

ZERO ENTROPY PERMUTATIONS

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ABSTRACT. The entropy of a permutation is the (topological) entropy of the “connect-the-dots” map determined by it. We give matrix- and graph-theoretic, geometric, and dynamical characterizations of zero entropy permutations, as well as a procedure for constructing all of them. We also include some information about the number of zero entropy permutations.

INTRODUCTION

This paper deals with “combinatorial dynamics on an interval”. A *map of an interval* is a continuous map from a nondegenerate, compact interval to itself. For such a map f , let $P(f)$ denote the set of *periodic points* of f , i.e., points p such that

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$f^k(p) = p$ for some $k \geq 1$. (Here $f^2 = f \circ f$, etc.) The least such k is called the *period* of p .

If f has finitely many periodic points, then $f|_{P(f)}$ is given by a permutation π of $\{1, \dots, n\}$, where $n = \#P(f)$. (Here $\#$ denotes cardinality.) If $P(f) = \{p_1, \dots, p_n\}$ with $p_1 < \dots < p_n$, then each $p_{\pi(i)} = f(p_i)$. It is occasionally useful to consider permutations of finite, totally ordered sets other than those of the form $\{1, \dots, n\}$. For permutations, as for maps, we use exponentiation to denote iterated composition.

A map f of an interval with finitely many periodic points has zero (topological) entropy [B-1]. With entropy of permutations appropriately defined, the associated permutation $f|_{P(f)}$ also has zero entropy. Conversely, we will show (Theorem 3.1) that if π has zero entropy, then there is a map f of an interval with finitely many periodic points such that π is a subpermutation of the permutation given by $f|_{P(f)}$. In [LR], J. Llibre and A. Reventós described the geometric structure of a class of zero entropy permutations. In this paper, we give matrix- and graph-theoretic, geometric, and dynamical characterizations of zero entropy permutations, and a method of constructing all of them. We also give a crude result about the number of zero entropy permutations.

We would be remiss if we failed to point out that it follows from Sharkovskii's Theorem and standard entropy estimates (see e.g., [BF], [BGM]) that the lengths of the subcycles (i.e., cyclic subpermutations) of a zero entropy permutation are powers of 2, and every subcycle is simple [B-2]. As we shall see, the converses of these statements are false.

Related questions concerning periodic orbits of zero entropy tree maps are studied in [Bl-1] and [Bl-2].

1. MATRIX- AND GRAPH-THEORETIC CHARACTERIZATIONS

Let π be a permutation of $\{1, \dots, n\}$. A map f of an interval has a *representative* of π if there exist $p_1 < \dots < p_n$ such that each $f(p_i) = p_{\pi(i)}$. The *entropy* of π , denoted $\text{ent}(\pi)$, is the infimum of the topological entropies [AKM], also denoted $\text{ent}(\cdot)$, of all maps of an interval which have a representative of π . It is well-known ([ALM, Theorem 4.4.5], [BC, Proposition VIII.19]) that $\text{ent}(\pi) = \text{ent}(L_\pi) = \text{ent}(\tilde{L}_\pi)$, where $L_\pi : [1, n] \rightarrow [1, n]$ is the *canonical π -linear map*, which is obtained by “connecting the dots” of $\{(i, \pi(i)) : i = 1, \dots, n\}$ by straight lines, and $\tilde{L}_\pi : [1, n] \rightarrow [1, n]$ is the *canonical π -adjusted map*, which is obtained from L_π by replacing some of the straight lines by the graphs of quadratic functions. See [MN, Section 1] for details.

The combinatorial dynamics with which we are concerned can be studied by means of S. Baldwin’s forcing relation [Ba]. A permutation π *forces* a permutation θ if every map of an interval which has a representative of π also has a representative of θ . π and θ need not be permutations of the same number of elements. The reason for looking at π -adjusted maps instead of just π -linear maps is given by the following lemma. See also Theorem 3.1.

Lemma 1.1. [MN, Corollary 1.16] *π forces θ if and only if the canonical π -adjusted map \tilde{L}_π has a representative of θ .*

To determine the entropy of a permutation π of $\{1, \dots, n\}$, we use its *Markov graph*. It is the directed graph G with $n - 1$ vertices, v_1, \dots, v_{n-1} , and an edge $v_i \rightarrow v_j$ if and only if $L_\pi[i, i + 1] \supseteq [j, j + 1]$ — equivalently, $\pi(i) \leq j < j + 1 \leq \pi(i + 1)$ or $\pi(i + 1) \leq j < j + 1 \leq \pi(i)$. It is well-known (see, e.g., [ALM], [BC, Proposition VIII.19]) that $\text{ent}(L_\pi)$ is the logarithm of the spectral radius (the largest of the absolute values of the

eigenvalues) of the *adjacency matrix* of π , the $(n-1) \times (n-1)$ matrix A given by the equivalent definition of G : $A_{ij} = 1$ if $\pi(i) \leq j < j+1 \leq \pi(i+1)$ or $\pi(i+1) \leq j < j+1 \leq \pi(i)$, $A_{ij} = 0$ otherwise. The Perron-Frobenius Theorem [S] says that for a nonnegative matrix, the spectral radius is itself an eigenvalue. The mapping of permutations to adjacency matrices is one-to-one, except that both permutations of $\{1, 2\}$ have adjacency matrix (1).

A *path* in a directed graph with no multiple edges is a sequence of vertices, each connected to the next by an edge. Equivalently, a path is a sequence of edges, each ending where the next begins. The *length* of a path is the number of (not necessarily distinct) edges in it. A *loop* is a closed path; it is *prime* if it has no proper subloops; it is *nonrepetitive* if it is not the repetition of a proper subloop. (Note that we are not using the standard graph-theoretic meaning of loop — a closed path of length one.) Prime loops are called primitive in [MN]; nonrepetitive loops are called prime in [MN], simple in [ALM]. $\text{Tr}(\cdot)$ denotes the trace of a matrix.

Lemma 1.2. *Let A be an $n \times n$ 0–1 matrix. A is the adjacency matrix of a directed graph G . The following statements are equivalent.*

- (1) *The spectral radius of A is 0 or 1.*
- (2) *Each vertex of G belongs to at most one prime loop.*
- (3) *$\{\text{Tr}(A^k) : k \geq 1\}$ is bounded.*
- (4) *$\text{Tr}(A^k) \leq n$ for all $k \geq 1$.*
- (5) *$(A^k)_{ii} \leq 1$ for $i = 1, \dots, n$ and all $k \geq 1$.*
- (6) *$(A^k)_{ii} \leq 1$ for $i = 1, \dots, n$ and $k = 1, \dots, 2n - 1$.*

Proof. $(A^k)_{ij}$ is the number of paths of length k starting at vertex i and ending at vertex j , and by [S], the spectral radius

of A is $\limsup \sqrt[k]{\text{Tr}(A^k)}$. It is then easy to prove the equivalence of (1)–(5). Clearly (5) implies (6). To prove that (6) implies (2), suppose that there are two prime loops in G with vertex i in common. If k is the sum of the lengths of these loops, then $(A^k)_{ii} \geq 2$. The largest such k is $2n - 1$, which occurs when G has one loop of length n and one of length $n - 1$. \square

The following theorem describes the structure of the Markov graph of a zero entropy permutation. Implicit in the theorem is an algorithm for determining whether or not a permutation has zero entropy.

Theorem 1.3. *Let π be a permutation of $\{1, \dots, n\}$, let A be its adjacency matrix, and let G be its Markov graph. The following statements are equivalent.*

- (1) $\text{ent}(\pi) = 0$.
- (2) $(A^k)_{ii} \leq 1$ for $i = 1, \dots, n - 1$ and all $k \geq 1$.
- (3) $(A^k)_{ii} \leq 1$ for $i = 1, \dots, n - 1$ and $k = 1, \dots, 2n - 3$.
- (4) *Distinct prime loops in G have no vertices in common.*
- (5) *Each vertex of G belongs to a unique prime loop.*
- (6) *Each nonrepetitive loop in G is prime.*
- (7) *If α and α' are distinct prime loops in G and there is a path from some vertex in α to some vertex in α' , then there is no path from any vertex in α' to any vertex in α .*

Proof. The equivalence of (1)–(4) follows from Lemma 1.2, and (5) implies (4) is immediate.

(4) implies (5). Let $1 \leq i \leq n - 1$. Since π is a permutation, there exists $k \geq 1$ such that $\pi^k(i) = i$ and $\pi^k(i + 1) = i + 1$. Therefore, each vertex belongs to a loop, hence to a prime loop.

(4) implies (6). Suppose there exists a nonrepetitive loop α which is not prime. Then α is the concatenation of two distinct loops, β and β' . There is a common vertex v of β and β' , edges

$v \rightarrow w$ of β and $v \rightarrow w'$ of β' , such that $w \neq w'$. These edges are contained in prime loops and these prime loops must be distinct. This contradicts (4).

(6) implies (4). Suppose α and α' are distinct prime loops which have a vertex v in common. Let β be the concatenation of α and α' . Since v appears exactly once in α and exactly once in α' , β is nonrepetitive. Since it is clearly not prime, this contradicts (6).

(4) implies (7). Let α and α' be distinct prime loops. Let v and w be vertices in α , v' and w' vertices in α' , and suppose there are paths from v to v' and from w' to w . Since every edge which is in some loop is also in a prime loop, there is a prime loop passing through some vertex u in α which also passes through some vertex which is not in α . Then u belongs to two distinct prime loops. This contradicts (4).

(7) implies (4). Suppose distinct prime loops α and α' have a vertex v in common. Set $w = w' = v' = v$. There is a path from v to v' and one from w' to w . This contradicts (7). \square

2. GEOMETRIC CHARACTERIZATION

Let π be a permutation of a finite, totally ordered set S . (It may be helpful to think of S as $\{1, \dots, n\}$.) An *interval* is a set of at least two consecutive members of S . An interval is called a *block* in [MN].

An interval I is (π -) *shrinkable* if for every $k \geq 0$, $\pi^k(I)$ is an interval and π is monotone on $\pi^k(I)$, and any two intervals in $\{\pi^k(I) : k \geq 0\}$ are either disjoint or identical. A *shrinking* of π is a permutation θ defined as follows. Let π be a permutation of S , let I be a π -shrinkable interval, and let $K \geq 1$ be least such that $\pi^K(I) = I$. By standard “ f -covering” arguments [B-3], there is an \tilde{L}_π -periodic point $p \in [\min I, \max I]$ of period K .

Let $T = (S - \{\pi^k(I)\}) \cup \{\tilde{L}_\pi^k(p)\}$ and $\theta = \tilde{L}_\pi|_T$.

Remark. The \tilde{L}_π -periodic points in $[\min I, \max I]$ have \tilde{L}_π -period either K or $2K$. Thus p may be chosen to be in S except if $\#I$ is even and the orientations σ_k (+1 for increasing, -1 for decreasing) of π on $\pi^k(I)$ satisfy $\sigma_0\sigma_1\cdots\sigma_{K-1} = -1$.

The reverse of a shrinking is an expansion. An *expansion* of θ is a permutation π defined as follows. Let θ be a permutation of T , let $m \geq 2$, let $t_0 \in T$, let $K \geq 1$ be least such that $\theta^K(t_0) = t_0$, and for $k = 0, \dots, K-1$, let $\sigma_k = +1$ or -1 . Let

$$S = ((T - \{\theta^k(t_0)\}) \times \{0\}) \cup (\{\theta^k(t_0)\} \times \{1, \dots, m\}),$$

ordered lexicographically. Define $\pi(t, 0) = (\theta(t), 0)$ if $t \notin \{\theta^k(t_0)\}$, and for $k = 0, \dots, K-1$,

$$\pi(\theta^k(t_0), i) = \begin{cases} (\theta^{k+1}(t_0), i) & \text{if } \sigma_k = +1 \\ (\theta^{k+1}(t_0), m+1-i) & \text{if } \sigma_k = -1. \end{cases}$$

Lemma 2.1. *Let π and θ be permutations. Then θ is a shrinking of π if and only if π is an expansion of θ . In this case, $\text{ent}(\theta) = \text{ent}(\pi)$.*

Proof. Let θ be a shrinking of π . Let I be the interval shrunk to obtain θ , let $m = \#I$, let I be shrunk to t_0 , let $K \geq 1$ be least such that $\pi^K(I) = I$, and for $k = 0, \dots, K-1$, let $\sigma_k = +1$ or -1 denote the orientation of π on $\pi^k(I)$. With these choices, π is an expansion of θ . It is even easier to verify that if π is an expansion of θ , then θ is a shrinking of π .

Finally, we show that if π is an expansion of θ , then $\text{ent}(\pi) = \text{ent}(\theta)$. Let A be the adjacency matrix of the Markov graph G of π , and let B and H be the corresponding objects for θ . Then G contains a copy of H . Let the names of the vertices of H be

the same as the names of the corresponding vertices of the copy. For each vertex v of H , $(A^k)_{vv} = (B^k)_{vv}$ for all $k \geq 0$, and for each vertex v of G but not of the copy of H , $(A^k)_{vv} \leq 1$ for all $k \geq 0$. Therefore, the spectral radii of A and B are the same, and so $\text{ent}(\pi) = \text{ent}(\theta)$. \square

Theorem 2.2. *Let π be a permutation. The following statements are equivalent.*

- (1) $\text{ent}(\pi) = 0$.
- (2) π can be reduced to the one-point permutation by a sequence of shrinkings.
- (3) π can be built from the one-point permutation by a sequence of expansions.

Proof. It follows immediately from Lemma 2.1 that (2) and (3) are equivalent and that (2) implies (1). We show that (1) implies (2).

By Lemma 2.1, it suffices to show that if π is a zero entropy permutation, then there is a π -shrinkable interval. By (5) and (7) of Theorem 1.3, there is a prime loop α in the Markov graph such that any edge which starts at a vertex of α belongs to α .

Let K be least such that for some vertex v_i of α , $\pi^K \{i, i+1\} \cap \{i, i+1\} \neq \emptyset$. Then $\pi^K(i) = i+1$ if and only if $\pi^K(i+1) = i$. Thus there are three cases.

- (1) i and $i+1$ are fixed by π^K .
- (2) i and $i+1$ are interchanged by π^K .
- (3) One element of $\{i, i+1\}$ is fixed by π^K and the other is not.

In the first two cases, $\{i, i+1\}$ is a π -shrinkable interval. In the third case, we may assume that $\pi^K(i) = i$ and $\pi^K(i+1) \neq i+1$. Then $\pi^K(i+1) = i-1$, $\pi^K(i-1) = i+1$, and the three-point set $\{i-1, i, i+1\}$ is a π -shrinkable interval. \square

Implicit in the proof of Theorem 2.2 is a procedure (actually, a number of procedures) for constructing all zero entropy permutations. Because there are choices to be made — which point to expand, how big to make the expanded interval, etc. — Theorem 2.2 does not define a canonical procedure.

On the other hand, there is a canonical procedure for reducing a zero entropy permutation to the one-point permutation. Specifically, let π be a zero entropy permutation of $\{1, \dots, n\}$. Let v_1, \dots, v_{n-1} be the vertices of the Markov graph of π and define a permutation θ of $\{v_1, \dots, v_{n-1}\}$ as follows. Set $\theta(v_i) = v_j$, where $v_i \rightarrow v_j$ is the unique edge of the unique prime loop containing v_i which starts at v_i . Note that θ need not be a shrinking of π .

Proposition 2.3. *Let π be a zero entropy permutation and let θ be defined as above. Then $\text{ent}(\theta) = 0$.*

Proof. It can be verified that there is a map $f : [1, n] \rightarrow [1, n]$ satisfying

- $f(i) = \pi(i)$ for $i = 1, \dots, n$.
- f is monotone on each interval $[i, i + 1]$.
- For each prime loop $v_{i_1} \rightarrow \dots \rightarrow v_{i_k} \rightarrow v_{i_1}$ in the Markov graph, there is an f -periodic orbit $\{p_1, \dots, p_k\}$ such that each $p_j \in (i_j, i_j + 1)$.

Then $\text{ent}(f) = \text{ent}(L_\pi) = 0$ (the first equality by [BC, Proposition VIII.19]) and f has a representative of θ . Thus $\text{ent}(\theta) = 0$. \square

Remark. π need not force θ . For example, let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$. Then $\theta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ and we may take $f : [1, 4] \rightarrow [1, 4]$ to be defined by $f(1) = 2, f(2) = 1, f(2\frac{1}{2}) = 2\frac{1}{2}, f(3\frac{1}{2}) = 3\frac{1}{2}, f(4) = 5, f(5) = 4$, and f is linear on the intervals $[1, 2], [2, 2\frac{1}{2}]$, etc.

3. DYNAMICAL CHARACTERIZATION

It follows from Sharkovskii's Theorem and a result of M. Misiurewicz and W. Szlenk [MS] that a permutation has zero entropy if it forces only finitely many cycles. Theorem 3.1 below is a refinement of that result.

Theorem 3.1. *Let π be a permutation. The following statements are equivalent.*

- (1) $\text{ent}(\pi) = 0$.
- (2) *The canonical π -adjusted map \tilde{L}_π has only finitely many periodic points.*
- (3) *π forces only finitely many cycles.*
- (4) *π forces only cycles whose periods are no greater than the period of the largest subcycle of π .*

Proof. Since \tilde{L}_π has only finitely many representatives of any cycle forced by π , it follows that (2) and (3) are equivalent.

Since (4) obviously implies (3), and (3) implies (1) by the remark above, we need show only that (1) implies (4). Let π be a zero entropy permutation of $\{1, \dots, n\}$ which forces a cycle θ . By [MN, Corollary 1.16], \tilde{L}_π has a representative Q of θ . If $Q \cap \{1, \dots, n\} \neq \emptyset$, then θ is a subcycle of π . So suppose that $Q \cap \{1, \dots, n\} = \emptyset$. Let $k = \#Q$ and let

$$(\Lambda) \quad v_{\lambda_0} \rightarrow v_{\lambda_1} \rightarrow \cdots \rightarrow v_{\lambda_{k-1}} \rightarrow v_{\lambda_k}$$

with $\lambda_0 = \lambda_k$ be the loop of length k in the Markov graph of π such that $q = \min Q \in \bigcap_{i=0}^{k-1} \tilde{L}_\pi^{-i}[\lambda_i, \lambda_i + 1]$.

Since $Q \cap \{1, \dots, n\} = \emptyset$, it follows from the proof of [MN, Lemma 1.8] that (Λ) is nonrepetitive. Then, by Theorem 1.3, (Λ) is prime and is the shortest loop in the Markov graph through the vertex v_{λ_0} . Let $m \geq 1$ be least such that $\pi^m(\lambda_0) =$

λ_0 and $\pi^m(\lambda_0 + 1) = \lambda_0 + 1$. Since every period is a power of 2, m is the period of λ_0 and/or $\lambda_0 + 1$. Since there is a loop of length m in the Markov graph through the vertex v_{λ_0} , it follows that $k \leq m$. Thus k is no greater than the period of the largest subcycle of π . \square

4. COUNTING ZERO ENTROPY PERMUTATIONS

We would like to count, at least asymptotically, the number z_n of zero entropy permutations of $\{1, \dots, n\}$. Recall that $\text{explim sup } \frac{1}{n} \log s_n$ measures the “exponential growth rate” of a sequence of real numbers $s_n \geq 1$. (For convenience, we set $\log 0 = 0$.)

The exact number z'_n of zero entropy cycles of $\{1, \dots, n\}$ is known [Be, Lemma 2.1]. It is $z'_n = 0$ if n is not a power of 2, $z'_{2^k} = 2^{2^k - k - 1}$. Thus $\limsup \frac{1}{n} \log z_n \geq \log 2$.

Lemma 4.1. $\lim \frac{1}{n} \log z_n = \sup \frac{1}{n} \log z_n$.

Proof. The sequence (z_n) is “super-multiplicative”, i.e., $z_{m+n} \geq z_m z_n$. Therefore (cf., sub-multiplicative sequences, which show up in the definition of entropy), $\lim \frac{1}{n} \log z_n = \sup \frac{1}{n} \log z_n$. \square

Theorem 4.2. $\log 3 \leq \lim \frac{1}{n} \log z_n \leq \log 16$.

We prove the inequalities in separate lemmas.

Lemma 4.3. $\lim \frac{1}{n} \log z_n \geq \log 3$.

Proof. A permutation π has *no crossed orbits* if whenever j lies between i and $\pi(i)$, then so does $\pi(j)$. Permutations with no crossed orbits have zero entropy and the subcycles of such permutations have period 1 or 2. Let y_n be the number of permutations of $\{1, \dots, n\}$ which have no crossed orbits. Then $y_{3n} \geq c_n \binom{3n}{n}$, where c_n is the number of permutations of $\{1, \dots, 2n\}$

with no crossed orbits and every subcycle of period 2. ($\binom{3n}{n}$ is the number of ways of placing the n subcycles of period 1 in $\{1, \dots, 3n\}$.) Setting $c_0 = 1$, we have $c_{n+1} = \sum_{k=0}^n c_k c_{n-k}$. (For θ a permutation of $\{1, \dots, 2n+2\}$ with no crossed orbits and every subcycle of period 2, consider $\theta_L = \theta|_{\{2, \dots, \theta(1)-1\}}$ and $\theta_R = \theta|_{\{\theta(1)+1, \dots, 2n+2\}}$.) Then $c_n = \frac{1}{n+1} \binom{2n}{n}$, the n th Catalan number, $y_{3n} \geq \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n}$, and

$$\lim \frac{1}{n} \log z_n \geq \sup \frac{1}{3n} \log y_{3n} \geq \log 3. \quad \square$$

Lemma 4.4. $\lim \frac{1}{n} \log z_n \leq \log 16$.

Proof. By Theorem 2.1, every zero entropy permutation can be built from the one-point permutation by a sequence of expansions. We may require that each expanded interval has either two or three points and that the sequences are subject to an “expand or die” rule: each orbit is either “living” or “dead”, the living orbits may be expanded or killed, and the dead orbits must remain dead.

We show how a zero entropy permutation may be obtained from a sequence with entries in $\{1, 2+, 2-, 3+, 3-\}$. It will be clear from the description which follows that different zero entropy permutations cannot be obtained from the same sequence. (A zero entropy permutation may be obtained from more than one sequence, but that fact is irrelevant since we are finding an upper bound for $\lim \frac{1}{n} \log z_n$.) The sequence should be thought of as a set of instructions, telling how to build the permutation from the one-point permutation, by saying what do with the living points, considering these points in increasing order.

- 1 means “kill the point”.

- $2+$ (resp. $2-$) means “replace the point by two points, mapping them in increasing (resp. decreasing) order to the (necessarily two) points which replace the image of the point”.
- $3+$ (resp. $3-$) means the same as $2+$ (resp. $2-$), except that the point is replaced by three points.

For example, let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 5 & 3 & 2 & 6 & 1 \end{pmatrix}$. π determines the sequence

$$(3-, 1, 2+, 1, 2-, 1, 2+, 2-).$$

The “translation scheme” may be easier to follow if we group the entries, placing a semicolon (;) between sets corresponding to simultaneous expansions and “deaths”.

$$(3-; 1, 2+, 1; 2-, 1; 2+, 2-)$$

(Grouping is not really necessary, as it is implicit in the sequence itself.) Here $3-$ says replace 1 by $\{1a, 1b, 1c\}$ where $1a < 1b < 1c$, $1a \mapsto 1c$, $1b \mapsto 1b$, and $1c \mapsto 1a$. Relabelling, we get $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. $(1, 2+, 1)$ says kill 1 and 3 and replace 2 by $\{2a, 2b\}$, where $2a < 2b$, $2a \mapsto 2a$, and $2b \mapsto 2b$. Relabelling, we get $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$. To see the meaning of $(2-, 1)$, note that at this stage, the living points are 2 and 3. Thus $(2-, 1)$ says replace 2 by $\{2a, 2b\}$, where $2a < 2b$, $2a \mapsto 2b$, and $2b \mapsto 2a$, and kill 3. Relabelling, we get $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$. Finally, $(2+, 2-)$ says replace 2 by $\{2a, 2b\}$ and 3 by $\{3a, 3b\}$, where $2a < 2b < 3a < 3b$, $2a \mapsto 3a$, $2b \mapsto 3b$, $3a \mapsto 2b$, and $3b \mapsto 2a$. Relabelling gives π .

Let w_n denote the number of finite sequences with entries in $\{1, 2+, 2-, 3+, 3-\}$ which determine zero entropy permutations of $\{1, \dots, n\}$ by the procedure just described. It follows from Theorem 2.2 that this procedure yields all zero entropy permutations, and hence $z_n \leq w_n$. We will show that $w_n \leq 16^n$.

Suppose a sequence B with entries in $\{1, 2+, 2-, 3+, 3-\}$ determines a zero entropy permutation of S . Write $S = A \cup D$, where A is the set of living points and D is the set of dead points. To avoid notational clutter, let $\#S = s$, etc. Thus $s = a + d$. We show by induction on s that a sequence which determines a zero entropy permutation of S with d dead points has length at most $s + d$.

For $s = 1$, there is nothing to prove. Suppose that the statement is true for all sequences which determine zero entropy permutations of sets S' , where $s' < s$. Let π be a zero entropy permutation of S and write $S = A \cup D$ as above. Let B' be the finite sequence obtained by removing the final group of entries from B . In case B has only one group of entries, the statement follows directly, so we may assume this is not the case. Then B' determines a zero entropy permutation π' of $S' = A' \cup D'$, where A' and D' are as above. Then $s' < s$, $D' \subseteq D$, and $A' = A'_2 \cup A'_3 \cup A'_0$, where

$$\begin{aligned} A'_2 &= \{i \in A' : i \text{ will be expanded to two points}\} \\ A'_3 &= \{i \in A' : i \text{ will be expanded to three points}\} \\ A'_0 &= \{i \in A' : i \text{ will be killed}\}. \end{aligned}$$

Since B has a' more entries than B' , B has length at most $(s' + d') + a' = 2s'$. But

$$2s' = 2(a' + d') \leq (2a'_2 + 3a'_3 + a'_0 + d') + (a'_0 + d') = s + d.$$

Thus a sequence which determines a zero entropy permutation of $\{1, \dots, n\}$ with d dead points has length at most $n + d$, with d entries “1” and the remainder chosen from $\{2+, 2-, 3+, 3-\}$. There are at most $\binom{d+m}{d}$ ways of placing the entries “1” in the

sequence, where $d, m \leq n$. With the entries “1” placed, there are at most 4^n ways of selecting the entries from $\{2+, 2-, 3+, 3-\}$. Since $\binom{d+m}{d} \leq \binom{2n}{n} \leq 4^n$, it follows that $w_n \leq 16^n$. Therefore, $\lim \frac{1}{n} \log z_n \leq \log 16$. \square

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