = \{a\}$. Moreover, if p, q are positive integers such that $p/q \in (\rho_f, 1/2]$ then we may choose x to be periodic with over-rotation pair $orp(x) = (p, q)$.*
2. *If x is an admissible point, then $I_f(x) \subset I_f = [\rho_f, 1/2]$.*
3. *If f is piecewise-monotone and $\rho_f \neq 0$ then there exists an invariant measure μ such that f is minimal on the support of μ and there exists a point x, generic for μ and such that $I_{f,\chi}(x) = \{\rho_f\}$.*

Theorems BM1 and B1 guarantee the existence of a periodic orbit of any over-rotation pair (p, q) with rational $p/q \in int\, I_f$. By Theorem B1 there is also a point whose over-rotation set coincides with a given irrational number from $int\, I_f$. The remaining case of the left endpoint of I_f is resolved for piecewise-monotone maps in Theorem B1(3). It is easy to see that the piecewise-monotone assumption is necessary here. Indeed, consider the following example. Let $f : [0,1] \to [0,1]$ be a map with a sequence of concatenated invariant intervals I_j such that $\bigcup I_j = [0,1)$. Then 1 is clearly an f-fixed point. Suppose that $f|_{I_j}$ is piecewise-monotone for any j and such that the over-rotation intervals of $f|_{I_j}$ grow to their union $(\rho_f, 1/2]$. Then there exists no admissible point x for which $I_{f,\chi}(x)$ contains ρ_f. Thus in Theorem B1(3) above the piecewise-monotone assumption is necessary.

In a recent paper by Jozef Bobok [Bo] the case covered in Theorem B1(3) is studied in great detail and depth resulting into a much more precise claim. The results of [Bo] which complement and further develop Theorem B1 are summarized in Theorem Bo1 below. Recall that a dynamical system is said to be *strictly ergodic* if it has a unique invariant measure. To state Theorem Bo1 in full generality we need a couple of notions on which we will elaborate in Subsection 0.2.

A *pattern* is a cyclic permutation of the set $T_n = \{1, 2, \dots, n\}$. A pattern π *forces* a pattern θ if a continuous interval map f which exhibits π also exhibits θ. By [Ba] forcing is a partial ordering. One can talk about the *over-rotation pair* $orp(\pi)$ and the *over-rotation number* $\rho(\pi)$ of a pattern π. We call a pattern π an *over-twist pattern* (or just an *over-twist*) if it does not force other patterns of the same over-rotation number.

Theorem Bo1 [Bo]. *Let a point x and a measure μ be as defined in Theorem* B1(3). *Then the map $f|_{\omega(x)}$ is strictly ergodic with μ being the unique invariant measure of $f|_{\omega(x)}$. Moreover, if ρ_f is rational then x is periodic and the pattern of the orbit of x is an over-twist pattern of over-rotation number ρ_f.*

Thus, numerical information about a map, compressed to I_f, implies various types of the limit behavior of points reflected by their rotation numbers. Can one say more about the dynamics of a map f if I_f contains some number a? This question is addressed in the present paper for unimodal maps.

0.2 Patterns and unimodal maps. We need definitions from *combinatorial dynamics* ([ALM2]). A map f has a *horseshoe* if there are points a, b, c such that either $f(c) \le a = f(a) < b < c \le f(b)$ or $f(c) \ge a = f(a) > b > c \ge f(b)$. It is easy to see ([BM0]) that if a map has a horseshoe then it has periodic points of all possible over-rotation numbers. A map (not even necessarily one-to-one) of the set T_n into itself is called a *non-cyclic pattern.* If an interval map f on its cycle P is conjugate to a pattern π by an increasing map then P is a *representative* of π in f and f *exhibits* π on P; if f is *monotone (linear)* on each complementary to P interval, we say that f is *P-monotone (P-linear)*([MN]).

A pattern π is said to have a *block structure* if there is a collection of pairwise disjoint segments $I_0, \dots, I_k$ with $\pi(T_n \cap I_j) = T_n \cap I_{j+1}, \pi(T_n \cap I_k) = T_n \cap I_0$; the intersections of T_n with intervals I_j are called *blocks* of π. A pattern without a block structure is said to be *irreducible.* If we identify blocks, we get a new pattern

π', and then π is said to have a block structure *over* π'. A pattern π *forces* a pattern θ if a continuous interval map f which exhibits π also exhibits θ. By [Ba] forcing is a partial ordering. If π has a block structure over a pattern θ, then π forces θ. By [MN] for each pattern π there exists an irreducible pattern π' over which π has block structure (in particular, π' is forced by π).

One can talk about the *over-rotation pair* $orp(\pi)$ and the *over-rotation number* $\rho(\pi)$ of a pattern π. We call a pattern π an *over-twist pattern* (or just an *over-twist*) if it does not force other patterns of the same over-rotation number. Theorem BM2 and the properties of forcing imply the existence of over-twist patterns of a given rational over-rotation number between 0 and 1: it implies that a map which has a periodic point of rational over-rotation number ρ exhibits an over-twist pattern of rotation number ρ. By Theorem BM2 an over-twist pattern has a coprime rotation pair; in particular, over-twists of rotation number $1/2$ are of period 2, so from now on we consider over-twists of over-rotation numbers distinct from $1/2$. Combining this with Theorem Bo1 and Theorem B1, we come up with the following way of describing periodic dynamics of a piecewise monotone interval map f: *if f has the over-rotation interval I_f then for any rational number $\rho \in I_f$ there exists an f-periodic point x whose orbit exhibits an over-twist pattern of over-rotation number ρ.* This explains why studying over-twist patterns is important.

In the setting of (non-cyclic) patterns it is useful to consider an interpretation of over-rotation numbers which is close to symbolic dynamics. The following construction is a key ingredient of one-dimensional combinatorial dynamics. Let π be a (non-cyclic) pattern, P be a finite set with a map $f : P \to P$ of (non-cyclic) pattern π and f be a P-linear map. Say that a component I of $[0,1] \setminus P$ π-*covers* another such component J if $J \subset f(I)$. Construct the oriented graph G_π whose vertices are components of $[0,1] \setminus P$ and whose edges (arrows) go from I to J if and only if I π-covers J. Clearly, G_π does not depend on the actual choice of P and the definition is correct.

A cycle (and the pattern it represents) is *divergent* if it has points $x < y$ such that $f(x) < x$ and $f(y) > y$. A cycle (pattern) that is not divergent will be called *convergent*. It is well-known that a pattern does not force a horseshoe if and only if it is convergent (the main ideas of the proof date back to the original paper by Sharkovskiĭ [S]). Suppose that π is a convergent pattern and that P is a periodic orbit of pattern π. Let f be a P-linear map. Then f has a unique fixed point a. Consider the set $Q = P \cup \{a\}$ and denote its pattern by π'. We will work with the oriented graph $G_{\pi'}$.

Suppose that there is a real-valued function ψ defined on arrows of $G_{\pi'}$. This is a classical situation of one-dimensional symbolic dynamics. It is well-known [ALM2] then that the maximal and the minimal averages of ψ along all possible paths (with growing lengths) in $G_{\pi'}$ are assumed on periodic sequences. In particular, if the values of ψ on arrows are all rational, then the maximum and the minimum of those averages are rational. We choose a specific function ψ as follows. Associate to each arrow in $G_{\pi'}$ the number 1 if it corresponds to the movement of points from the right of a to the left of a. Otherwise associate 0 to the arrow. As explained above, this yields rational maximum and rational minimum of limits of averages of ψ taken along all possible paths (with growing lengths) in $G_{\pi'}$, and these extrema are assumed on periodic sequences.

Now we define unimodal maps. Let $I = [0, 1]$. A continuous map $f : I \to I$ is *unimodal* if there is $c \in (0, 1)$ such that $f(c) = 1, f(1) = 0$, f increases on $[0, c]$ and decreases on $[c, 1]$ (a unique critical point of a unimodal map g is denoted by $c_g \equiv c$). Suppose there is a fixed point $b \in [0, c]$. Then the map has a horseshoe (indeed, $0 = f^2c \leq fb = b < c < fc = 1$) which by the above implies that f has periodic points of all over-rotation pairs, $I_f = [0, 1/2]$, and all the points to the left of b are attracted by fixed points. To avoid considering this trivial case, from now on we assume that there are no fixed points in $[0, c)$. Given a (unimodal) interval map f we consider its over-rotation interval $I_f = [\rho_f, 1/2]$. In this paper we study ρ_f for unimodal maps. By the previous paragraph if c is periodic or preperiodic, then ρ_f is rational. Moreover, similarly to the previous paragraph it follows that if c is attracted to a periodic point of f then still ρ_f is rational. The main idea of the paper is to study ρ_f by first constructing a discontinuous lifting F whose over-rotation numbers coincide with those of f and then study F in the spirit of [M3].

Fix a unimodal map f. Next we briefly describe the main notions of *kneading theory*, due to Milnor and Thurston [MT]. For each point $x \in [0, 1]$ we define its *itinerary* as the sequence $i(x) = i_0(x)i_1(x)\dots$ of symbols L, C or R so that $i_j(x) = L$ if $f^j(x) < c$, $i_j(x) = C$ if $f^j(x) = c$, and $i_j(x) = R$ if $f^j(x) > c$. Define the order of the symbols to be $L < C < R$. Now, suppose that $A = a_0 \dots$ and $B = b_1 \dots$ are two sequences of symbols L, C or R. Define the order among them as follows. Choose the smallest j with $a_j \neq b_j$. Then we set $A \succ B$ if there is an even number of R's among $a_0, \dots, a_{n-1}$ and $a_n > b_n$ or if there is an odd number of R's among $a_0, \dots, a_{n-1}$ and $a_n < b_n$. It is shown in [MT] that $x > y$ implies $i(x) \succeq i(y)$ and $i(x) \succ i(y)$ implies $x > y$.

An itinerary A is said to be *shift maximal* if $A \succeq \sigma^j(A)$ for any non-negative j where σ is the left shift. The *kneading sequence* of f is the itinerary $\nu(f) = i(f(c))$. Clearly, $\nu(f)$ is shift maximal. By [MT], $\nu(f) \succeq A \succeq \sigma(\nu(f))$ if and only if there exists a point x such that $i(x) = A$. Therefore, the over-rotation interval of a unimodal map f is determined by its kneading sequence and we can talk about the over-rotation interval I_ν of a kneading sequence ν. In fact, by definition and properties of kneading sequences if $\nu_2 \succ \nu_1$ and f_1, f_2 are respective unimodal maps, then all patterns exhibited by f_1-cycles are also exhibited by f_2-cycles. In particular, then $I_{\nu_2} \subset I_{\nu_2}$.

0.3 Main results. Let us proceed in a more detailed manner. Consider the shift ζ_ρ by ρ on $[0, 1)$ modulo 1 assuming $\rho \leq 1/2$. Define the kneading sequence $\nu_\rho =$

$(\nu_\rho(0), \nu_\rho(1), \dots)$ as follows:

$$\nu_\rho(n) = \begin{cases} C & \text{if } \zeta_\rho^{n+1}(\rho) = 0, \\ R & \text{if } 0 < \zeta_\rho^{n+1}(\rho) < 2\rho \\ L & \text{if } 2\rho \le \zeta_\rho^{n+1}(\rho) \end{cases}$$

Clearly, we have $\nu_\rho = (R, L, \dots)$ (except for $\nu_{1/2} = (R, C, R, C, \dots)$). We show that ν_ρ is a kneading sequence of a unimodal map (this can be done formally, but we prefer a more geometrical approach). Note that ν_ρ is periodic if ρ is rational and non-periodic otherwise. Let $\rho = p/q$, p, q coprime. It is easy to see that the corresponding to ν_ρ pattern γ_ρ is given by the cyclic permutation φ of the set of points $\{0, 1/q, \dots, (q-1)/q\}$ where φ is defined as follows:

(1) $\varphi(j/q) = j/q + p/q$ for $0 \le j \le q - 2p - 1$,
(2) $\varphi(j/q) = (2q - 2p - 1 - j)/q$ for $q - 2p \le j \le q - p - 1$,
(3) $\varphi(j/q) = (q - 1 - j)/q$ for $q - p \le j \le q - 1$.

In other words, here is what the pattern $\gamma_{p/q}$ does with q points $x_0, \dots, x_{q-1}$ of the periodic orbit. The first $q - 2p$ points from the left are shifted to the right by p points. The next p points are "flipped" (i.e. the orientation is reversed, but the points which are adjacent remain adjacent) all the way to the right. Finally, the last p points of the orbit are flipped all the way to the left. As we will see in what follows *the pattern γ_ρ is the only unimodal twist pattern of rotation number ρ* in the case of a rational ρ. Note that the same fact follows from the results of [BK] where different methods are used.

Let us now introduce a non-cyclic pattern γ'_μ for $\mu < 1/2$.

(1) $D_\mu = \{0, 1/q, \dots, \frac{q-1}{q}, a', a\}$ where $\frac{q-2p-1}{q} < a' < \frac{q-2p}{q}, \frac{q-p}{q} < a < \frac{q-p+1}{q}$; also let $j = \gamma^{-1}(q - 2p)$.
(2) $\gamma'_\mu : D_\mu \to D_\mu$.
(3) $\gamma'_\mu(i/q) = \gamma_\mu(i/q)$ for any $i/q \ne j/q, a, a'$.
(4) $\gamma'_\mu(j/q) = a', \gamma'_\mu(a') = \gamma'_\mu(a) = a$.

As follows from the definition, the pattern γ'_ρ can be easily obtained from the pattern γ_ρ. Indeed, consider the pattern γ_ρ with added points a', a at the appropriate places. Change the map on the first preimage of c so that it maps to a' (and then, of course, to a). This gives the pattern γ'_ρ.

Also suppose that $\gamma'_{1/2} : \{0, 1/4, 1/2, 3/4\} \to \{0, 1/4, 2/4, 3/4\}$ is defined as follows: $\gamma'_{1/2}(0) = 1/2, \gamma'_{1/2}(1/4) = 3/4, \gamma'_{1/2}(1/2) = 1/2, \gamma'_{1/2}(3/4) = 0$. Clearly, the kneading invariant ν'_μ corresponding to γ'_μ is obtained from ν_μ as follows: $\nu'_\mu = (\nu_\mu(1), \dots, \nu_\mu(q-2), L, R, R, R, \dots)$, i.e. in the end of ν'_μ there stands an infinite string of R-s. Let also $\nu'_\mu = \nu_\mu$ if μ is irrational. Finally let us denote the kneading sequence of a unimodal map f by $\nu(f)$.

Theorem 2.4. *Let f be a unimodal map. Then $I_f = [\mu, 1/2]$ iff $\nu'_\mu \succ \nu(f) \succ \nu_\mu$; in particular if μ is irrational then $I_f = [\mu, 1/2]$ iff $\nu_\mu = \nu(f)$.*

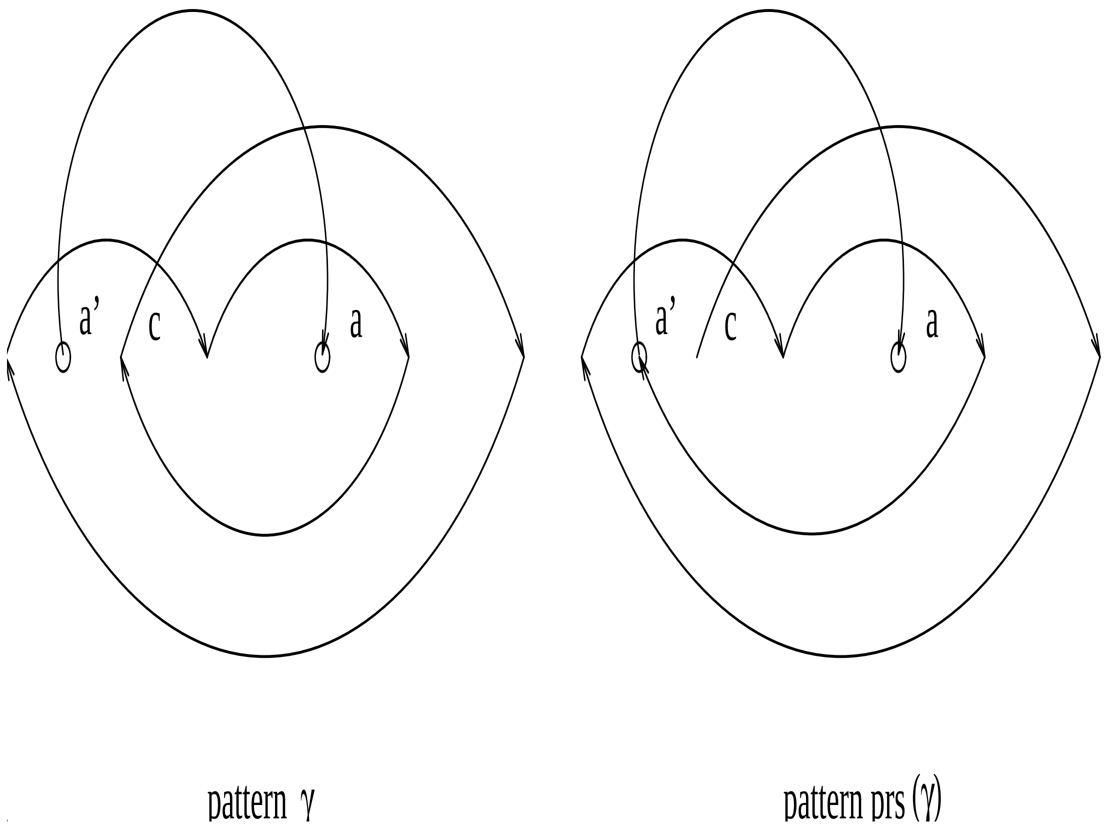

FIGURE 0.1. Patterns $\gamma_{2/5}$ and $\gamma'_{2/5}$

In fact the left endpoint ρ_f of the rotation interval of a unimodal map f was also introduced as a topological invariant of the map by J.-M. Gambaudo and C. Tresser in [GT] (see [BM1] where the connection between the rotation interval and the invariant introduced in [GT] is established). The authors study the behavior of ρ_{f_ν} for families of maps $f_\nu : [-1, 1] \to [-1, 1], f_\nu(x) = 1 - \nu|x|^\mu$ where $\mu \le 1$ and prove for these families the decreasing of ρ_{f_ν} which in our terms means the growth of the rotation interval I_{f_ν}. This fact can be deduced from the results of Section 3 which deals with some ways to compare rotation intervals of various interval maps. By Theorem 0.1 the set of periods is defined by the rotation interval if the latter is not degenerate, so we get a method of comparing both rotation intervals and sets of periods of maps. As an application we prove the monotone growth of rotation interval for some one-parameter families of unimodal maps.

Let $\mathcal{S} = \{f : f$ be a convex map of the interval $[0, 1]$ into itself with a unique turning point c_f which is a local maximum such that $f|[0, c_f]$ and $f|[c_f, 1]$ are C^1-maps, $f(c_f) > c_f$ and $f(0) = f(1) = 0\}$. Let us make some simple remarks. First, we only require that f be continuously differentiable at c_f from the left and the right separately. In fact, one-sided derivatives of f at c_f do not necessarily vanish. Also, the assumption $f(c_f) > c_f$ is needed to avoid considering trivial cases and can be made without loss of generality. Under this assumption there is no fixed point in $(0, c_f]$ and there is a unique fixed point $a_f \in (c_f, 1]$. For any $x \neq c_f$ there is a well-defined point $x'_f \neq x$ such that $f(x) = f(x'_f)$; also let $c'_f = c_f$. Also, $\lambda(K)$ denotes the Lebesgue measure of a measurable set K.

Lemma 3.2. *The following statements are true.*

(1) *Let* $|f'| \geq |g'|, c_f = c_g = c$ *and* $\dfrac{|c - a_f|}{|c - a'_f|} \geq \dfrac{|c - a_g|}{|c - a'_g|}$ *where* $f, g \in \mathcal{S}$. *Then* $I_f \supset I_g$.
(2) *Let* $g \in \mathcal{S}$ *and* $|g'(x)(x - c_g)| \leq |g'(x'_g)(x'_g - c_g)|$ *for any* $x \geq c_g$. *Then* $I_{\nu g} \supset I_g$ *for any* $\nu > 1$.

Another close result deals with maps from the class $\mathcal{G} \subset \mathcal{S}, \mathcal{G} = \{g : g \in \mathcal{S}$ is a polynomial map of $[0, 1]$ into itself of degree no more than $3\}$.

Lemma 3.7. *Let* $f \geq g$ *and* $f, g \in \mathcal{G}$. *Then* $I_f \supset I_g$.

For the sake of convenience we sum up the results of Lemmas 3.2 and 3.7 in Theorem 3.8 dealing with one-parameter families of interval maps.

Theorem 3.8. *Let* $f_\nu, \nu \in [b, d]$ *be a one-parameter family of interval maps such that one of the following properties holds.*

(1) $f_\nu \in \mathcal{S}$ *for any* ν*; also, if* $\nu > \mu$ *then* $|f'_\nu| \geq |f'_\mu|$ *and* $\dfrac{|c_{f_\nu} - a_{f_\nu}|}{|c_{f_\nu} - a'_{f_\nu}|} \geq \dfrac{|c_{f_\mu} - a_{f_\mu}|}{|c_{f_\mu} - a'_{f_\mu}|}$.
(2) $f = f_b \in \mathcal{S}$, $|f'(x)(x - c_f)| \leq |f'(x'_f)(x'_f - c_f)|$ *for any* $x \geq c_f$ *and* $f_\nu = \nu f$.
(3) $f_\nu \in \mathcal{G}$ *for any* ν *and* $f_\nu \geq f_\mu$ *if* $\nu > \mu$.

Then $I_{f_\nu} \supset I_{f_\mu}$ *if* $\nu > \mu$ *and so if* f_b *has an odd periodic point then* $P(f_\nu) \supset P(f_\mu)$ *if* $\nu > \mu$.

1. Preliminaries

We need some well-known tools. Let $I_0, \dots$ be intervals such that $f(I_j) \supset I_{j+1}$ for $0 \leq j$; then we say that $I_0, \dots$ is an *f-chain* or simply a *chain* of intervals. If a finite chain of intervals $I_0, \dots, I_{k-1}$ is such that $f(I_{k-1}) \supset I_0$ then we call $I_0, \dots, I_{k-1}$ an *f-loop* or simply a *loop* of intervals.

Lemma ALM [ALM2]. *The following statements are true.*

(1) *Let* $I_0, \dots, I_k$ *be a finite chain of intervals. Then there is an interval* M_k *such that* $f^j(M_1) \subset I_j$ *for* $0 \leq j \leq k - 1$ *and* $f^k(M_k) = I_k$.
(2) *Let* $I_0, \dots$ *be an infinite chain of intervals. Then there is a nested sequence of intervals* M_k *defined as in* (1) *whose intersection is an interval* M *such that* $f^j(M) \subset I_j$ *for all* j.
(3) *Let* $I_0, \dots, I_k$ *be a loop of intervals. Then there is a periodic point* x *such that* $f^j(x) \in I_j$ *for* $0 \leq j \leq k - 1$ *and* $f^k(x) = x$.

Let $\mathcal{U}$ be the set of all piecewise-monotone interval maps g with one fixed point (denoted $a_g = a$). Fix $f \in \mathcal{U}$; then $f(x) > x$ for any $x < a$ and $f(x) < x$ for any $x > a$. Call an interval I *admissible* if one of its endpoints is a. Call a chain (a loop) of admissible intervals $I_0, I_1, \dots$ *admissible*; if $I_0, \dots, I_{k-1}$ is an admissible loop then $k > 1$ since the image of an admissible interval cannot contain this interval. For any admissible loop $\bar{\alpha} = \{I_0, \dots, I_{k-1}\}$ call the pair of numbers $(p/2, k) = orp(\bar{\alpha})$ the *over-rotation pair* of $\bar{\alpha}$ where p is the number of indices $0 \le s \le k-1$ with I_s and I_{s+1} located on opposite sides of a. It is easy to see, that this definition is consistent with the definition of over-rotation pair given above. Observe also, that for an admissible loop the number p is always even (as the interval has to come back to where the loop starts). Call the number $\rho(\bar{\alpha}) = p/2k$ the *over-rotation number* of $\bar{\alpha}$. A sequence $\{y_1, \dots, y_l\}$ is called *non-repetitive* if it cannot be represented as several repetitions of a smaller sequence. Define a function φ_a on all admissible intervals so that $\varphi_a([b,a]) = 0$ if $b < a$ and $\varphi_a([a,d]) = 1$ if $a < d$. Finally, given a set A and a point x we say that $A \le x$ if for any $y \in A$ we have $y \le x$.

Lemma BM3 [BM0]. *Let $f \in \mathcal{U}$ and $\bar{\alpha} = \{I_0, \dots, I_{k-1}\}$ be an admissible loop of non-degenerate intervals. Then there are the following possibilities.*

(1) *Let k be even and for each j the intervals I_j and I_{j+1} are such that either $I_j \le a \le I_{j+1}$ or $I_j \ge a \ge I_{j+1}$. Then f has a point x of period 2.*
(2) *If the first possibility fails, then there is a periodic point $x \in I_0$ such that $x \ne a, f^j(x) \in I_j (0 \le j \le k-1), f^k(x) = x$ and so $\rho(x) = \rho(\bar{\alpha})$. If the sequence $\{\varphi_a(I_0), \dots, \varphi_a(I_{k-1})\}$ is non-repetitive, then $orp(x) = orp(\bar{\alpha})$. Moreover, x can be found so that the following holds: for every y from the orbit of x there exists no z such that $y >_a z$ and $f(y) = f(z)$.*

Any point x with the properties from Lemma BM3 is said to be *generated* by $\bar{\alpha}$.

Lemma 1.1. *Let $f \in \mathcal{U}$, let a point $c < a$ be such that $f(c) > a$ and $f^n(c) \le c$ for some number n. Suppose that among the points $c, f(c), \dots, f^{n-1}(c)$ there are p iterates x with x and $f(x)$ lying on opposite sides of a; then the following holds.*

(1) *p is even and $p/2n \in I_f$.*
(2) *If $f^{n+1}(c) \le a$ then for any r, s with $r/s = p/2n$ there is a periodic point of over-rotation pair (r, s).*
(3) *If $f^{n+1}(c) \le a$ and for some N the f^N-image of interval $[f^{n+1}(c), a]$ covers c then a small left semi-neighborhood of $p/2n$ is contained in I_f.*

Proof. (1) As c and $f^n(c)$ are on the same side of a, it follows that $p = 2p''$ is even. Consider the admissible loop $[c,a], [f(c),a], \dots, [f^{n-1}(c),a]$. By Lemma 1.8. it generates a periodic point x of over-rotation number $p/2n = p''/n$.

(2) Since $f(c) > a$, $f^n(c) \le c$ and $f^{n+1}(c) \le a$ then there is a point $a' \in [f^n(c), c)$ such that $f(a') = a$. By Lemma ALM(1) there is an interval $M \subset [c,a]$ with $fM \subset [f(c),a], \dots, f^nM = [a',a]$ and an interval $M' \subset [a',c]$ with $fM \subset [f(c),a], \dots, f^nM' = [a',a]$. Thus, $f^nM \cap f^nM' \supset M \cup M'$.

Now, let p'' and n be coprime. By Lemma ALM there is a periodic point $x \in M$ with $f^n(x) = x$ and $\rho(x) = p''/n$. Since p'' and n are coprime, then $orp(x) = (p'', n)$. Using standard arguments, for a given $r > 1$ we can find a periodic point $y \in M$ of period rn with $f^n(y) \in M, \dots, f^{(r-1)n}(y) \in M, f^{rn}(y) = y$. Then the construction implies that $orp(y) = (rp'', rn)$. This proves (2) in the case when p'', n are coprime.

Now, let $p'' = tp', n = tn'$ where p', n' are coprime and $t \geq 2$. By Lemma ALM there is a periodic point $x \in M$ of period n; clearly, it has over-rotation pair (sp', sn') for some s. By Theorem BM2 and by [MN], the pattern π represented by $orb(x)$ forces an irreducible pattern π' of over-rotation pair (p', n'). Suppose that π does not have a block structure over π'. Then by [MN] π forces the existence of patterns of all possible over-rotation pairs $(qp', qn'), q \geq 1$ as desired. Thus, we may assume that π, the pattern of the orbit of x, has block structure over π'.

Hence the points from the orbit of x enter intervals $[0, a]$ and $[a, 1]$ periodically with the period n'. On the other hand, by the construction $f^j(c)$ and $f^j(x) \in f^j(M)$ are located on the same side of a for every $j \leq n$. Thus, $f^i(c)$ and $f^{n'}(f^i(c))$ lie to the same side of a for all $0 \leq i \leq n - n'$. Consider possible locations of $f^{n'}(c)$. By the above $f^{n'}(c) < a$. Suppose that $f^{n'}(c) < a'$. Then, as in the first paragraph of the proof of (2), by Lemma ALM(1) we can construct two intervals N, N' (similar to M, M') with $N \subset [c, a], fN \subset [f(c), a], \dots, f^n N = [a', a]$ and $N' \subset [a', c], fN' \subset [f(c), a], \dots, f^n N' = [a', a]$. This implies that for any $r > 1$ we can find a periodic point $y \in N$ with $f^{n'}(y) \in N, \dots, f^{(r-1)n'}(y) \in N, f^{rn'}(y) = y$ so that $orp(y) = (rp', rn')$ as desired. Thus we may assume that $f^{n'}(c) > a'$.

Choose the greatest image $d = f^{n'j}(c)$ of c under powers of $f^{n'}$ such that $a' < d$. Then $n'j \leq n - n'$ (because $f^n(c) < a'$ by the construction) and $f^{n'}(d) \leq a' < d$. By the periodicity with which the orbit of c enters intervals $[0, a]$ and $[a, 1]$ we see that among points $d, f(d), \dots, f^{n'}(d) = f^n(c) \leq a'$ there are p' points lying to the right of a. By repeating the construction from the first paragraph of the proof of (2) and using Lemma ALM(1) one can find intervals N, N' with $N \subset [d, a], fN \subset [f(d), a], \dots, f^n N = [a', a]$ and $N' \subset [a', d], fN' \subset [f(d), a], \dots, f^n N' = [a', a]$. So repeating the arguments from the first paragraph of the proof we can find points of all over-rotation pairs (rp', rn').

(3) Let $b \neq a$. If $b < a$ then $f_r(b) = \max\{f(z) : z \in [b, a]\}$; if $b > a$ then $f_r(b) = \min\{f(z) : z \in [a, b]\}$. Clearly f_r maps points from $[0, a]$ into $[a, 1]$ and vice versa. Then $[a, f_r(b)] = f[b, a] \cap [a, 1]$ if $b < a$ and $[f_r(b), a] = f[a, b]$ if $a < b$. Also, since $f \in \mathcal{U}$ then $f([b, a]) \subset [b, f_r(b)]$ if $b < a$ and $f([a, b]) \subset [f_r(b), b]$ if $a < b$. In the situation of the lemma there is the smallest N such that $f_r^N(f^{n+1}(c)) \leq c$. Moreover, by the properties of f_r we see that $N = 2m$ is even. Hence by the definition of f_r we see that the following is an admissible loop:

$$[c, a], [f(c), a], \dots, [f^{n+1}(c), a], [f_r(f^{n+1}(c)), a], \dots, [f_r^{2m-1}((f^{n+1}(c)), a].$$

A direct computation shows that its over-rotation number is $\frac{p/2+m}{n+2m} < p/2n$. Since by Lemma BM3 there exists a periodic point with the over-rotation number $\frac{p/2+m}{n+2m}$, then the proof is complete. □

Finally we state the results of [BM3]. One of them gives a criterion for a pattern to be an over-twist pattern. To state this criterion we need to define a *code*, i.e. a special function which maps points of either a periodic orbit or of a pattern to the reals. We also need a few other definitions.

Recall, that a cycle (and the pattern it represents) is *divergent* if it has points $x < y$ such that $f(x) < x$ and $f(y) > y$, and that a cycle (pattern) that is not divergent is called *convergent*. Clearly, a convergent pattern has a unique complementary interval U such that its left endpoint is mapped to the right and its right endpoint is mapped to the left. If this pattern is exhibited by a map f, then this interval contains a fixed point, always denoted by a. Given two points x, y of the pattern we say that $x >_a y$ if x and y are located on the same side of U and x is farther away from U than y.

Similar notation is used for periodic orbits of interval maps which exhibit convergent patterns. There is an equivalent way to define convergent patterns. Namely, if f is a P-monotone map for a cycle P then P is convergent if and only if $f \in \mathcal{U}$. If $f \in \mathcal{U}$ and a is the fixed point of f then we write $x >_a y$ if points x, y are located on the same side of a and x is farther away from a than y. (This notation is similar to the one used for convergent patterns.)

Let P be a cycle of $f \in \mathcal{U}$ and φ be a function defined as 1 to the right of a and zero elsewhere. Following [BK], we introduce the *code* for P as follows. The code is a function $L : P \to \mathbb{R}$, defined by $L(x) = 0$ for the leftmost point x of P and then by induction we have $L(f(y)) = L(y) + \rho - \varphi(y)$, where ρ is the over-rotation number of P. When we get back to x along the orbit P, we add ρ n times (n is the period of P), and we subtract the sum of φ along P, which is $n\rho$, so we have a sum of 0. Therefore, the definition is correct.

Clearly, we can also speak of codes for patterns. If $f \in \mathcal{U}$ (or if the pattern in question is convergent), we say that the code for P is *monotone* if for any $x, y \in P$, $x >_a y$ implies $L(x) < L(y)$.

Theorem BM4 [BM3]. *A pattern is over-twist if and only if it is convergent and has monotone code.*

2. Unimodal over-twist patterns

Let us describe unimodal over-twist patterns. Our aim is to show that the pattern γ_ρ defined in Subsection 0.3 is the unique unimodal over-twist pattern of over-rotation number ρ.

Lemma 2.1. *Suppose that $\rho = p/q$ is such that $0 < \rho \le 1/2$. Then the only unimodal over-twist pattern of over-rotation number ρ is the pattern γ_ρ. Thus, a unimodal map f has the over-rotation interval $[\mu, 1/2]$ with $\mu \le \rho$ if and only if f has a periodic orbit of pattern γ_ρ.*

Proof. Suppose that τ is a unimodal over-twist pattern of over-rotation number $\rho = p/q < 1/2$. Consider a unimodal map $f : [0, 1] \to [0, 1]$ with the unique critical point c, unique fixed point a, and the unique f-preimage a' of a. Assume that

c is periodic and its orbit P exhibits the pattern τ. Then by Lemma BM4 there are no points of P in $[a', c)$. Moreover, by Theorem BM2 c is of period q. By definition of over-rotation pair, there are p points of P in the interval $(a, 1]$. This implies that there are p points of P in the interval $[c, a]$ too and, moreover, that $f(P \cap [c, a]) = P \cap [a, 1]$. In fact, the map f simply flips all points of $P \cap [c, a]$ to the other side of a. Clearly, then there are $q - 2p$ points of P in $[0, c)$.

Let us compute out codes of some points of P. By definition $L(0) = 0$. This implies that $L(1) = 1 - \rho$ and $L(c) = 1 - 2\rho = (q - 2p)/q$. Since the values of the code on points of P are fractions with denominator p and the code on $P \cap [0, c) = \{x_0 = 0 < x_1 < \cdots < x_{q-2p-1}\}$ is monotonically increasing, we see that on all the $q - 2p$ points of $P \cap [0, c) = P \cap [0, a']$ the code equals $L(x_0) = 0, L(x_1) = 1/q, \dots, L(x_{q-2p-1}) = (q - 2p - 1)/q$. This (and the definition of the code) immediately implies that $\tau = \gamma_\rho$ as desired.

Now, by Theorem Bo1, Theorem B1 and Theorem BM2 it follows that if a unimodal map f has the over-rotation interval $I_f = [\mu, 1/2]$ and $\mu \le \rho$ then f has a periodic orbit of pattern γ_ρ. On the other hand, it is clear that if f has a periodic orbit of pattern γ_ρ then $I_f = [\mu, 1/2]$ with $\mu \le \rho$. This completes the proof of the lemma. □

Lemma 2.1 gives the least (the weakest) kneading invariant of f implying the fact that $\rho \in I_f$. It also follows from Lemma 2.1 and Theorem BM2 that the kneading invariant $\nu_{p/q}$ of the over-twist pattern of rational rotation number p/q depends monotonically on p/q: if $s < t$ then $\nu_s \succ \nu_t$. This fact can be also easily checked directly. However, using kneading sequences is much more to the point if we concentrate upon the question of the greatest (strongest) kneading sequence (pattern) which gives the over-rotation interval precisely coinciding with $[\rho, 1/2]$.

Lemma 2.2. *A unimodal kneading sequence ν is such that $I_\nu = [p/q, 1/2]$ if and only if $\nu'_{p/q} \succeq \nu \succeq \nu_{p/q}$.*

Proof. By Lemma 2.1 if $\nu_{p/q} \le \nu$ then $[p/q] \subset I_\nu$. Let us show that $I_{\nu'_{p/q}} = [p/q, 1/2]$. Indeed, assume that f is a unimodal map such that its unique critical point c eventually maps to its unique fixed point a so that the orbit of c exhibits the pattern $\gamma'_{p/q}$. We need to show that $I_f = [p/q, 1/2]$. By Lemma 2.1 it suffices to prove that for any periodic orbit Q of f we have that $\rho(Q) \ge p/q$.

There are two ways this claim can be proven. One of them uses kneading sequences. Indeed, suppose that for some cycle Q of f we have that $\rho(Q) < p/q$. By Lemma 2.1 we may assume that Q has an over-twist pattern. For convenience we can choose a very big r and then choose Q of period $rq + 1$ and over-rotation number $rp/(rq + 1)$. Then it follows from the description of the pattern $\gamma'_{p/q}$ and from the properties of Q given by Lemma 2.1, that $i(x) \succ \nu(f)$, a contradiction with properties of kneading sequences.

However, we will give a more direct proof of the claim. Assume that P is a periodic orbit of pattern $\gamma_{p/q}$ with $\min P = 0$ and $\max P = 1$. Let f be the P-linear map. Construct the oriented graph G_P whose vertices correspond to the closures of the components of $[0,1] \setminus \{P \cup a_f \cup a'_f\}$ and whose arrows connect a vertex J and a vertex I if and only if $f(J) \supset I$ (this is a standard construction in one-dimensional dynamics [ALM2]; as always, a_f is the fixed point of f and a'_f is its f-preimage). Assign numerical value 1 to all arrows which come out of segments-vertices located to the right of a, and assign 0 to all arrows which come out of segments-vertices located to the left of a. Denote the just defined function on arrows by ψ. Then it follows that the over-rotation number of P is the minimum of averages of assigned values of ψ along all possible loops in the graph G_P.

Now, suppose that Q is the orbit of (non-cyclic) pattern $\gamma'_{p/q}$ with $\min Q = 0$ and $\max Q = 1$; denote the map acting on Q by g. We may assume that points of Q coincide with points of P except for the fact that the point z of P which maps to c_f by f will be mapped to a'_f by g (the map g will act the same way on a'_f, a_f as f). This allows us to use notation a, a', c without references to the map f. Extend g to the Q-linear map. Then construct the oriented graph whose vertices correspond to the closures of the components of $[0,1] \setminus \{Q \cup a_g \cup a'_g\}$ and whose arrows connect a vertex J and a vertex I if and only if $g(J) \supset I$ (a_g is the fixed point of g and a'_g is its g-preimage). Clearly, the sets of vertices of the graphs G_P and G_Q are the same.

Assign numerical value 1 to all arrows of G_Q which come out of segments-vertices located to the right of a_g and 0 to all arrows of G_Q which come out of segments-vertices located to the left of a_g. It follows that the graphs G_P and G_Q almost coincide, except for arrows of either graph which come into the segment-vertex $[a', c]$. Now, suppose that g has a periodic orbit Z of over-rotation number less than p/q. By Lemma BM3 we may assume that Z avoids $[a, c]$. However this implies that the loop of arrows in G_Q which corresponds to Z avoids arrows which come into the segment-vertex $[a', c]$. Hence this loop of arrows consists of the arrows common for both G_P and G_Q. This is a contradiction as all loops of arrows in G_P must produce averages of ψ which are greater than or equal to p/q.

It remains to show that if a kneading sequence ν is such that $\nu \succ \nu'_{p/q}$ then $I_\nu = [t, 1/2]$ and $t < p/q$. Suppose that $\nu \succ \nu'_{p/q}$ and denote a unimodal map which exhibits this itinerary by h. We can find a point d, say, to the right of c_h so that except for the first moment the h-itinerary of d and the g-itinerary of c_g under the above defined map g are the same up to the point when c maps to a' by g^q and d maps slightly to the left of a' by h^q; now Lemma 1.1(3) immediately implies that I_h contains not only p/q but also its small neighborhood, a contradiction. □

Lemma 2.2 gives the precise description of unimodal maps which have the over-rotation interval $[p/q, 1/2]$ through their kneading sequences. However, we still do not have the description of unimodal maps which have the over-rotation intervals with *irrational* left endpoint. We will describe them in Lemma 2.3. In the proof we use a special construction which relates unimodal maps to irrational rotations of the circle.

Lemma 2.3. *Let f be a unimodal map. If μ is irrational then $I_f = [\mu, 1/2]$ iff*

$\nu_\mu = \nu(f)$.

Proof. Consider a unimodal map f such that $I_f = [\rho_f, 1/2]$ and ρ_f is irrational. Then, clearly, the kneading sequence of f is not periodic or preperiodic. It is well-known that then we may assume that f has no wandering intervals. If f had such wandering intervals, they could be collapsed to points which would not change the over-rotation interval. Similarly, we may assume that f has no non-trivial periodic intervals. Otherwise either the intervals could be collapsed to points, or they contain c_f which implies that ρ_f is rational. This in turn implies that c_f is approached from either side by periodic or preperiodic points.

Choose any periodic or preperiodic point z and then choose the point x in the orbit of z so that $f(x)$ is greater than any point of the orbit of z. Consider a *truncation* f_x of f defined as follows: for every y we have that $f_x(y) = f(y)$ if $f(y) \le f(x)$ and $f_x(y) = f(x)$ if $f(y) > f(x)$. Then f_x has a preperiodic critical point which implies that it has the over-rotation interval with rational left endpoint. Hence this interval is strictly smaller than I_f. We conclude that there are kneading sequences which are smaller than ν_f and generate smaller over-rotation intervals that are arbitrarily close to I_f. A similar analysis shows that the change in I_f can be achieved also if we increase ν_f. In this respect, the over-rotation interval depends upon kneading sequence sensitively at ν_f.

In fact, for a given irrational number $\rho < 1/2$ there is a unique kneading sequence ν_ρ such that $I(\nu_\rho) = [\rho, 1/2]$. This kneading sequence is defined earlier in Subsection 0.3 and is closely related to circle rotations (i.e., to shifts by $\rho \mod 1$ considered on $[0, 1)$. Thus, the kneading sequences ν_ρ can be characterized as the $\succ$-smallest such that the corresponding over-rotation interval $I(\nu_\rho)$ contains the number ρ.

Let us verify the claims made in the previous paragraph. Unlike in the case of rational numbers ρ where we relied upon interval techniques, for irrational numbers ρ we develop a new approach directly relating unimodal maps f to circle maps. Here we use a special discontinuous lifting of f to a discontinuous degree one map of the real line, studied in the spirit of [M3]. The construction is described below. One can easily see that using this construction we can treat both rational and irrational cases. However we chose to use interval tools to tackle the rational case to show how different methods can work with unimodal maps.

Let us fix a unimodal map f with the rotation interval $[\mu, 1/2], \mu < 1/2$; it implies that $f(0) < a$ because otherwise we have that $f([0, a]) \subset [a, 1], f([a, 1]) \subset [0, a]$ and thus $I_f = \{1/2\}$, a contradiction. As we have already considered the case when μ is rational, we may assume that μ is irrational. Then the construction is as follows.

First we define a discontinuous conjugacy $\sigma : [0, 1] \to [0, 1]$ as the identity on $[0, a)$ and the symmetry (flip) with respect to $(a + 1)/2$ on $[a, 1]$, so that $\sigma(x) = x$ if $0 \le x < a$ and $\sigma(x) = a + 1 - x$ if $a \le x \le 1$; note that $\sigma^{-1} = \sigma$. Define now $g : [0, 1] \to [0, 1]$ by $g = \sigma \circ f \circ \sigma$ with the following two changes: $g(a') = a$, not 1, and $g(1) = a$, not 1 (see Figure 2.1). Note that g-image of $[0, 1]$ is $[0, 1)$, i.e. it does not contain 1.

1) On the interval $[0, a']$ we have $g(x) = f(x)$, thus the graph of g on $[0, a']$ is

the same as that of f; in particular $g(0) = f(0)$ and $g(a') = a$.

2) On the interval (a', a) we have $g(x) = \sigma(f(x)) = a + 1 - f(x)$. We obtain the graph of g by flipping the corresponding piece of f in the vertical direction symmetrically with respect to the horizontal line $y = (a+1)/2$; in particular, $g(c) = a$, and g-images of points that are close to a from the left will approach 1 and g-images of points close to a' from the right will approach 1.

3) On the interval $[a, 1]$ we have $g(x) = f(\sigma(x)) = f(a + 1 - x)$, thus the graph of g on $[a, 1]$ is obtained by flipping the corresponding piece of the graph of f in the horizontal direction symmetrically with respect to the line $x = (a+1)/2$; in particular $g(a) = 0, g(1) = a$.

Clearly we can define over-rotation numbers for g like it is done for f (e.g., counting how many times the g-orbit of a point enters the interval $[a, 1]$); moreover, the same way the over-rotation pairs for g-periodic orbits may be introduced. Moreover, σ conjugates f and g on orbits which avoid a and c so that the over-rotation sets on these orbits coincide; the same can be said about over-rotation pairs of periodic points. Consider the remaining orbits. First assume that c is neither periodic nor mapped into a by some iterate of f; it means that g-orbit of 0 never passes through a or 1 and so σ conjugates f on the f-orbit of 0 with g on the g-orbit of 0 keeping the rotation sets. Since $g(c) = a, g(a) = 0$ we now see that in fact due to the conjugacy σ the rotation sets of f and g are the same. Similarly we can easily check that the over-rotation numbers of points for g and of f coincide in the remaining cases (when c is periodic or c is a preimage of a) too. We conclude that $I_g = I_f = [\mu, 1/2]$.

Now let us specify a lifting F of a map f. We do it by setting on $[0, 1)$

$$F(x) = \begin{cases} g(x) = f(x) & \text{if } 0 \le x \le a', \\ g(x) = a + 1 - f(x) & \text{if } a' < x < a, \\ g(x) + 1 = f(a + 1 - x) + 1 & \text{if } a \le x < 1, \end{cases}$$

and then as usual: if $x = k + y$ with $y \in [0, 1)$ then $F(x) = k + F(y)$. The map F is a *degree one lifting* map of the real line into itself or an *old* map (see [M3]). Obviously by the construction the sets of classical rotation numbers and pairs of F coincide with the sets of over-rotation numbers and pairs of g, and therefore with the sets of over-rotation numbers and pairs of f, so the classical rotation set I_F coincides with I_f. An example can be found on Figure 2.1.

A famous nice tool for studying of old maps is so-called "pouring water from below or above" (see [M1-M3]); the definitions can be given in general situation but we only explain how one can use "pouring water" in our particular case. Let $d \neq 0$ be such that $f(0) = f(d)$). Consider the continuous map $G : \mathbb{R} \to \mathbb{R}$ of degree one which depends on F (i.e. in the end on f) and is defined on every

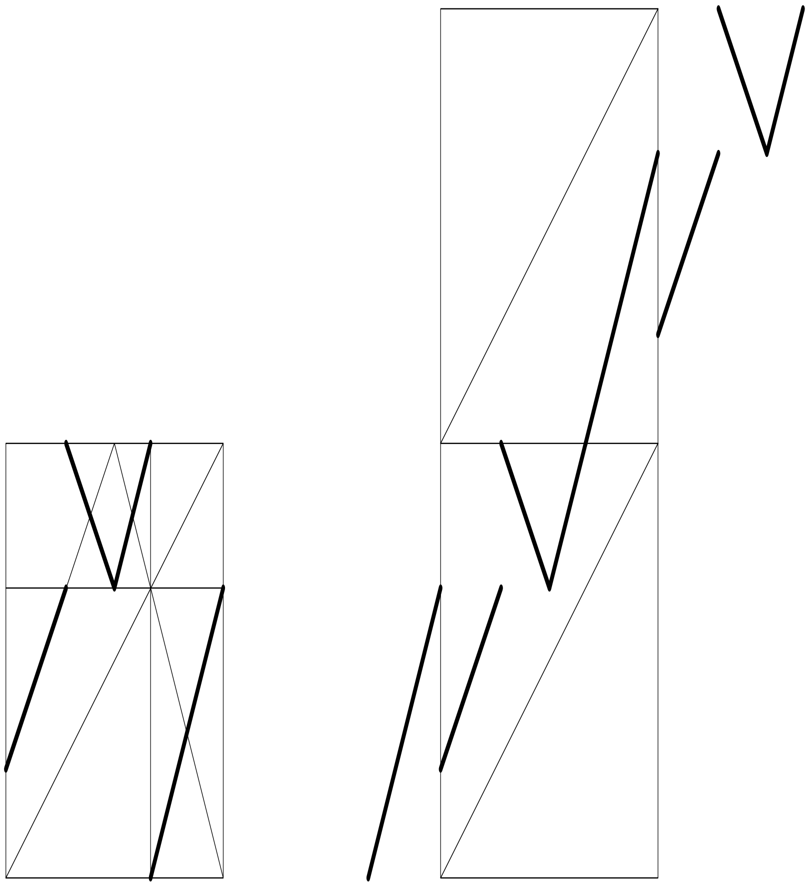

FIGURE 2.1. Functions g (left) and F (right)

interval $[n, n+1]$ as follows:

$$G(x) = \begin{cases} F(x) & \text{if } n \le x \le n + a', \\ n + a & \text{if } n + a' < x \le n + c \\ F(x) & \text{if } n + c < x \le n + a + 1 - d \\ F(n+1) = n + f(0) & \text{if } n + a + 1 - d < x < n + 1 \end{cases}$$

The connection between $F|[n, n+1]$ and $G|[n, n+1]$ is obvious: (1) $F = G$ except for two intervals, $(n + a', c)$ and $(n + a + 1 - d, n + 1)$ on which G is a constant; (2) $G \le F$; (3) G is continuous. Clearly the construction is possible for a unimodal map with homtervals; let us see though what consequences the absence of homtervals implies for G. First, it follows that the only flat spots of G are intervals $(n + a', c)$

and $(n+a+1-d, n+1)$. Now, suppose there is an interval $I \in [0,1]$ such that $G^m(I)$ is disjoint from intervals $(n+a', c)$ and $(n+a+1-d, n+1)$ for any m and n. The connection between f and G then implies that $f^m|I$ is monotone for any m which is impossible since f has no homtervals. So for any interval I there are m and n such that $G^m(I)$ is not disjoint from $(n+a', c) \cup (n+a+1-d, n+1)$. In other words the set A of all points X which avoid all intervals $(n+a', c)$ and $(n+a+1-d, n+1)$ is nowhere dense.

The properties of continuous monotone old maps of real line into itself are studied in [M1-M3], [ALM2] (for the sake of convenience in what follows we mostly refer to [ALM2] with respect to related questions). It is proven there that there is a unique number ρ such that for any $Z \in \mathbb{R}$ we have $(1/n)G^n(Z) \to \rho$ and that there is a point $Z' \in [0,1)$ whose G-orbit avoids intervals $(n+a', c)$ and $(n+a+1-d, n+1)$ for all n. Therefore due to the connection between f and G we have $I_G(Z') = \rho = I_f(Z')$ which implies that $\mu \leq \rho$. On the other hand $\rho \leq \mu$ since $G \leq F$ and G is monotone; indeed, for any point $X \in \mathbb{R}$ we have that $G(X) \leq F(X)$ and moreover, $G^n(X) \leq F^n(X)$ implies (after we apply G to both sides of the inequality) that $G^{n+1}(X) \leq G(F^n(X)) \leq F^{n+1}(X)$. Hence when we take the limits of $G^n(X)/n$ and $F^n(X)/n$ we see that ρ is less than or equal to all numbers from the over-rotation interval of f, i.e. that $\rho \leq \mu$ (see, e.g., [ALM2]). So, $\rho = \mu$.

Let $\pi : \mathbb{R} \to S^1$ be the usual projection of $\mathbb{R}$ onto S^1 such that $[0,1)$ is mapped onto the circle 1-to-1. Then π semiconjugates G to a continuous monotone map $\widetilde{f} : S^1 \to S^1$ of degree 1 of the same rotation number μ. Moreover, $\widetilde{f}$ has exactly two arcs which it collapses to points and otherwise is strictly monotone. These arcs are $J = \pi[a+1-d, 1]$ and $V = \pi[a', c]$. Clearly, $\pi(Z')$ is a point of the circle which avoids interiors of these arcs. The closure H of the orbit of $\pi(Z')$ in the circle is then also disjoint from the interiors of J and V. It is well-known, that the induced map on H can be semiconjugate to the interval rotation by the angle μ by means of collapsing complementary to H arcs. Moreover, this implies that H is minimal.

Let us show that H must contain $\pi(c)$ (so that $c \in \pi^{-1}(H)$). Indeed, suppose otherwise. Then it follows from the construction that there is an invariant closed set $H' \subset [0,1]$ which corresponds, through the construction, to the set H in the circle. Using the above introduced notation we see that H' consists of points which do not enter (a', c) and (a, d). In other words, points of H' avoid (a', c) and those of them which belong to $(a, 1)$ have images located non-strictly to the left of $f(0)$. By construction, points x' of H' have the same over-rotation numbers as the limits of $G^n(x)/n$ for the points $x \in H$, i.e. ρ.

If, by way of contradiction, $\pi(c)$ does not belong to H, then it follows that c does not belong to H'. Hence, as was explained right after the proof of Lemma 2.2 we can find periodic and preperiodic points x as close to c as we wish so that these points will never have eventual images greater than $f(x)$. Given such point x we can construct a well-defined truncation f_x of f so that all the points which never enter the open segment $(f(x), f(c))$ have the same orbits under both f and f_x. Moreover, we can choose x arbitrarily close to c. In particular, we can do this so that $f|_{H'} = f_x|_{H'}$. Then on the one hand the over-rotation interval of f_x must be such that $\rho_{f_x} > \mu = \rho$ is rational and hence $\rho_{f_x} > \mu$, on the other hand points of H' produce over-rotation numbers $\rho = \mu < \rho_{f_x}$, a contradiction. Hence $c \in H'$ as desired. Since H is minimal, it follows that so is H'. In particular, $H' = \omega(c)$. Again by construction this implies that the kneading sequence of f coincides with the kneading sequence ν_μ defined in Subsection 0.3.

To complete the proof of the lemma, it remains to show that for any irrational number μ there exists a unimodal map f which has the over-rotation interval $[\mu, 1/2]$. To construct such a map we reverse the construction. First, we construct a monotone map $\widetilde{f}$ of the circle to itself which has two "flat spots", i.e. two arcs $V = [a', c]$ and $J = [a + 1 - d, 1]$ which $\widetilde{f}$ collapses to points and $0 < a' < c < a < a + 1 - d < 1$ are such points of the circle that $\widetilde{f}(a') = a$ and $\widetilde{f}(a) = 1$. Moreover, the point $0 = 1$ of the circle never enters $V \cup J$ and is such that the order of points in its $\widetilde{f}$-orbit is the same as the order of points in the orbit of a point of the circle under the irrational rotation by the angle μ (it is easy to see that this is possible). This implies that the rotation interval of $\widetilde{f}$ (as defined in [ALM2]) is degenerate and coincides with μ. It remains now to reverse the construction in order to see that the corresponding to $\widetilde{f}$ unimodal map f exists. Reversing the arguments from above we also see that $I_f = [\mu, 1/2]$ as desired. By definition and by construction the kneading sequence of f coincides with the kneading sequence ν_μ defined in Subsection 0.3. This completes the proof of the lemma. □

It is now easy to see that if we put together Lemmas 2.1, 2.2.and 2.3 we get Theorem 2.4.

Theorem 2.4. *Let f be a unimodal map. Then $I_f = [\mu, 1/2]$ iff $\nu'_\mu \ge \nu(f) \ge \nu_\mu$; in particular if μ is irrational then $I_f = [\mu, 1/2]$ iff $\nu_\mu = \nu(f)$.*

The construction from Lemma 2.3 shows that if f is a unimodal map such that $I_f = [\mu, 1/2]$ and μ is irrational then $f|_{\omega(c)}$ is semiconjugate to the irrational rotation by the angle μ by a map constructed in the proof of Lemma 2.3. More precisely, to construct this semiconjugacy we need to apply the map σ from the proof of Lemma 2.3 to $\omega(c)$, then transport $\sigma(\omega(c))$ to circle using the standard projection π of the interval onto the circle, and then consider the map induced by f on the set $\pi(\sigma(\omega(c)))$. The resulting map of a closed subset of the circle will be the map $\widetilde{f}$ from the proof of Lemma 2.3 and will be such that the order of points in the orbit of any point of this set is the same as the order of points under the (irrational) rotation by the angle μ.

On the other hand, it can be verified directly that the same holds if the orbit of c is periodic and exhibits an over-twist pattern. That is, in this case we can construct the map $\widetilde{f}$ as well; clearly, it will be defined on a finite set of points, cyclically permuted by $\widetilde{f}$. Lemma 2.1 would imply that then the map $\widetilde{f}$ permutes the points of this set as the (rational) rotation by μ prescribes. Thus, the construction from Lemma 2.3 yields the description of over-twist patterns too.

In fact, the same construction shows how we can figure out the number ρ_f for a given unimodal map f. Apply the construction from Lemma 2.3 and construct the map $\widetilde{f} : S^1 \to S^1$; moreover, let us use the notation from the proof of Lemma 2.3. Then again, as in the proof of Lemma 2.3, by [ALM2] there exists a point $Z' = \pi(z')$ which avoids V and J so that z' avoids $[a', c]$ and $[a, d]$. By [ALM2] the limit ρ of $G^n(Z')/n$ equals the limit of $G^n(x)/n$ for all x; this common limit ρ is called the *rotation number* of $\widetilde{f}$. By construction it follows that the over-rotation number of z' equals ρ. Similar to the proof of Lemma 2.3 one can show that then $I_f = [\rho, 1/2]$ which implies that $\rho = \rho_f$.

Hence the algorithm of finding ρ_f is as follows. Take the following three intervals: $K_1 = [0, a'], K_2 = [c, a]$ and $K_3 = [d, 1]$ where d is chosen so that $f(0) = f(d)$. These intervals have images which complement each other to the whole $[0, 1]$ and intersect only at the images of their endpoints. Moreover, on each interval the map is monotone. This implies that there exist points x with orbits contained in the union K of intervals K_1, K_2 and K_3. In fact one such point is a fixed point a. Moreover, we can also take the point a' or, more generally, other points which travel within $K = K_1 \cup K_2 \cup K_3$ and eventually map to a.

However we need other points which travel inside K. To discard a and its preimages, we choose a point $d' \in [c, a]$ such that $f(d') = d$. Then set $K_2' = [c, d']$, and consider all points x which travel inside $K' = K_1 \cup K_2' \cup K_3$. Since $K' \subset f(K')$, such points exist. Moreover, by construction it follows that each such point can play the role of z' exhibits the minimal over-rotation number ρ_f in I_f. This gives a useful algorithm of finding ρ_f as well as orbits on which ρ_f is realized as the over-rotation number.

3. How to compare rotation intervals of maps?

It is sometimes important to be able not only to estimate or compute various characteristics of maps but also to compare them for different maps without in fact estimating or computing. It turns out that the problem of comparing over-rotation intervals of two continuous interval maps (not necessarily piecewise monotone) can be approached from the point of view related to the tools introduced in [B1] which involve such notions as chains and loops of admissible intervals (see Lemmas ALM, BM3 and 1.1). In fact a very simple geometrical condition allows to compare the over-rotation intervals of two maps.

Suppose that $I \subset J$ and $f : J \to J$ and $g : I \to I$ are two maps. Let g have a fixed point a and $[f(x), a] \supset [g(x), a]$ for any $x \in I$. Then we say that f is more *repellent from* a than g. Then if $g(x) < a$ then $f(x) < a$, and if $g(x) > a$ then $f(x) > a$. Hence if $g(y) = a$ and y is not a local extremum of g, then arbitrarily close to y there are points mapped by g (and therefore, f) both to the left and to the right of a. This implies that $f(y) = a$ too. In particular, if we assume that a itself is not a local extremum of g, then $f(a) = a$.

There is also another remark worth making. So far in the definition we compare how maps f and g repel *all* the points of the interval I from a. It is sometimes useful to compare the repelling of the points of some specific set $B \subset I$ only. So if $B \subset I$ and we know that $[f(x), a] \supset [g(x), a]$ for any $x \in B$ then we say that f is more *repellent from a on B* than g. In fact we deal with the following specific set B. Let g be a map with a unique fixed point a which is not a local extremum of g. For any point x let $\psi_g(x)$ be the closest to a point in $[x, a]$ (or in $[a, x]$ if $a < x$) such that $g(x) = g(\psi_g(x))$.The set B on which we compare the repelling properties of f and g is the set $\psi_g(I)$. In the statement of Lemma 3.1 we use the notation introduced above.

Lemma 3.1. *The following statements are true.*

(1) *Let $f : J \to J$ be more repellent from a than $g : I \to I \subset J$ on the set $B = \psi_g(I)$ (a is a unique fixed point of g inside I which is not a local extremum of g). Then $I_f \supset I_g$; in particular if g has a point of odd period then $P(f) \supset P(g)$.*

(2) *Let $g : [0, 1] \to [0, 1]$ and $f : [u, v] \to [u, v] \subset [0, 1]$ be two maps. Suppose that $g(0) = g(1) = 0, g|[0, c]$ is increasing, $g|[c, 1]$ is decreasing, $a \in (c, 1]$ is a unique g-fixed point in $(0, 1]$, a' is a unique g-preimage of a in $[0, c]$, $[g^2(c), g(c)] \subset [u, v]$ and $f \le g$ on $[g^2(c), a']$, $f \ge g$ on $[c, a]$, $f \le g$ on $[a, g(c)]$. Then f is more repellent from a than g on the set $B = \psi_g([g^2(c), g(c)])$ and so $I_f \supset I_g$.*

Proof. (1) Note that by the arguments we presented before stating Lemma 3.1 a is an f-fixed point. Let x be a periodic point of g of rotation pair (p, q). Consider a g-loop $[\psi_g(x), a], [\psi_g(g(x)), a], \dots, [\psi_g(g^{q-1}(x)), a]$ and prove that it is an f-loop too. To this end it suffices to show that $[f(\psi_g(g^i(x))), a] \supset [\psi_g(g^{i+1}(x)), a]$, and indeed it follows immediately from the properties of f and g and the construction of the function ψ_g. Therefore by Lemma BM3 the map f also has a periodic point of rotation number p/q which proves that $I_f \supset I_g$ as desired. This easily implies the second part of the statement (1) of the lemma.

(2) Clearly ψ_g is the identity map on $[0, a') \cup [c, 1]$. On the other hand for any $x \in [a', c)$ we have $\psi_g(x) = x'_g$ where x'_g is a unique point such that $g(x'_g) = g(x), x'_g \ne x$. Hence $B = \psi_g([g^2(c), g(c)]) = [g^2(c), a'] \cup [c, g(c)]$ and clearly f is more repellent from a on B than g. Due to the fact that all periodic points of g but the point 0 are contained in a g-invariant interval $[g^2(c), g(c)]$ it completes the proof. □

We apply Lemma 3.1 to some pairs of maps and eventually to one-parameter families of unimodal maps. Usually considered families satisfy the conditions of pointwise growth or even are formed by multiples of the same map; in other words the usual assumptions differ from those of Lemma 3.1. However “changing coordinates” (i.e properly conjugating one of the maps involved) one can try to see if two maps from a given family fit into the situation of Lemma 3.1. We work with a few classes of maps having a single turning point which for the sake of the definiteness is assumed to be maximum from now on. Also without loss of generality we work from now on with interval maps which map 0 into itself and 1 into 0.

Let $\mathcal{S} = \{f : f$ is a convex (concave down) map of the interval $[0, 1]$ into itself with a unique turning point c_f which is maximum such that $f|[0, c_f]$ and $f|[c_f, 1]$ are C^1-maps, $f(c_f) > c_f$ and $f(0) = f(1) = 0\}$ (for convenience we repeat here the definition given in Introduction). We can assume $f(c_f) > c_f$ without loss of generality; under this assumption by the convexity of f there is no fixed point in $(0, c_f]$ and there is a unique fixed point $a_f \in (c_f, 1]$. For any $x \neq c_f$ there is a well defined point $x'_f \neq x$ such that $f(x) = f(x'_f)$; also let $c'_f = c_f$.

Lemma 3.2. *The following statements are true.*

(1) *Let $f, g \in \mathcal{S}$ be such maps that $|f'| \geq |g'|$ and $\dfrac{|c_f - a_f|}{|c_f - a'_f|} \geq \dfrac{|c_g - a_g|}{|c_g - a'_g|}$. Then $I_f \supset I_g$.*

(2) *Let $g \in \mathcal{S}$ and $|g'(x)(x - c_g)| \leq |g'(x'_g)(x'_g - c_g)|$ for any $x \geq c_g$. Then $I_{\nu g} \supset I_g$ for any $\nu > 1$.*

Proof. (1) We can assume that the set of periodic points $P(g)$ of g is not $\{1\}$. Observe that $c_g = c_f = c$ which follows from $|f'| \geq |g'|$ since $0 = |f'(c_f)| \geq |g'(c_f)|$. Notice also that since $|f'| \geq |g'|$ then $a_g \leq a_f$. Indeed, $|f'| \geq |g'|$ implies $\lambda(g[a_f, 1]) \leq \lambda(f[a_f, 1])$ (recall that by $\lambda(A)$ we denote the Lebesgue measure of a set A); since $f[a_f, 1] = [0, a_f]$ and $g(1) = 0$ we conclude that $g[a_f, 1] \subset [0, a_f]$ and so $g(a_f) \leq a_f$ which implies that $a_g \leq a_f$. Let φ be a *linear* contraction of $[0, 1]$ towards the point c onto its image I' which maps c into c and a_f into $a_g = a$. The coefficient of contraction then is $q = \dfrac{|c - a_g|}{|c - a_f|} \leq 1$ so that $\varphi'(u) = q$ for each y. The map φ conjugates f and $h : I' \to I'$; moreover, by construction $g(a) = h(a) = a$. Let us show that $|h'(x)| \geq |g'(x)|$ for any $x \in I'$. Indeed, points are attracted by φ closer to c but remain to the same side of c. Then by the properties of f and g and by the choice of φ we have

$$|h'(x)| = q \cdot |f'(\varphi^{-1}(x))| \cdot q^{-1} = |f'(\varphi^{-1}(x))| \geq |g'(\varphi^{-1}(x))| \geq |g'(x)|$$

Let us show that h is more repellent from a on $[c, h(c)]$ than g. Indeed, let $a \leq y \leq h(c)$. Then $\lambda(h[a, y]) = a - h(y) \geq \lambda(g[a, y]) = a - g(y)$ which implies that $h(y) \leq g(y)$. Similarly for $c \leq y \leq a$ one can show that $h(y) \geq g(y)$. In particular it implies that $[g^2(c), g(c)] \subset [h^2(c), h(c)]$. Let us check that h is more repellent from a than g on $[g^2(c), a'_g]$. First we show that $a'_g = a' \leq a'_h$. Indeed, using the fact that $c = c_h = c_f = c_g$ and $a = a_g = a_h$ we can write $\dfrac{|c - a|}{|c - a'_h|} = \dfrac{|c - a_f|}{|c - a'_f|} \geq \dfrac{|c - a|}{|c - a'|}$ (the last inequality is actually given in the conditions of the lemma as $a = a_g$ and $a' = a'_g$) which implies that indeed $a' \leq a'_h$. Hence $h(a') \leq h(a'_h) = a = g(a')$. Since $|h'| \geq |g'|$ we have $\lambda(h[z, a']) \geq \lambda(g[z, a'])$ for any $z \in [g^2(c), a']$ which in turn implies that $h(z) \leq g(z)$. This allows to apply Lemma 3.1 and completes the proof of statement (1). Figure 3.1 illustrates the proof. On this figure as well as on some

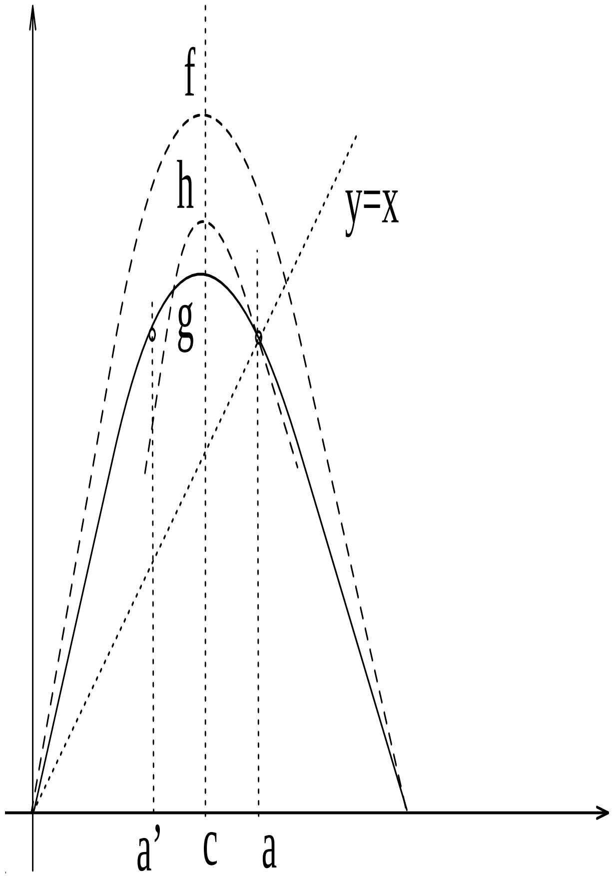

FIGURE 3.1. Functions g (continuous line) and f, h (dashed line)

of the following figures the graphs of functions are given in continuous or dashed lines while additional dotted lines are playing explanatory role.

(2) This statement is in fact a corollary of statement (1). Indeed, let $\nu g = f, c_g = c_f = c, a_g = a, a'_g = a'$; also for any z let $z' = z'_g = z'_f = \tau(z)$. To apply statement (1) we need to check that $\frac{|c-a_f|}{|c-a'_f|} \geq \frac{|c-a|}{|c-a'|}$. By the proven in the beginning of the proof of (1), $a_f > a$. Thus it is enough to show that for any pair of points x, y such that $c < z \leq y$ we have $\frac{|c-y|}{|c-y'|} \geq \frac{|c-z|}{|c-z'|}$. Let us show that the function $\psi(x) = \ln(x-c)/(c-x') = \ln(x-c) - \ln(c-\tau(x))$ is increasing for $z \in (c, 1]$. Notice that $\tau'(x) = g'(x)/g'(x')$; therefore $\psi'(x) = \frac{1}{x-c} - \frac{-\tau'(x)}{c-\tau(x)} = \frac{1}{x-c} + \frac{g'(x)}{g'(x')(c-x')} \geq 0$ (the last inequality follows from $|g'(x)(x-c)| \leq |g'(x')(x'-c)|$). $\square$

Call a map g *even* if $g(x) = g(1-x)$ for any $x \in [0, 1]$. Clearly Lemma 3.2(1) holds if both f and g are even and $|f'| \geq |g'|$ because then $\frac{|c_f - a_f|}{|c_f - a'_f|} = \frac{|c_g - a_g|}{|c_g - a'_g|} = 1$

automatically. E.g., this applies if f and g are from the quadratic family. Similarly, Lemma 3.2(2) holds if g is even.

For the sake of convenience let us call all polynomials of degree no more than 3 "cubic". Another result close to that of Lemma 3.2 deals with maps from the class $\mathcal{G} \subset \mathcal{S}, \mathcal{G} = \{g : g \in \mathcal{S}$ is a cubic polynomial map of $[0, 1]$ into itself $\}$. Our purpose is to prove an analog of Lemma 3.2 for maps from $\mathcal{G}$. It is worth mentioning here that in fact the assumption of convexity could be somewhat weakened; some statements are proven for maps from the family $\mathcal{H} = \{f : f$ is a cubic map of $[0, 1]$ into $\mathbb{R}_+ \cup 0$ with a unique critical point in $[0, 1]$ such that $f(0) = f(1) = 0\}$. We combine geometrical trivia about cubic maps in Lemma 3.3; also, some of the statements in what follows are proven for the sake of completeness.

Lemma 3.3. *Let $f \neq g$ be non-trivial cubic polynomials.*

1. *The graphs of f and g have no more than three common points and they cannot have two common points at which their derivatives are equal.*
2. *If there are three different points $x < y < z$ at which $f = g$ then at none of them $f' = g'$ and $f - g$ has one sign on $(-\infty, x) \cup (y, z)$ and the other on $(x, y) \cup (z, \infty)$.*
3. *If there are exactly two points $x < y$ between, say, a and b at which $f = g$ then either* (a) *$f'(x) \neq g'(x), f'(y) \neq g'(y)$ and $f - g$ has one sign on $(a, x) \cup (y, b)$ and the other on (x, y), or* (b) *$f'(x) = g'(x), f'(y) \neq g'(y)$ and $f - g$ has one sign on $(a, x) \cup (x, y)$ and the other on (y, b), or* (c) *$f'(x) \neq g'(x), f'(y) = g'(y)$ and $f - g$ has one sign on (a, x) and the other on $(x, y) \cup (y, b)$.*
4. *Let $a < b, f(a) < g(a), f(b) < g(b)$. Then either* (a) *$f(x) < g(x)$ for any $x \in (a, b)$, or* (b) *there is a single $x \in (a, b)$ such that $f(x) = g(x)$, $f(y) < g(y)$ for $y \in (a, b) \setminus \{x\}$ and in fact $f'(x) = g'(x)$, or* (c) *there are two points $u < v$ in (a, b) such that $f(y) < g(y)$ for $y \in (a, u)$, $f(y) > g(y)$ for $y \in (u, v)$, $f(y) < g(y)$ for $y \in (v, b)$ and $f'(u) \neq g'(u), f'(v) \neq g'(v)$.*
5. *If $f \geq g$ on $[a, b]$ and $f(a) = g(a), f(b) = g(b)$ then in fact $f > g$ on (a, b).*

Proof. Statements (1), (2), (3) follow immediately from the fact that f and g are cubic. Statement (4) follows from (2) and (3). Statement (5) follows from (2). □

The next lemma studies properties of conjugacies of cubic maps.

Lemma 3.4. *Let $f \geq g, f \neq g$ be two maps from $\mathcal{H}$. Let ψ be a linear non-strict contraction with the fixed point 0 which maps $[0, 1]$ onto $[0, v], v \leq 1$ and conjugates f to a map $h : [0, v] \to \mathbb{R}_+ \cup 0$. Then the following holds:*

1. *if $h \leq g$ in a small right semi-neighborhood of 0 then $h'(0) = f'(0) = g'(0)$ and either $v = 1, f = g = h$, or $h < g$ on $(0, v]$;*
2. *if* (1) *does not hold then either* (a) *$v = 1$ and $h = f > g$ inside $(0, 1)$, or* (b) *$v < 1$ and there exists a point $u \in (0, v)$ such that $h(x) > g(x)$ if $x \in (0, u)$, $h(x) < g(x)$ if $x \in (u, v]$ and $h'(u) \leq g'(u)$.*

Proof. Note that by the construction $h(v) = 0 < g(v)$ if $v < 1$.

(1) By the assumption $h'(0) \leq g'(0)$. On the other hand $f \geq g$ and so $f'(0) \geq g'(0)$. Since f is conjugate to h by a linear map ψ we see that $f'(0) = h'(0)$ and so $h'(0) \geq g'(0)$. Thus in fact $h'(0) = g'(0) = f'(0)$. If now $v = 1$ we immediately get $f = g = h$. Suppose that $v < 1$. Then $h(v) = 0 < g(v)$. Let us show that then $h < g$ on $(0, v]$. Indeed, otherwise either there are two points between 0 and v at which $g = h$ (as $h'(0) = g'(0)$ and $h(0) = g(0)$, this is impossible for cubic maps), or there is one such point between 0 and v and at that point both $g = h$ and $g' = h'$ (which is impossible for similar reasons). Hence $h < g$ on $(0, v]$.

(2) First assume that $v = 1$ and so $h = f$; then by Lemma 3.3(5) we have possibility (2)(a). So let $v < 1$ and consider possibilities concerning common points of the graphs of h and g on $(0, v]$. If there are no such points then $h(v) < g(v)$ implies $h(x) < g(x)$ for all $x \in (0, v]$ and by (1) we get possibility (2)(b) from the lemma. Suppose there are such points. Then by Lemma 3.3(1) there are no more than two of them. Consider a few cases.

(i) Let $0 < s < t < v$ be such that $h(s) = g(s)$ and $h(t) = g(t)$. By Lemma 3.3.(2) then the fact that $h(v) < g(v)$ implies that $h(x) < g(x)$ if $x \in (0, s)$. Hence by (1) $h'(0) = g'(0)$ which contradicts Lemma 3.3(2).

(ii) Let $u \in (0, v]$ be the only point in this interval such that $h(u) = g(u)$. Then $h < g$ on $(u, v]$. If $h < g$ on $(0, u)$ then clearly $h'(u) = g'(u)$ and by (1) $h'(0) = g'(0)$. Since g and h are both cubic but not identical, this is impossible. Hence $h > g$ on $(0, u)$. It follows then that in fact $h'(u) \leq g'(u)$ and we get possibility (b) from the lemma. This completes the proof. □

In the next lemma we study some geometrical properties of maps from $\mathcal{H}$.

Lemma 3.5. *Let $f \geq g, f \neq g$ be two maps from $\mathcal{H}$. Then the following holds.*

1. *Let the line $y = \gamma x, \gamma > 0$ intersect the graph of g at a point $(x, \gamma x), x > 0$ and the graph of f at a point $(z, \gamma z), z > x$. Moreover, suppose that $g'(x) \leq 0$. Then $f'(z) \leq g'(x)$ and moreover $x/x'_g < z/z'_f$.*
2. *$g(c_g)/c_g < f(c_f)/c_f$*

Proof. (1) Consider the linear contraction ψ with a fixed point 0 which maps z into x and conjugates f and a map h. Let $v = \psi(1) < 1$. Then $h(x) = g(x)$ so neither Lemma 3.4(1) nor Lemma 3.4(2)(a) holds. Thus, Lemma 3.4(2)(b) holds which implies that $g > h$ on (x, v) and $g < h$ on $(0, x)$. Moreover, by the same lemma $h'(x) = f'(z) \leq g'(x)$. The last inequality implies that, since $g'(x) \leq 0$ (i.e. $x \geq c_g$), then $f'(z) \leq 0$ (i.e. $z \geq c_f$).

Let $x'_g = x', x'_h = x'', z'_f = z'$. Then $x' \leq x, x'' < x$. By construction $g(x') = g(x) = h(x) = h(x'') = \gamma x$. Since $g < h$ on $(0, x)$ then $g(x'') < h(x'') = g(x)$ which implies $x'' < x'$ since otherwise $x' \leq x'' < x$ and so $g(x'') \geq g(x)$. Hence $x/x' = x/x'_g < x/x'' = x/x'_h = z/z'_f$. Figure 3.2 illustrates the arguments.

(2) Consider the line α which connects the points $(0, 0)$ and $(c_g, g(c_g))$. Then the point $(c_g, f(c_g))$ is strictly above this line. This implies that the graph of f and

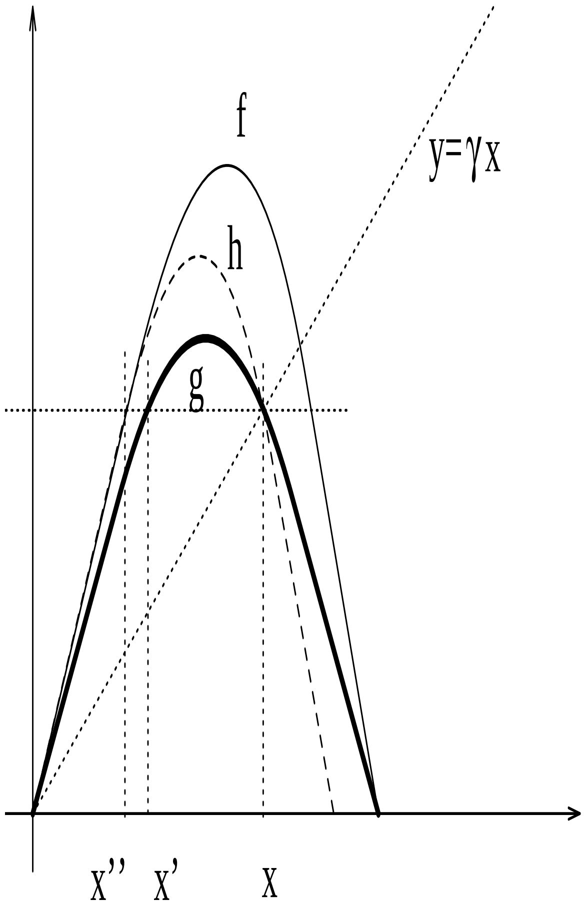

FIGURE 3.2. Functions g (bold line), f (regular line) and h (dashed line) from Lemma 3.5.

the line α intersect strictly to the right of c_g. Suppose that the point of intersection is (z, y). Let us apply statement (1) to this situation. Then $f'(z) \leq g'(c_g) = 0$ and so $z \geq c_f$. Since $f(c_f) \geq f(z) = y$ we see that the point $(c_f, f(c_f))$ lies to the left and above the point (z, y). Clearly it implies that $g(c_g)/c_g \leq f(c_f)/c_f$. $\square$

The next lemma is important for the proof of the analog of Lemma 3.2 for maps from $\mathcal{G}$.

Lemma 3.6. *Let $f, g \in \mathcal{H}$ and g have a fixed point $a_g \in [c_g, 1]$. If $f \geq g, f \neq g$ then there is a unique f-fixed point $a_f \geq c_f$ and $a_f > a_g$, $a_g/a'_g < a_f/a'_f$ and $f'(a_f) \leq g'(a_g)$.*

Proof. By Lemma 3.3(5) $f(x) > g(x)$ if $x \in (0, 1)$. Consider the line $y = x$. Clearly this line intersects the graph of f at a point (z, w) where $z > a_g$ is an f-fixed point. Then by Lemma 3.5 $f'(z) \leq g'(a_g) \leq 0$ and so $z \geq c_f$. On the other hand obviously if there exists an f-fixed point $a_f \in [c_g, 1]$ then it is unique. Thus, $z = a_f$, and by the above $a_f > a_g$. Observe that f moves a_g strictly to the right

$(f(a_g) > g(a_g) = a_g)$. It remains to apply Lemma 3.5(1) to the line $y = x$ and points (a_g, a_g) on the graph of g and (a_f, a_f) on the graph of f. □

Now we are ready to prove the analog of Lemma 3.2 for maps from $\mathcal{G}$.

Lemma 3.7. *Let $f \geq g$ and $f, g \in \mathcal{G}$. Then $I_f \supset I_g$.*

Proof. The idea of the proof is the same as in Lemma 3.2. We may assume that $f \neq g$ and that there is a fixed point of g in $[c_g, 1]$. Then by Lemma 3.6 $a_f > a_g$. Consider a contraction φ which has a fixed point 0 and maps a_f to $a_g = a$ (thus, $\varphi(x) = (a_g/a_f)x$). Let φ conjugate f to a map $h : [0, \varphi(1)] \to [0, \varphi(1)]$; by Lemma 3.6 $\varphi(a'_f) = a'_f(a_g/a_f) < a'_g$. Consider now the linear contraction ψ which has a fixed point a and maps $\varphi(a'_f)$ to a'_g. Then $\zeta = \psi \circ \varphi$ is a linear map such that $\zeta(a_f) = a_g = a, \zeta(a'_f) = a'_g = a', \zeta([0, 1]) = [u, v] \subset [0, 1]$. Let $w : [u, v] \to [u, v]$ be a map to which f is conjugate by ζ. We prove that Lemma 3.1 is applicable to the maps w and g which will then imply the required. Figure 3.3 illustrates the proof.

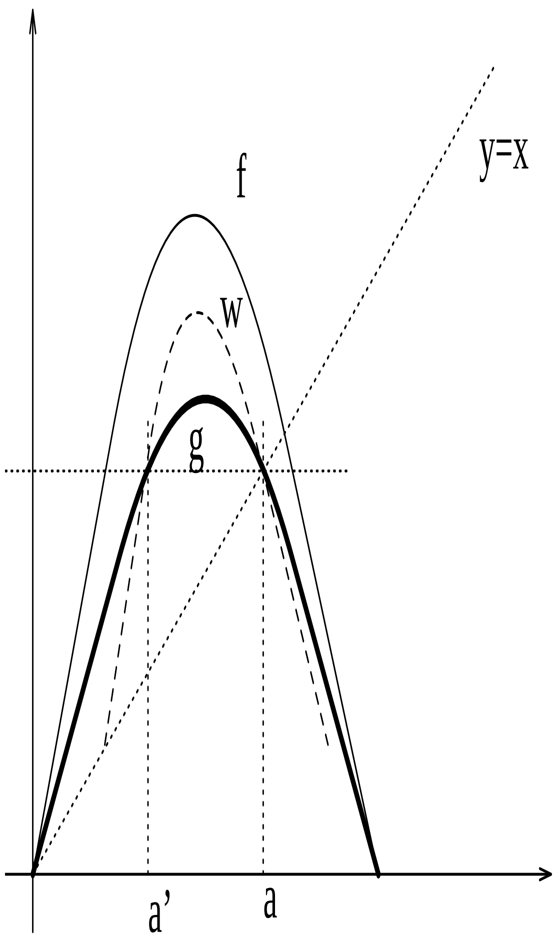

FIGURE 3.3. Functions g (bold line), f (regular line) and w (dashed line) from Lemma 3.7

Indeed, the convexity of g implies that $w(u) = u < g(u)$. Now, the segment of straight line $\mathbf{k}$ connecting points (a, a) and $(\varphi(1), 0)$ is located below the graph of g on the interval $[a, \varphi(1)]$ because g is convex. At the same time it is clear that the linear map ψ conjugates h and w; since ψ is linear then a point $(\psi(\varphi(1)), w(\psi(\varphi(1)))) = (v, w(v))$ lies on $\mathbf{k}$ and thus is still below the graph of g which means that $w(v) < g(v)$. These are the only two times we rely upon the convexity in the proof. Compare w and g on $[u, v]$. By Lemma 3.3(4) we have $w(x) < g(x)$ if $x \in [u, a')$, $w(x) > g(x)$ if $x \in (a', a)$ and $w(x) < g(x)$ if $x \in (a, v]$. This allows to apply Lemma 3.2 which completes the proof. □

For the sake of convenience we sum up the results of Lemmas 3.2 and 3.7 in Theorem 3.8 dealing with one-parameter families of interval maps.

Theorem 3.8. *Let $f_\nu, \nu \in [b, d]$ be a one-parameter family of interval maps such that one of the following properties holds.*

(1) *$f_\nu \in \mathcal{S}$ for any ν; also, if $\nu > \mu$ then $|f'_\nu| \geq |f'_\mu|$ and $\dfrac{|c_{f_\nu} - a_{f_\nu}|}{|c_{f_\nu} - a'_{f_\nu}|} \geq \dfrac{|c_{f_\mu} - a_{f_\mu}|}{|c_{f_\mu} - a'_{f_\mu}|}$.*

(2) *$f = f_b \in \mathcal{S}$, $|f'(x)(x - c_f)| \leq |f'(x'_f)(x'_f - c_f)|$ for any $x \geq c_f$ and $f_\nu = \nu f$.*

(3) *$f_\nu \in \mathcal{G}$ for any ν and $f_\nu \geq f_\mu$ if $\nu > \mu$.*

Then $I_{f_\nu} \supset I_{f_\mu}$ if $\nu > \mu$; in particular, if f_b has an odd periodic point then $P(f_\nu) \supset P(f_\mu)$ if $\nu > \mu$.

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