QMI Lesson 8: The Chain Rule & Marginal Analysis

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Motivation for the Chain Rule

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since $f(x_o) = y$.

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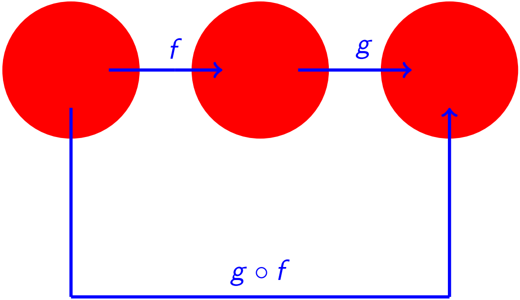
Motivation for the Chain Rule: Image



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We may write equivalently

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $h(x) = y = g(u)$ *and* $u = f(x)$.

The Generalized Power Rule

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Theorem (The Generalized Power Rule)

If $h(x) = [f(x)]^n$ *where* $n \in \mathbb{R} \setminus \{0\}$, *then*

$$h'(x) = n[f(x)]^{n-1} \cdot f'(x).$$

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$$2(x^2+1)^2(-2x+1)[3x(-2x+1) - 2(x^2+1)] =$$

$$2(x^2+1)^2(-2x+1)[-8x^2+3x-2].$$

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A Tip to Avoid the Quotient Rule

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These concepts form the basis of marginal analysis.

Definitions

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This function is used to approximate the true marginal cost. It is the first derivative of the cost function.

The marginal cost function does not give **exactly** the marginal cost, but it is a good approximation in most smooth, i.e. differentiable, cases.

Example

The cost of producing ovens is given by

$$C(x) = 7084 + 150x - 0.25x^2 \qquad (0 \leq x \leq 644).$$

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Well, $C(251) - C(250) = 24.75$.
What is the rate of change at $x = 250$?
Well $C'(250) = 150 - 0.5(250) = 25$.

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Definition (Marginal Average Cost)

The marginal average cost function, denoted $\bar{C}'(x)$ is given by

$$\bar{C}'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right].$$

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Notice, the marginal average cost is *always* negative! So, producing an extra unit always lowers the average cost, meaning that $\bar{C}(x)$ must be a decreasing function. Moreover, notice that $\lim_{x \to \infty} \bar{C}(x) = 20$.

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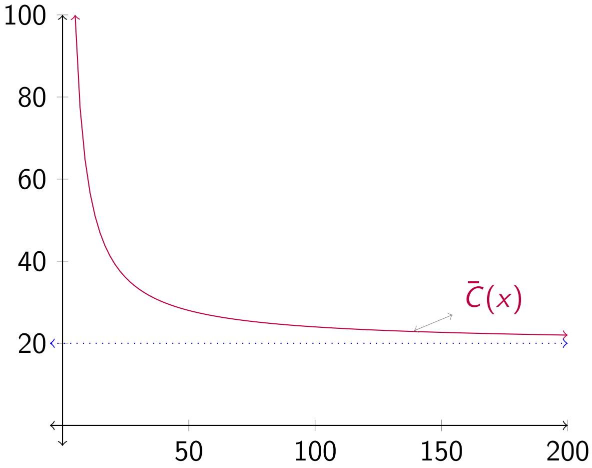
Notice, the marginal average cost is *always* negative! So, producing an extra unit always lowers the average cost, meaning that $\bar{C}(x)$ must be a decreasing function. Moreover, notice that $\lim_{x\to\infty} \bar{C}(x) = 20$. This makes sense, because the fixed cost of producing any units (\$400) becomes "swallowed up" in the variable cost of producing large x number of units.

Example: Graph



Example

The daily total cost of producing x DVD players is given by $C(x) = 0.0001x^3 - 0.08x^2 + 40x + 5000$ (in dollars). Find $\bar{C}(x)$, $\bar{C}'(x)$, and $\bar{C}'(500)$. Then discuss the economic implications of these results, using the graph of $\bar{C}(x)$ to help interpret results

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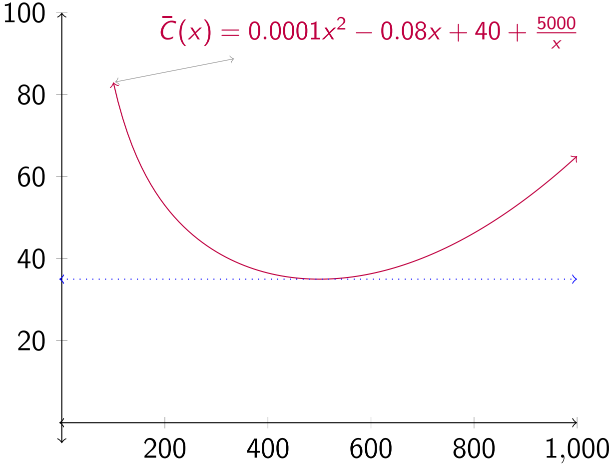
And $\bar{C}'(x) = 0.0002x - 0.08 - \frac{5000}{x^2}$.

The graph of $\bar{C}(x)$ follows with analysis of the results.

Example: Graph



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Example

Suppose the relationship between the price p in dollars of a loudspeaker and the quantity demanded x is given by $p(x) = -0.02x + 400$ $(0 \leq x \leq 20000)$. Find the revenue function, the marginal revenue function, compute $R'(2000)$, and interpret your results.

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$R(x) = xp(x) = -0.02x^2 + 400x$. And $R'(x) = -0.04x + 400$. Thus, $R'(2000) = -0.04(2000) + 400 = 320$. Thus, the actual revenue realized by the sale of the 2001st loudspeaker is approximately \$320.

Definition

Definition (Marginal Profit)

If $P(x) = R(x) - C(x)$ is a revenue function with $R(x)$ and $C(x)$ being revenue and cost functions respectively, then the marginal profit function is given by $P'(x) = R'(x) - C'(x)$.

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Using the previous example, say also that the cost of producing x loudspeakers is given by $C(x) = 100x + 200000$. Find P, P', compute $P'(2000)$, and interpret your results.

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This notion can be generalized to the derivative of a function in the following way. The relative change of f with respect to x at x is

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which is the ratio of the relative rate of change of f to the relative rate of change of p. The *negative* of this quantity is called the elasticity of demand by economists.

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If f is a differentiable demand function defined by $x = f(p)$, then the elasticity of demand at price p is given by $E(p) = -\frac{pf'(p)}{f(p)}$.

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$E(p) < 1$ defines inelastic demand.

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This means a small (positive) relative change in price results in a smaller relative (negative) change in quantity demanded for a price corresponding to an inelastic demand. What are the corresponding interpretations of unitary and elastic demands?

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So, for a price at which demand is elastic, $R'(p)$ would be negative. Thus, $R(p)$ would be decreasing at that price, and a small increase in p would result in a decrease in R. How could you correspondingly interpret unitary and inelastic demands?

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Assignment

Read 3.5-3.7. Do problems 16, 32, 52, 64, 78, 90 in 3.3 and 4, 12, 28, 32, 36 in 3.4.