Patchy Solutions of Hamilton Jacobi Bellman Partial Differential Equations

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This paper is dedicated to our esteemed colleague and good friend Giorgi Picci on the occasion of his sixty fifth birthday.

1 Hamilton Jacobi Bellman PDEs

Consider the optimal control problem of minimizing the integral

$$\int_0^\infty l(x,u)\ dt \tag{1}$$

of a Lagrangian $l(x,u)$ subject to the controlled dynamics

$$\begin{aligned} \dot{x} &= f(x,u) \\ x(0) &= x^0 \end{aligned} \tag{2}$$

where f, l are smooth and l is strictly convex in $u \in I\!R^m$ for all $x \in I\!R^n$.

Suppose the dynamics and Lagrangian have Taylor series expansions about $x = 0, u = 0$ of the form

$$\dot{x} = Fx + Gu + f^{[2]}(x,u) + f^{[3]}(x,u) + \dots \tag{3}$$

$$l(x,u) = \frac{1}{2}\left(x'Qx + u'Ru\right) + l^{[3]}(x,u) + l^{[4]}(x,u) + \dots \tag{4}$$

where $^{[d]}$ indicates terms of degree d in the power series. We shall say that the optimal control problem is nice if F, G is stabilizable and $Q^{\frac{1}{2}}, F$ is detectable.

A special case of this optimal control problem is the linear quadratic regulator (LQR) where one seeks to minimize a quadratic cost

$$\int_0^\infty \frac{1}{2}\left(x'Qx + u'Ru\right)\ dt$$

subject to linear dynamics

$$\dot{x} = Fx + Gu$$

If this is nice then there is a unique nonnegative definite solution to the algebraic Riccati equation

$$0 = F'P + PF + Q - PGR^{-1}G'P \tag{5}$$

that gives the optimal cost

$$\pi(x^0) = \frac{1}{2}(x^0)'Px^0 = \min \int_0^\infty \frac{1}{2}\left(x'Qx + u'Ru\right)\, dt \tag{6}$$

Furthermore the optimal control is given in feedback form

$$u(t) = \kappa(x(t)) = Kx(t)$$

where

$$K = -R^{-1}G'P \tag{7}$$

and the closed loop dynamics

$$\dot{x} = (F + GK)\, x \tag{8}$$

is exponentially stable.

Returning to the nonlinear problem, it is well-known that if it admits a smooth optimal cost $\pi(x)$ and a smooth optimal feedback $u = \kappa(x)$ locally around $x = 0$ then they must satisfy the Hamilton Jacobi Bellman (HJB) PDE

$$\begin{aligned} 0 &= \min_u \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \\ \kappa(x) &= \arg\min_u \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \end{aligned}$$

We shall assume that $\frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u)$ is strictly convex in u locally around $x = 0, u = 0$ then the HJB PDE can be rewritten as

$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial x}(x) f(x,\kappa(x)) + l(x,\kappa(x)) \\ 0 &= \frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial u}(x,\kappa(x)) + \frac{\partial l}{\partial u}(x.\kappa(x)) \end{aligned} \tag{9}$$

Al'brecht [1] has shown that for nice optimal control problems, the Hamilton Jacobi Bellman PDE can be approximately solved by Taylor series methods locally around the origin. Lukes [14] showed that under suitable conditions this series expansion converges to the true solution. The method has been implemented on examples by Garrard and Jordan [7], Yoshida and Loparo [22], Spencer, Timlin, Sain and Dyke [20] and others. We have implemented it in the Nonlinear Systems Toolbox [11], a MATLAB based package.

Assume the dynamics and Lagrangian have power series expansions (3, 4). We assume that the unknowns, the optimal cost and optimal feedback, have similar expansions.

$$\begin{aligned} \pi(x) &= \tfrac{1}{2}x'Px + \pi^{[3]}(x) + \pi^{[4]}(x) + \dots \\ \kappa(x) &= Kx + \kappa^{[2]}(x) + \kappa^{[3]}(x) + \dots \end{aligned} \tag{10}$$

We plug these into the HJB PDE (9) and extract terms of lowest degree to obtain the equations

$$\begin{aligned} 0 &= x'\left(F'P + PF + Q - K'RK\right)x \\ 0 &= x'\left(PG + K'R\right) \end{aligned}$$

Notice the first equation is quadratic x and the second is linear in x. More importantly the first equation is linear in the unknown P but quadratic in the unknown K while the second is linear in both the unknowns. They lead to the familiar equations (5, 7) .

Having found P, K, we extract the next lowest terms from (9) and obtain

$$\begin{aligned} 0 &= \frac{\partial \pi^{[3]}}{\partial x}(x)(F + GK)x + x'Pf^{[2]}(x, Kx) + l^{[3]}(x, Kx) \\ 0 &= \frac{\partial \pi^{[3]}}{\partial x}(x)G + x'P\frac{\partial f^{[2]}}{\partial u}(x, Kx) + \frac{\partial l^{[3]}}{\partial u}(x, Kx) + \left(\kappa^{[2]}(x)\right)' R \end{aligned} \tag{11}$$

Notice several things. The first equation is cubic in x and the second is quadratic. The equations involve the previously computed P, K. The unknowns $\pi^{[3]}(x)$; $\kappa^{[2]}(x)$ appear linearly in these equations. The equations are triangular, $\kappa^{[2]}(x)$ does not appear in the first one. If we can solve the first for $\pi^{[3]}(x)$ then clearly we can solve the second for $\kappa^{[2]}(x)$ as R is assumed to be invertible.

To decide the solvability of the first we study the linear operator

$$\pi^{[3]}(x) \mapsto \frac{\partial \pi^{[3]}}{\partial x}(x)(F + GK)x \tag{12}$$

from cubic polynomials to cubic polynomials. Its eigenvalues are of the form $\lambda_i + \lambda_j + \lambda_k$ where $\lambda_i, \lambda_j, \lambda_k$ are eigenvalues of $F + GK$. A cubic resonance occurs when such a sum equals zero. But all the eigenvalues of $F + GK$ are in the open left half plane so there are no cubic resonances. Hence the linear operator (12) is invertible and (11) is solvable.

The higher degree terms are found in a similar fashion. Suppose that $\overline{\pi}(x)$ and $\overline{\kappa}(x)$ are the expansions of the optimal cost and optimal feedback through degrees d and $d-1$ respectively. We wish to find the next terms $\pi^{[d+1]}(x)$ and $\kappa^{[d]}(x)$. We plug $\overline{\pi}(x) + \pi^{[d+1]}(x)$ and $\overline{\kappa}(x) + \kappa^{[d]}(x)$ into the HJB PDEs (9) and extract terms of degrees $d+1$ and d respectively to obtain

$$
\begin{aligned}
0 &= \frac{\partial \pi^{[d+1]}}{\partial x}(x)\,(F+GK)\,x + \left(\frac{\partial \overline{\pi}}{\partial x}(x) f(x,\overline{\kappa}(x))\right)^{[d+1]} + x' P G \kappa^{[d]}(x) \\
&\quad + \left(l(x,\overline{\kappa}(x))\right)^{[d+1]} + x' K' R \kappa^{[d]}(x) \\
0 &= \frac{\partial \pi^{[d+1]}}{\partial x}(x) G + \left(\frac{\partial \overline{\pi}}{\partial x}(x) \frac{\partial f}{\partial u}(x,\overline{\kappa}(x))\right)^{[d]} \\
&\quad + \left(\frac{\partial l}{\partial u}(x,\overline{\kappa}(x))\right)^{[d]} + \left(\kappa^{[d]}(x)\right)' R
\end{aligned}
$$

where $(\cdot)^{[d]}$ is the degree d part of the enclosed.

Because of (7) $\kappa^{[d]}(x)$ drops out of the first of these equations yielding

$$
\begin{aligned}
0 = \frac{\partial \pi^{[d+1]}}{\partial x}(x)\,(F+GK)\,x + \left(\frac{\partial \overline{\pi}}{\partial x}(x) f(x,\overline{\kappa}(x))\right)^{[d+1]} \\
+ \left(l(x,\overline{\kappa}(x))\right)^{[d+1]}
\end{aligned}
\tag{13}
$$

Consider the linear operator from degree $d+1$ polynomials to degree $d+1$ polynomials

$$
\pi^{[d+1]}(x) \mapsto \frac{\partial \pi^{[d+1]}}{\partial x}(x)\,(F+GK)\,x
$$

Its eigenvalues are of the form $\lambda_{i_1} + \ldots + \lambda_{i_{d+1}}$ where λ_j is an eigenvalue of $F + GK$. A resonance of degree $d+1$ occurs when such a sum equals zero. But all the eigenvalues of $F+GK$ are in the open left half plane so there are no resonances of degree $d+1$ and we can solve (13) for $\pi^{[d+1]}(x)$. Then the second equation can be solved for $\kappa^{[d]}(x)$

$$
\kappa^{[d]}(x) = -R^{-1}\left(\frac{\partial \pi^{[d+1]}}{\partial x}(x) G + \left(\frac{\partial \overline{\pi}}{\partial x}(x)\frac{\partial f}{\partial u}(x,\overline{\kappa}(x))\right)^{[d]} + \left(\frac{\partial l}{\partial u}(x,\overline{\kappa}(x))\right)^{[d]}\right)'
\tag{14}
$$

We have developed MATLAB based software to compute the series solutions to the HJB PDE [11]. In principle the computation can be carried out to any degree in any number of variables but there are practical limitations in execution time and memory. This is the familiar curse of dimensionality. There are $n+d-1$ choose d monomials of degree d in n variables. Still the software is quite fast. For example we are able to solve an HJB PDE in six states and one control to degree six in the optimal cost and degree five in optimal feedback in less than 30 seconds on a five year old laptop (500 MHz) with limited memory (512 MB). There are 462 monomials of degree 6 in 6 variables.

The main problem with the power series approach is that is local in nature. The power series solution to the HJB PDE is very close to the true solution in some neighborhood of the origin. Increasing the degree of the approximation may increase the accuracy but does not necessarily yield a larger domain of validity of the approximation. Complicating this is the fact that in general HJB PDEs do not have globally smooth solutions. The underlying optimal control problem may have conjugate points or focal points. It is for this reason that the theory of viscosity solutions was developed [4], [5].

2 Other Approaches

There are several other approaches to solving HJB PDEs, and a large literature, for example see [3], [6], [13], [9], [10], [16], [18], [19], [21] and their references. One approach is to discretize the underlying optimal control problem and convert it into a nonlinear program in discrete time and space. But the curse of dimensionality rears its ugly head. Consider the optimal control problem generating the above mentioned HJB PDE. If each of the six states is discretized into 10 levels then there would 1,000,000 discrete states.

Other approaches involve discretizing the HJB PDE with subtle tricks so that the algorithm converges to its viscosity solution. This also suffers from the curse of dimensionality. The fast sweeping and marching method (Tsitsiklis [21], Osher et al. [16], [9], [10] and Sethian [19]) are ways to lessen this curse. It takes advantage of the fact that an HJB PDE has characteristics. These are the closed loop optimal state trajectories that converge to the origin as $t\theta_0\infty$. The fast marching method grows the solution out from the origin discrete state by discrete state in reverse time by computing the solution at new discrete states that are on the boundary of the already computed solution.

3 New Approach

The new approach that we are proposing is a extension of the power series method of Al'brecht [1], the Cauchy-Kovalevskaya technique [8], the fast marching method [21], [19] and the patchy technique of Ancona and Bressan [2]. It is similar to that of Navasca and Krener [15]. Suppose we have computed a power series solution to some degree $d+1$ of an HJB PDE in a neighborhood of the origin by the method of Al'brecht. We verify that this power series solution is valid in some sublevel set of the computed optimal cost function by checking how well it satisfies the HJB PDE on the level set that is its boundary. At the very least it should be a valid Lyapunov function for the dynamics with the computed optimal feedback on the sublevel set. Also the computed closed loop dynamics should point inward on the boundary of the sublevel set, in other words, the computed backward characteristics of the HJB PDE should radiate outward. This sublevel set is called the zeroth patch.

Then we pick a point on the boundary of the zeroth patch and assume the optimal cost and optimal feedback have a power series expansion around that point. We already know the partial derivatives of these in directions tangent to the boundary of the patch. Using a technique similar to that of Cauchy-Kovalevskaya, we can compute the other partial derivatives from the HJB PDE because we have assumed that the computed closed loop dynamics is not tangent to the level set, it points inward. In this way we compute the solution in a patch that overlaps the zeroth patch. Call this the first patch. Again we can estimate the size of this patch by how well the computed solution satisfies the HJB PDE.

It is not essential that the dynamics f and Lagrangian l be smooth at the boundary of zeroth patch (or other patches). If they are not smooth at the boundary we use their derivatives to the outside of the zeroth patch. This is a form of upwind differentiation We do assume that they are smooth at the origin but they can have discontinuities or corners elsewhere. If they do, we choose the patches so that these occur at patch boundaries. In this way it is an upwinding scheme because the closed loop dynamics, the characteristic curves of the PDE point inward on the boundary of the zeroth patch. When computing the solution on the second patch we use the derivative information in the backward characteristic direction.

Then we choose another point that is on the boundary of the zeroth patch but not in the first patch and repeat the process. In this way we grow a series of patches encircling the sublevel set. The validity of the computed solution on each patch is verified via how well it solves the HJB PDE. On the boundary between adjacent patches we may have two possible closed loop vector fields. If the angle between them is obtuse, the two trajectories are diverging, then there is no problem and we can choose either when on the boundary between the patches. If the angle is acute then there may be a sliding regime and another patch in between may be needed. Another possibility is to blend the computed costs across the patch boundary. This will cause a blending of the computed feedback. (These are research questions.)

After the original sublevel set has been completely encircled by new patches we have piecewise smooth approximations to the optimal cost and optimal feedback. We choose a higher sublevel set of the computed cost that is valid for all the patches and repeat the process.

The patches are ordered and the approximate solution to the problem at x is defined to be the approximate solution in the lowest ordered patch containing x.

The patches can also be defined a priori, this would simplify the method but might lead to unsatisfactory solutions if they are chosen too large or long computation times if they are chosen too small.

Of course there is the problem of shocks caused by conjugate or focal points. The assumptions that we make ensure that these do occur at the origin, the true solution is smooth around there. But that does not mean they will not occur elsewhere. When possible we will choose the patches so that they occur at patch boundaries. Not a lot is known about the types of singularities that can occur and how they affect the optimal feedback. One of the goals of our future research project is to better understand these issues.

We expect most of the time to compute the expansions to degree four for the optimal cost and degree three for the optimal feedback. But if the dynamics and/or Lagrangian is not sufficiently smooth we might compute to degrees two and one respectively.

As we noted before in many engineering problems stability of the closed loop dynamics is the principle goal. There may be considerable freedom in choosing the Lagrangian and so a smooth Lagrangian may be chosen. In many problems there are state and/or control constraints. Then the Lagrangian can be chosen so that the solution does not violate the constraints.

In the following sections we discuss the method in more detail.

4 One Dimensional HJB PDEs

For simplicity we consider an optimal control problem (1, 2) where the state dimension $n = 1$ and the control dimension $m = 1$. Occasionally to simplify the calculations we shall assume that the dynamics is affine in the control and the Lagrangian is quadratic in the control

$$\begin{aligned} f(x,u) &= f(x) + g(x)u \\ l(x,u) &= q(x) + s(x)u + \tfrac{1}{2}r(x)u^2 \end{aligned} \tag{1}$$

with $r(x) > 0$. The method works for more general $f,\ l$ but it is more complicated. In any case we shall assume that $l(x,u) = 0$ iff $x = 0, u = 0$

We assume that the degree $d+1$ polynomial $\pi^0(x)$ and the degree d polynomial $\kappa^0(x)$, computed by the power series method of Al'brecht described above, approximately solves this problem in a neighborhood of $x = 0$. We plug the power series expansions of π^0, κ^0 into the right side of the first HJB equation with the exact dynamics f and exact Lagrangian l and compute the local error

$$\rho^0(x) = \frac{\partial \pi^0}{\partial x}(x) f(x, \kappa^0(x)) + l(x.\kappa^0(x)) \tag{2}$$

or relative local error

$$\rho^0_r(x) = \frac{\rho^0(x)}{\pi^0(x)} \tag{3}$$

Of course the local error and some of its derivative will (nearly) vanish at $x = 0$ but it will generally be nonzero for $x \neq 0$. Suppose $\rho^0_r(x)$ is small on some interval $[0, x^1]$ then we accept the power series solution $\pi^0(x), \kappa^0(x)$, on this interval. We would like to continue the solution to the right of x^1. Let $\pi^1(x), \kappa^1(x)$ denote this continued solution. We have an approximation to the optimal cost $\pi^0(x^1)$ and optimal feedback $\kappa^0(x^1)$ at x^1, we accept the former by setting $\pi^1(x^1) = \pi^0(x^1)$ but not the latter. We shall compute $u^1 = \kappa^1(x^1)$.

We evaluate the HJB PDE (9) at x^1 using the assumption (1) to obtain

$$0 = \frac{\partial \pi^1}{\partial x}(x^1) f(x^1, u^1) + q(x^1) + s(x^1)u^1 + \frac{1}{2} r(x^1) \left(u^1\right)^2 \tag{4}$$

$$0 = \frac{\partial \pi^1}{\partial x}(x^1) g(x^1) + s(x^1) + r(x^1)u^1 \tag{5}$$

We can solve the second equation for u^1 and plug it into the first to obtain a quadratic in $\frac{\partial \pi^1}{\partial x}(x^1)$. We set u^1 to be the root nearer to $\kappa^0(x^1)$. In this way we find $\frac{\partial \pi^1}{\partial x}(x^1)$ and u^1.

If assumption (1) does not hold then we must solve a coupled pair of nonlinear equations for the unknowns $\frac{\partial \pi^1}{\partial x}(x^1)$ and u^1. This can be done by a couple of iterations of Newton's method as we already have good starting guesses, $\frac{\partial \pi^0}{\partial x}(x^1)$ and $\kappa^0(x^1)$.

Since we assumed that $l(x,u) = 0$ iff $x = 0, u = 0$ we conclude from (4) that $f(x^1, u^1) \neq 0$.

To find $\frac{\partial^2 \pi^1}{\partial x^2}(x^1)$ and $\frac{\partial \kappa^1}{\partial x}(x^1)$ we proceed as follows. Differentiate the HJB PDEs (9) with respect to x at x^1 to obtain

$$\begin{aligned} 0 &= \frac{\partial^2 \pi^1}{\partial x^2}(x^1) f(x^1,u^1) + \frac{\partial \pi^1}{\partial x}(x^1)\left(\frac{\partial f}{\partial x}(x^1,u^1) + \frac{\partial f}{\partial u}(x^1,u^1)\frac{\partial \kappa}{\partial x}(x^1)\right) \\ &\quad + \frac{\partial l}{\partial x}(x^1,u^1) + \frac{\partial l}{\partial u}(x^1,u^1)\frac{\partial \kappa}{\partial x}(x^1) \end{aligned} \tag{6}$$

$$\begin{aligned} 0 &= \frac{\partial^2 \pi^1}{\partial x^2}(x^1)\frac{\partial f}{\partial u}(x^1,u^1) + \frac{\partial \pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial x \partial u}(x^1,u^1) + \frac{\partial^2 l}{\partial x \partial u}(x^1,u^1) \\ &\quad + \left(\frac{\partial \pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + \frac{\partial^2 l}{\partial u^2}(x^1,u^1)\right)\frac{\partial \kappa^1}{\partial x}(x^1) \end{aligned} \tag{7}$$

Because of (4), the first equation (6) reduces to

$$0 = \frac{\partial^2 \pi^1}{\partial x^2}(x^1) f(x^1,u^1) + \frac{\partial \pi^1}{\partial x}(x^1)\frac{\partial f}{\partial x}(x^1,u^1) + \frac{\partial l}{\partial x}(x^1,u^1) \tag{8}$$

Notice the unknown $\frac{\partial \kappa^1}{\partial x}(x^1)$ does not appear in this equation so we can easily solve for the unknown $\frac{\partial^2 \pi^1}{\partial x^2}(x^1)$ since $f(x^1,u^1) \neq 0$. Because of the assumptions (1) the second equation reduces to

$$\begin{aligned} 0 = &\frac{\partial^2 \pi^1}{\partial x^2}(x^1) g(x^1) + \frac{\partial \pi^1}{\partial x}(x^1)\frac{\partial g}{\partial x}(x^1) \\ &+ \frac{\partial s}{\partial x}(x^1) + \frac{\partial r}{\partial x}(x^1)u^1 + r(x^1)\frac{\partial \kappa^1}{\partial x}(x^1) \end{aligned}$$

By assumption $r(x^1) > 0$ so we can solve the second equation for other unknown $\frac{\partial \kappa^1}{\partial x}(x^1)$.

To find the next unknowns $\frac{\partial^3 \pi^1}{\partial x^3}(x^1)$ and $\frac{\partial^2 \kappa^1}{\partial x^2}(x^1)$ we proceed in a similar fashion. We differentiate HJB PDEs (9) twice with respect to x and evaluate at x^1 assuming (1) to obtain two equations,

$$\begin{aligned}
0 =& \frac{\partial^3\pi^1}{\partial x^3}(x^1)f(x^1,u^1) + 2\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial f}{\partial x}(x^1,u^1) \\
&+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial x^2}(x^1,u^1) + \frac{\partial^2 l}{\partial x^2}(x^1,u^1) \\
&+\left(\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial f}{\partial u}(x^1,u^1) + \frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial x\partial u}(x^1,u^1) + \frac{\partial^2 l}{\partial x\partial u}(x^1,u^1)\right)\frac{\partial\kappa^1}{\partial x}(x^1)
\end{aligned}$$

$$\begin{aligned}
0 =& \frac{\partial^3\pi^1}{\partial x^3}(x^1)\frac{\partial f}{\partial u}(x^1,u^1) + 2\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^2 f}{\partial x\partial u}(x^1,u^1) \\
&+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial x^2\partial u}(x^1,u^1) + \frac{\partial^3 l}{\partial x^2\partial u}(x^1,u^1) \\
&+2\left(\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + \frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial x\partial u^2}(x^1,u^1) + \frac{\partial^3 l}{\partial x\partial u^2}(x^1,u^1)\right)\frac{\partial\kappa^1}{\partial x}(x^1) \\
&+\left(\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial u^3}(x^1,u^1) + \frac{\partial^3 l}{\partial u^3}(x^1,u^1)\right)\left(\frac{\partial\kappa^1}{\partial x}(x^1)\right)^2 \\
&+\left(\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + \frac{\partial^2 l}{\partial u^2}(x^1,u^1)\right)\frac{\partial^2\kappa^1}{\partial x^2}(x^1)
\end{aligned}$$

The unknown $\frac{\partial^2\kappa^1}{\partial x^2}(x^1)$ does not appear in the first equation because of (4). Since $f(x^1,u^1) \neq 0$ we can solve this equation for the unknown $\frac{\partial^3\pi^1}{\partial x^3}(x^1)$ The second is linear in both unknowns. Under the assumptions (1) the second equation reduces to

$$\begin{aligned}
0 =& \frac{\partial^3\pi^1}{\partial x^3}(x^1)g(x^1) + 2\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial g}{\partial x}(x^1) \\
&+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^2 g}{\partial x^2}(x^1) + \frac{\partial^2 s}{\partial x^2}(x^1) + \frac{\partial^2 r}{\partial x^2}(x^1)u^1 \\
&+2\frac{\partial r}{\partial x}(x^1)\frac{\partial\kappa^1}{\partial x}(x^1) + r(x^1)\frac{\partial^2\kappa^1}{\partial x^2}(x^1)
\end{aligned}$$

and because $r(x^1) > 0$ it is readily solvable for the other unknown $\frac{\partial^2\kappa^1}{\partial x^2}(x^1)$.

To find the next unknowns $\frac{\partial^4\pi^1}{\partial x^4}(x^1)$ and $\frac{\partial^3\kappa^1}{\partial x^3}(x^1)$ we differentiate HJB PDE (9) three times with respect to x and evaluate at x^1 assuming (1) to obtain the two equations,

$$
\begin{aligned}
0 = &\frac{\partial^4\pi^1}{\partial x^4}(x^1)f(x^1,u^1)) + 3\frac{\partial^3\pi^1}{\partial x^3}(x^1)\frac{\partial f}{\partial x}(x^1,u^1) + 3\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^2 f}{\partial x^2}(x^1,u^1)\\
&+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial x^3}(x^1,u^1) + \frac{\partial^3 l}{\partial x^3}(x^1,u^1)\\
&+2\left(\frac{\partial^3\pi^1}{\partial x^3}(x^1)\frac{\partial f}{\partial u}(x^1,u^1) + 2\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^2 f}{\partial x\partial u}(x^1,u^1)\right.\\
&\left.+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial x^2\partial u}(x^1,u^1) + \frac{\partial^3 l}{\partial x^2\partial u}(x^1,u^1)\right)\frac{\partial\kappa^1}{\partial x}(x^1)\\
&+\left(\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + \frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial x\partial u^2}(x^1,u^1)\right.\\
&\left.+\frac{\partial^3 l}{\partial x\partial u^2}(x^1,u^1)\right)\left(\frac{\partial\kappa^1}{\partial x}(x^1)\right)^2\\
&+\left(\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial f}{\partial u}(x^1,u^1) + \frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial x\partial u}(x^1,u^1)\right.\\
&\left.+\frac{\partial^2 l}{\partial x\partial u}(x^1,u^1)\right)\frac{\partial^2\kappa^1}{\partial x^2}(x^1)
\end{aligned}
$$

$$
\begin{aligned}
0 = &\frac{\partial^4\pi^1}{\partial x^4}(x^1)\frac{\partial f}{\partial u}(x^1,u^1) + 3\frac{\partial^3\pi^1}{\partial x^3}(x^1)\frac{\partial^2 f}{\partial x\partial u}(x^1,u^1) + 3\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^3 f}{\partial x^2\partial u}(x^1,u^1)\\
&+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^4 f}{\partial x^3\partial u}(x^1,u^1) + \frac{\partial^4 l}{\partial x^3\partial u}(x^1,u^1)\\
&+3\left(\frac{\partial^3\pi^1}{\partial x^3}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + 2\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^3 f}{\partial x\partial u^2}(x^1,u^1)\right.\\
&\left.+\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^4 f}{\partial x^2\partial u^2}(x^1,u^1) + \frac{\partial^4 l}{\partial x^2\partial u^2}(x^1,u^1)\right)\frac{\partial\kappa^1}{\partial x}(x^1)\\
&+3\left(\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + \frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial x\partial u^2}(x^1,u^1) + \frac{\partial^3 l}{\partial x\partial u^2}(x^1,u^1)\right)\frac{\partial^2\kappa^1}{\partial x^2}(x^1)\\
&+3\left(\frac{\partial^2\pi^1}{\partial x^2}(x^1)\frac{\partial^3 f}{\partial u^3}(x^1,u^1) + \frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^4 f}{\partial x\partial u^3}(x^1,u^1) + \frac{\partial^4 l}{\partial x\partial u^3}(x^1,u^1)\right)\left(\frac{\partial\kappa^1}{\partial x}(x^1)\right)^2\\
&+3\left(\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^3 f}{\partial u^3}(x^1,u^1) + \frac{\partial^3 l}{\partial u^3}(x^1,u^1)\right)\frac{\partial\kappa^1}{\partial x}(x^1)\frac{\partial^2\kappa^1}{\partial x^2}(x^1)\\
&+\left(\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^4 f}{\partial u^4}(x^1,u^1) + \frac{\partial^4 l}{\partial u^4}(x^1,u^1)\right)\left(\frac{\partial\kappa^1}{\partial x}(x^1)\right)^3\\
&+\left(\frac{\partial\pi^1}{\partial x}(x^1)\frac{\partial^2 f}{\partial u^2}(x^1,u^1) + \frac{\partial^2 l}{\partial u^2}(x^1,u^1)\right)\frac{\partial^3\kappa^1}{\partial x^3}(x^1)
\end{aligned}
$$

We expect to stop at degree four most of the time, The assumptions (1) greatly simplify the last equation,

$$
\begin{aligned}
0 = &\frac{\partial^4 \pi^1}{\partial x^4}(x^1) g(x^1) + 3\frac{\partial^3 \pi^1}{\partial x^3}(x^1)\frac{\partial g}{\partial x}(x^1) + 3\frac{\partial^2 \pi^1}{\partial x^2}(x^1)\frac{\partial^2 g}{\partial x^2}(x^1) \\
&+ \frac{\partial \pi^1}{\partial x}(x^1)\frac{\partial^3 g}{\partial x^3}(x^1) + \frac{\partial^3 s}{\partial x^3}(x^1) + \frac{\partial^3 r}{\partial x^3}(x^1) u^1 + r(x^1)\frac{\partial^3 \kappa^1}{\partial x^3}(x^1)
\end{aligned}
$$

Notice the similarities with Al'brecht's method. We successively solve for $\frac{\partial^{d+1}\pi}{\partial x^{d+1}}(x^1)$ and $\frac{\partial^{d}\kappa^1}{\partial x^{d}}(x^1)$ for $d = 0, 1, 2, \ldots$. At the lowest level the equations are coupled and if (1) holds we must solve a quadratic equation similar to a Riccati equation. At the higher levels the equations are linear and triangular in the unknowns.

Once we have computed a satisfactory approximate solution on the interval $[x^1, x^2]$ we can repeat the process and find an approximate solution to the right of x^2.

5 One Dimensional Example

Consider the simple LQR of minimizing

$$\frac{1}{2}\int_0^\infty z^2 + u^2 \, dt$$

subject to

$$\dot{z} = z + u$$

Here both z and u are one dimensional.

The Riccati equation (5) is

$$0 = 2P + 1 - P^2$$

and its unique nonnegative solution is $P = 1 + \sqrt{2}$. Therefore the optimal cost and optimal feedback are

$$
\begin{aligned}
\pi(z) &= \frac{1+\sqrt{2}}{2} z^2 \\
\kappa(z) &= -(1+\sqrt{2}) z
\end{aligned}
$$

The optimal closed loop dynamics is

$$\dot{z} = -\sqrt{2} z$$

After the change of coordinates

$$z = \sin x$$

then the LQR become the nonlinear optimal control problem of minimizing

$$\frac{1}{2}\int_0^\infty \sin^2 x + u^2 \, dt$$

subject to

$$\dot{x} = \frac{\sin x + u}{\cos x}$$

We know that the optimal cost and optimal feedback is

$$\pi(x) = \frac{1+\sqrt{2}}{2}\sin^2 x$$
$$\kappa(z) = -(1+\sqrt{2})\sin x$$

Notice that the optimal cost is even and the optimal feedback is odd. We can compare it with the solution computed by the method described above.

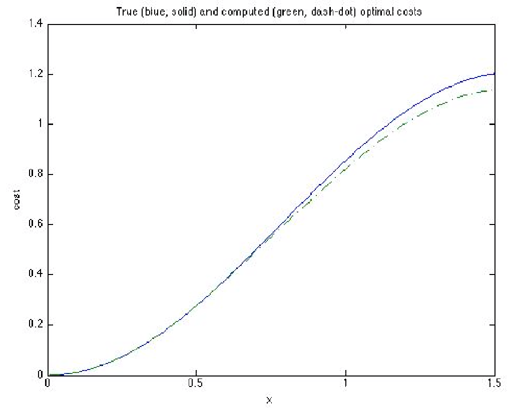

Fig. 1. True cost (solid) and the computed cost (dash-dot).

The computed solution on the interval $[0, 0.9]$ is the one of Al'brecht. As we compute the solution for larger x, the size of the patches decreases because the change of coordinates is becoming more nearly singular as we approach $\frac{\pi}{2}$. There are 15 patches. The relative error tolerance is 0.5.

6 HJB PDEs in Higher Dimensions

In this section we generalize the proposed scheme to higher dimensional state spaces $n \geq 1$. For notational simplicity we shall assume that the control is one dimensional $m = 1$, generalizing to higher control dimensions causes no conceptual difficulty. We also make the simplifying assumptions that the dynamics is affine in the control and the cost is quadratic in the control of the form

$$\dot{x} = f(x) + g(x)u$$
$$l(x, u) = q(x) + r(x)u^2/2$$

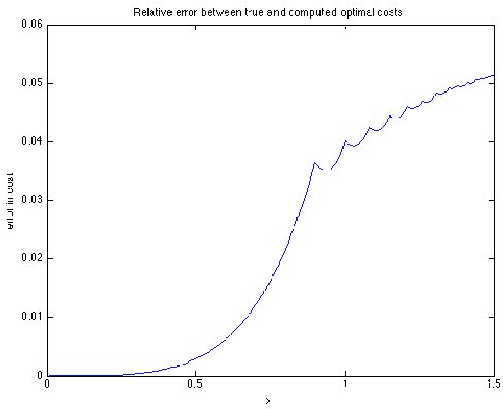

Fig. 2. Relative error between true cost and the computed cost.

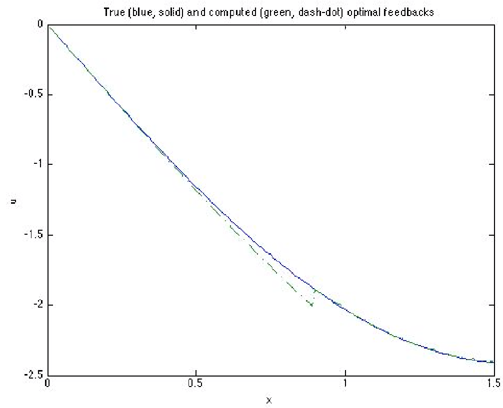

Fig. 3. True feedback (solid) and the computed feedback (dash-dot) .

The method does not require these assumptions but they do greatly simplify it.

Suppose we have computed the Al'brecht solution $\pi^0(x),\ \kappa^0(x)$ to the HJB PDE (9) in some neighborhood of the origin. We check the local error $\rho^0(x)$ (2) or relative local error (3) and decide that it is a reasonable solution in some sublevel set $\{x : \pi^0(x) \leq c\}$ which we call the zeroth patch $\mathcal{P}^0$. We choose x^1 on the level set $\pi(x^1) = c$ and seek to extend the solution in a patch around x^1. To do so we need to estimate the low degree partial derivatives of the optimal cost and optimal feedback at x^1.

We assume that the Al'brecht closed loop dynamics is transverse to the boundary of the sublevel set and points inward

$$\frac{\partial \pi^0}{\partial x}(x^1) f(x^1, \kappa^0(x^1)) < 0$$

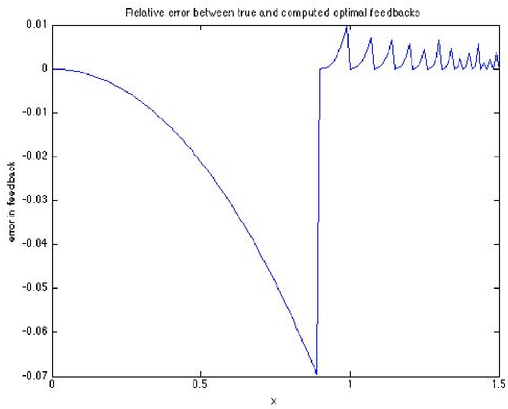

Fig. 4. Relative error between the true feedback and the computed feedback.

We accept that $\pi^1(x^1) = \pi^0(x^1)$ but we will compute a new $u^1 = \kappa^1(x^1)$ probably different from $\kappa^0(x^1)$.

The HJB equations become

$$0 = \frac{\partial \pi}{\partial x_\sigma}(x)\left(f_\sigma(x) + g_\sigma(x)\kappa(x)\right) + q(x) + r(x)\left(\kappa(x)\right)^2/2 \tag{1}$$

$$0 = \frac{\partial \pi}{\partial x_\sigma}(x) g_\sigma(x) + r(x)\kappa(x) \tag{2}$$

We choose the index k that maximizes

$$|f_k(x^1) + g_k(x^1)\kappa^0(x^1)|$$

For notational convenience we assume that $k = n$.

We assume that

$$\pi^1(x^1) = \pi^0(x^1)$$
$$\frac{\partial \pi^1}{\partial x_\sigma}(x^1) = \frac{\partial \pi^0}{\partial x_\sigma}(x^1)$$

for $1 \leq \sigma < n$. Then we can solve the second HJB equation for $\kappa(x^1)$ and plug it into the first to get a quadratic equation in the other unknown

$$0 = a\left(\frac{\partial \pi^1}{\partial x_n}(x^1)\right)^2 + b\frac{\partial \pi^1}{\partial x_n}(x^1) + c$$

where

$$
\begin{aligned}
a &= \frac{1}{2r(x^1)}(g_n(x^1))^2 \\
b &= \frac{1}{r(x^1)} g_n(x^1) \sum_{\sigma=1}^{n-1} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) g_\sigma(x) - f_n(x^1) \\
c &= \frac{1}{2r(x^1)} \sum_{\sigma=1}^{n-1} \sum_{\tau=1}^{n-1} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) g_\sigma(x^1) \frac{\partial \pi^1}{\partial x_\tau}(x^1) g_\tau(x^1) \\
&\quad -q(x^1) - \sum_{\sigma=1}^{n-1} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) f_\sigma(x^1)
\end{aligned}
$$

Assuming this equation has real roots, we set $\frac{\partial \pi^1}{\partial x_n}(x^1)$ to be the root closest to $\frac{\partial \pi^0}{\partial x_n}(x^1)$ and we solve for $\kappa(x^1)$,

$$
\kappa(x^1) = -\frac{1}{r(x^1)} \sum_{\sigma=1}^{n} \frac{\partial \pi^1}{\partial x_\sigma}(x^1) g_\sigma(x^1)
$$

The next unknowns in a power series expansion of the optimal cost and feedback around x^1 are $\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1)$ and $\frac{\partial \kappa^1}{\partial x_i}(x^1)$ for $1 \leq i \leq j \leq n$. We assume that

$$
\frac{\partial^2 \pi^1}{\partial x_i \partial x_j}(x^1) = \frac{\partial^2 \pi^0}{\partial x_i \partial x_j}(x^1)
$$

for $1 \leq i \leq j \leq n-1$ and we take the partials of (1, 2) with respect to x_i to obtain $2n$ equations

$$
\begin{aligned}
0 &= \frac{\partial^2 \pi^1}{\partial x_i \partial x_\sigma}(x^1) \left(f_\sigma(x^1) + g_\sigma(x^1) \kappa^1(x^1) \right) && (3) \\
&\quad + \frac{\partial \pi^1}{\partial x_\sigma}(x^1) \left(\frac{\partial f_\sigma}{\partial x_i}(x^1) + \frac{\partial g_\sigma}{\partial x_i}(x^1) \kappa^1(x^1) \right) \\
&\quad \frac{\partial q}{\partial x_i}(x^1) + \frac{1}{2} \frac{\partial r}{\partial x_i}(x^1)(\kappa^1(x^1))^2 \\
0 &= \frac{\partial^2 \pi^1}{\partial x_i \partial x_\sigma}(x^1) g_\sigma(x^1) + \frac{\partial \pi^1}{\partial x_\sigma}(x^1) \frac{\partial g_\sigma}{\partial x_i}(x^1) && (4) \\
&\quad + \frac{\partial r}{\partial x_i}(x^1) \kappa^1(x^1) + r(x^1) \frac{\partial \kappa^1}{\partial x_i}(x^1)
\end{aligned}
$$

for the remaining $2n$ unknowns. Because of the second HJB equation (2), the first n equations do not contain the unknowns $\frac{\partial \kappa^1}{\partial x_i}(x^1)$ for $1 \leq i \leq n$. Moreover the first n equations are decoupled and can be solved one by one

$$
\begin{aligned}
\frac{\partial^2\pi^1}{\partial x_i \partial x_n}(x^1) = & \frac{-1}{f_n(x^1)+g_n(x^1)\kappa^1(x^1)} \\
& \times \left(\sum_{\sigma=1}^{n-1} \frac{\partial^2\pi^1}{\partial x_i \partial x_\sigma}(x^1)\left(f_\sigma(x^1)+g_\sigma(x^1)\kappa^1(x^1)\right) \right. \\
& + \frac{\partial \pi^1}{\partial x_\sigma}(x^1)\left(\frac{\partial f_\sigma}{\partial x_i}(x^1)+\frac{\partial g_\sigma}{\partial x_i}(x^1)\kappa^1(x^1)\right) \\
& \left. \frac{\partial q}{\partial x_i}(x^1)+\frac{1}{2}\frac{\partial r}{\partial x_i}(x^1)(\kappa^1(x^1))^2 \right)
\end{aligned}
$$

We invoke the summation convention when the range of the sum is from 1 to n, otherwise we explicitly show the sum.

The remaining n equations are also solvable one by one,

$$
\frac{\partial \kappa^1}{\partial x_i}(x^1) = \frac{-1}{r(x^1)}\left(\frac{\partial^2\pi^1}{\partial x_i \partial x_\sigma}(x^1)g_\sigma(x^1)+\frac{\partial \pi^1}{\partial x_\sigma}(x^1)\frac{\partial g_\sigma}{\partial x_i}(x^1)+\frac{\partial r}{\partial x_i}(x^1)\kappa^1(x^1)\right)
$$

Next we find the third partials of π^1 at x^1. We assume that

$$
\frac{\partial^3\pi^1}{\partial x_i \partial x_j \partial x_k}(x^1) = \frac{\partial^3\pi^0}{\partial x_i \partial x_j \partial x_k}(x^1)
$$

for $1 \leq i \leq j \leq k \leq n-1$. Equations for the other third partials are obtained by differentiating the first HJB equation (1) with respect to x_i and x_j for $1 \leq i \leq j \leq n$ and evaluating at x^1 yielding

$$
\begin{aligned}
0 = & \frac{\partial^3\pi^1}{\partial x_i \partial x_j \partial x_\sigma}(x^1)\left(f_\sigma(x^1)+g_\sigma(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^2\pi^1}{\partial x_i \partial x_\sigma}(x^1)\left(\frac{\partial f_\sigma}{\partial x_j}(x^1)+\frac{\partial g_\sigma}{\partial x_j}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^2\pi^1}{\partial x_j \partial x_\sigma}(x^1)\left(\frac{\partial f_\sigma}{\partial x_i}(x^1)+\frac{\partial g_\sigma}{\partial x_i}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial \pi^1}{\partial x_\sigma}(x^1)\left(\frac{\partial^2 f_\sigma}{\partial x_i \partial x_j}(x^1)+\frac{\partial^2 g_\sigma}{\partial x_i \partial x_j}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^2 q}{\partial x_i \partial x_j}(x^1)+\frac{1}{2}\frac{\partial^2 r}{\partial x_i \partial x_j}(x^1)(\kappa^1(x^1))^2 \\
& - r(x^1)\frac{\partial \kappa^1}{\partial x_i}(x^1)\frac{\partial \kappa^1}{\partial x_j}(x^1)
\end{aligned} \tag{5}
$$

These are $(n+1)n/2$ equations in the $(n+1)n/2$ unknowns $\frac{\partial^3\pi^1}{\partial x_i \partial x_j \partial x_n}(x^1)$ for $1 \leq i \leq j \leq n$. They can be solved one by one in lexographic order. The unknowns $\frac{\partial^2\kappa^1}{\partial x_i \partial x_j}(x^1)$ do not appear because of (2) and they are simplified by (4).

Then we differentiate the second HJB equation (3) with respect to x_i and x_j for $1 \leq i \leq j \leq n$ to obtain the $(n+1)n/2$ equations

$$
\begin{aligned}
0 = {} & \frac{\partial^3 \pi^1}{\partial x_i \partial x_j \partial x_\sigma}(x^1) g_\sigma(x^1) + \frac{\partial^2 \pi^1}{\partial x_i \partial x_\sigma}(x^1) \frac{\partial g_\sigma}{\partial x_j}(x^1) \\
& + \frac{\partial^2 \pi^1}{\partial x_j \partial x_\sigma}(x^1) \frac{\partial g_\sigma}{\partial x_i}(x^1) + \frac{\partial \pi^1}{\partial x_\sigma}(x^1) \frac{\partial^2 g_\sigma}{\partial x_i \partial x_j}(x^1) \\
& + \frac{\partial^2 r}{\partial x_i \partial x_j}(x^1) \kappa^1(x^1) + \frac{\partial r}{\partial x_i}(x^1) \frac{\partial \kappa^1}{\partial x_j}(x^1) \\
& + \frac{\partial r}{\partial x_j}(x^1) \frac{\partial \kappa^1}{\partial x_i}(x^1) + r(x^1) \frac{\partial^2 \kappa^1}{\partial x_i \partial x_j}(x^1)
\end{aligned}
\tag{6}
$$

which can be solved one by one for the $(n+1)n/2$ unknowns $\frac{\partial^2 \kappa^1}{\partial x_i \partial x_j}(x^1)$, $1 \leq i \leq j \leq n$.

To find the fourth partials of π^1 at x^1, we assume that

$$
\frac{\partial^4 \pi^1}{\partial x_i \partial x_j \partial x_k \partial x_l}(x^1) = \frac{\partial^4 \pi^0}{\partial x_i \partial x_j \partial x_k \partial x_l}(x^1)
$$

for $1 \leq i \leq j \leq k \leq l \leq n-1$. We differentiate the first HJB equation (1) with respect to x_i, x_j, x_k to obtain

$$
\begin{aligned}
0 = {} & \frac{\partial^4 \pi^1}{\partial x_i \partial x_j \partial x_k \partial x_\sigma}(x^1)\left(f_\sigma(x^1) + g_\sigma(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^3 \pi^1}{\partial x_i \partial x_j \partial x_\sigma}(x^1)\left(\frac{\partial f_\sigma}{\partial x_k}(x^1) + \frac{\partial g_\sigma}{\partial x_k}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^3 \pi^1}{\partial x_i \partial x_k \partial x_\sigma}(x^1)\left(\frac{\partial f_\sigma}{\partial x_j}(x^1) + \frac{\partial g_\sigma}{\partial x_j}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^3 \pi^1}{\partial x_j \partial x_k \partial x_\sigma}(x^1)\left(\frac{\partial f_\sigma}{\partial x_i}(x^1) + \frac{\partial g_\sigma}{\partial x_i}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^2 \pi^1}{\partial x_i \partial x_\sigma}(x^1)\left(\frac{\partial^2 f_\sigma}{\partial x_j \partial x_k}(x^1) + \frac{\partial^2 g_\sigma}{\partial x_j \partial x_k}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^2 \pi^1}{\partial x_j \partial x_\sigma}(x^1)\left(\frac{\partial^2 f_\sigma}{\partial x_i \partial x_k}(x^1) + \frac{\partial^2 g_\sigma}{\partial x_i \partial x_k}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^2 \pi^1}{\partial x_k \partial x_\sigma}(x^1)\left(\frac{\partial^2 f_\sigma}{\partial x_i \partial x_j}(x^1) + \frac{\partial^2 g_\sigma}{\partial x_i \partial x_j}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial \pi^1}{\partial x_\sigma}(x^1)\left(\frac{\partial^3 f_\sigma}{\partial x_i \partial x_j \partial x_k}(x^1) + \frac{\partial^3 g_\sigma}{\partial x_i \partial x_j \partial x_k}(x^1)\kappa^1(x^1)\right) \\
& + \frac{\partial^3 q}{\partial x_i \partial x_j \partial x_k}(x^1) + \frac{1}{2}\frac{\partial^3 r}{\partial x_i \partial x_j \partial x_k}(x^1)(\kappa^1(x^1))^2 \\
& - \frac{\partial r}{\partial x_i}(x^1)\frac{\partial \kappa^1}{\partial x_j}(x^1)\frac{\partial \kappa^1}{\partial x_k}(x^1) \\
& - \frac{\partial r}{\partial x_j}(x^1)\frac{\partial \kappa^1}{\partial x_i}(x^1)\frac{\partial \kappa^1}{\partial x_k}(x^1) \\
& - \frac{\partial r}{\partial x_k}(x^1)\frac{\partial \kappa^1}{\partial x_i}(x^1)\frac{\partial \kappa^1}{\partial x_j}(x^1) \\
& - r(x^1)\frac{\partial^2 \kappa^1}{\partial x_i \partial x_j}(x^1)\frac{\partial \kappa^1}{\partial x_k}(x^1) \\
& - r(x^1)\frac{\partial^2 \kappa^1}{\partial x_i \partial x_k}(x^1)\frac{\partial \kappa^1}{\partial x_j}(x^1) \\
& - r(x^1)\frac{\partial^2 \kappa^1}{\partial x_j \partial x_k}(x^1)\frac{\partial \kappa^1}{\partial x_i}(x^1)
\end{aligned} \tag{7}
$$

These $(n+2)(n+1)n/6$ equations can be solved one by one in lexograhic order for the $(n+2)(n+1)n/6$ unknowns $\frac{\partial^4 \pi^1}{\partial x_i \partial x_j \partial x_k \partial x_n}(x^1)$ for $1 \le i \le j \le k \le n$. The unknowns $\frac{\partial^3 \kappa^1}{\partial x_i \partial x_j \partial x_k}(x^1)$ do not appear because of (2) and they are simplified by (4) and (7).

Then we differentiate the second HJB equation (3) with respect to x_i, x_j, x_k for $1 \le i \le j \le n$ to obtain the $(n+2)(n+1)n/6$ equations

$$
\begin{aligned}
0 = {} & \frac{\partial^4 \pi^1}{\partial x_i \partial x_j \partial x_k \partial x_\sigma}(x^1) g_\sigma(x^1) \\
& + \frac{\partial^3 \pi^1}{\partial x_i \partial x_j \partial x_\sigma}(x^1) \frac{\partial g_\sigma}{\partial x_k}(x^1) \\
& + \frac{\partial^3 \pi^1}{\partial x_i \partial x_k \partial x_\sigma}(x^1) \frac{\partial g_\sigma}{\partial x_j}(x^1) \\
& + \frac{\partial^3 \pi^1}{\partial x_j \partial x_k \partial x_\sigma}(x^1) \frac{\partial g_\sigma}{\partial x_i}(x^1) \\
& + \frac{\partial^2 \pi^1}{\partial x_i x_\sigma}(x^1) \frac{\partial^2 g_\sigma}{\partial x_j \partial x_k}(x^1) \\
& + \frac{\partial^2 \pi^1}{\partial x_j x_\sigma}(x^1) \frac{\partial^2 g_\sigma}{\partial x_i \partial x_k}(x^1) \\
& + \frac{\partial^2 \pi^1}{\partial x_k x_\sigma}(x^1) \frac{\partial^2 g_\sigma}{\partial x_i \partial x_j}(x^1) \\
& + \frac{\partial \pi^1}{\partial x_\sigma}(x^1) \frac{\partial^3 g_\sigma}{\partial x_i \partial x_j \partial x_k}(x^1) \\
& + \frac{\partial^3 r}{\partial x_i \partial x_j \partial x_k}(x^1) \kappa^1(x^1) \\
& + \frac{\partial^2 r}{\partial x_i \partial x_j}(x^1) \frac{\partial \kappa^1}{\partial x_k}(x^1) \\
& + \frac{\partial^2 r}{\partial x_i \partial x_k}(x^1) \frac{\partial \kappa^1}{\partial x_j}(x^1) \\
& + \frac{\partial^2 r}{\partial x_j \partial x_k}(x^1) \frac{\partial \kappa^1}{\partial x_i}(x^1) \\
& + \frac{\partial r}{\partial x_i}(x^1) \frac{\partial^2 \kappa^1}{\partial x_j \partial x_k}(x^1) \\
& + \frac{\partial r}{\partial x_j}(x^1) \frac{\partial^2 \kappa^1}{\partial x_i \partial x_k}(x^1) \\
& + \frac{\partial r}{\partial x_k}(x^1) \frac{\partial^2 \kappa^1}{\partial x_i \partial x_j}(x^1) \\
& + r(x^1) \frac{\partial^3 \kappa^1}{\partial x_i \partial x_j \partial x_k}(x^1)
\end{aligned} \tag{8}
$$

which can be solved one by one for the $(n+2)(n+1)n/6$ unknowns $\frac{\partial^3 \kappa^1}{\partial x_i \partial x_j \partial x_k}(x^1)$, $1 \leq i \leq j \leq k \leq n$.

7 Two Dimensional Example

We consider the optimal control problem of driving a planar pendulum of length 1 and mass 1 to the upright condition by a torque u at its pivot. The dynamics is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 + u\end{aligned}$$

We choose the Lagrangian

$$l(x,u) = \frac{1}{2}\left(|x|^2 + u^2\right)$$

We computed the Al'brecht solution around the origin to degree 4 in the cost and degree 3 in the optimal feedback. We accepted it on the sublevel set $\pi^0(x) \leq 0.5$. Then using the method described above we computed the solution at four points in the eigenspaces of the quadratic part of the cost where $\pi^0(x) = 0.5$. There is one in each quadrant. These outer solutions were also computed to degree 4 in the cost and degree 3 in the feedback.

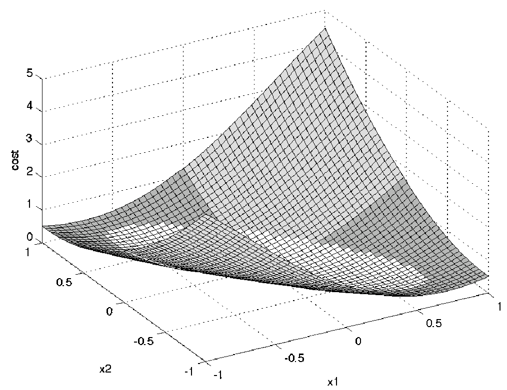

Fig. 5. Optimal cost computed on five patches. The outer patches are bounded in part by the axes.

8 Conclusion

We have sketched out a patchy approach to solving Hamilton Jacobi Bellman equations for nice optimal control problems and applied it to one and two dimensional examples. We were deliberately vague about some aspects of the proposed algorithm such as how to choose the boundary between outer patches. Further research is needed to clarify these issues and this can come only with extensive computation.

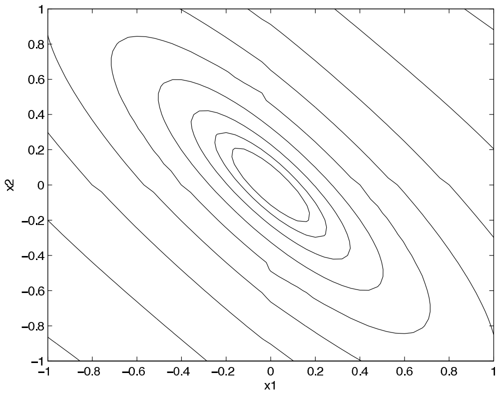

Fig. 6. Contour plot of five patch cost. The inner 4 contours are within the central patch. Notice that there is a slight mismatch of the outer contours when they meet at the axes.

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