The Patchy Cost and Feedback for the HJB PDE

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Abstract. In this paper, we describe our development of a higher-order method for solving the Hamilton-Jacobi-Bellman PDE by incorporating several techniques. There are the power series method of Albrecht, Cauchy-Kovalevskaya techniques, patchy methods of Ancona and Bressan and Navasca and Krener, the fast sweeping and marching methods, and regularization methods. We also demonstrate the method through an example.

1 Introduction

The Hamilton-Jacobi-Bellman (HJB) PDE arises ubiquitously in many algorithms for the control and estimation of nonlinear systems. The solutions to the HJB equations are necessary for nonlinear control and estimation. Yet current numerical methods are inadequate for approximating the solutions in dimension higher than three. The HJB PDE arises from optimal control problems which have many real-world applications of high dimension, namely in, engineering, economics and recently in biomedicine. In this paper, we describe our development of a higher-order method for solving the HJB PDE by incorporating several methods: power series methods of Albrecht [1], Cauchy-Kovalevsky techniques [7], patchy methods of Ancona and Bressan [2], Navasca and Krener [12, 14], the fast sweeping [8, 15, 9] and marching methods [17] and regularization methods [5].

Current numerical methods are infeasible for these types of problems due to the excessively large requirement in computation and memory. There are numer-

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ous techniques to solving HJB PDE, for example see [3], [6], [10], [8], [9], [15] [16], [17], [18] and their references therein. However, these methods suffer the *curse of dimensionality* since the computation grows exponentially in the dimension of the state variable x and control variable u. A standard approach is to introduce temporal and spatial discretizations of the entire optimal control problem which then becomes an inordinately large-scale nonlinear optimization program. There are also approximation methods for solving HJB using techniques borrowed from conservation laws; for example, see the paper of Osher and Shu [15]. These numerical solvers are quite efficient and could handle nonlinear and nonconvex Hamiltonians but they only hold for state variables in low dimension, in practice. The fast sweeping and marching method (Tsitsiklis [21], Osher et al. [15, 8, 9] and Sethian [17]) are ways to lessen this curse. It takes advantage of the fact that an HJB PDE has characteristics which are the closed loop optimal state trajectories that converge to the origin as $t \to \infty$.

1.1 The Hamilton-Jacobi-Bellman PDEs

The Hamilton-Jacobi-Bellman Partial Differential Equation arises in many control problems. Consider the infinite horizon optimal control problem of minimizing the cost

$$\int_t^\infty l(x,u)\ dt \tag{1}$$

subject to the dynamics

$$\dot{x} = f(x,u) \tag{2}$$

and initial condition

$$x(t) = x^0.$$

The state vector x is an n dimensional column vector, the control u is an m dimensional column vector and the dynamics $f(x,u)$ and Lagrangian $l(x,u)$ are assumed to be sufficiently smooth.

If the minimum exists and is a smooth function $\pi(x^0)$ of the initial condition then it satisfies the HJB PDE

$$\min_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\} = 0$$

and the optimal control $\kappa(x)$ satisfies

$$\kappa(x) = \arg\min_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\} \tag{3}$$

These are expressed in terms of the Hamiltonian

$$H(p,x,u) = pf(x,u) + l(x,u) \tag{4}$$

where the argument $p = \frac{\partial \pi}{\partial x}$ is an n dimensional row vector. The HJB PDE becomes

$$0 = \min_u H(\frac{\partial \pi}{\partial x}(x), x, u) \tag{5}$$

$$\kappa(x) = \arg\min_u H(\frac{\partial \pi}{\partial x}(x), x, u) \tag{6}$$

If the Hamiltonian $H(p, x, u)$ is strictly convex in u for all $p,\ x$. Then pair of HJB equations (5, 6) becomes

$$\frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) + l(x, \kappa(x)) = 0 \tag{7}$$

and

$$\frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) + \frac{\partial l}{\partial u}(x, \kappa(x)) = 0 \tag{8}$$

where the optimal cost $\pi(x)$ and optimal control $\kappa(x)$ are the unknown solutions.

If the dynamics and the cost have Taylor series expansion, then we can find power series solutions $\pi(x)$ and $\kappa(x)$ of the HJB PDE converging in a some local neighborhood containing zero through the method of Al'brecht [1].

2 The Patchy Method

In the paper [12], we have developed a method which is a higher order approach for locally approximating the solutions of the HJB PDE in two stages. In the first step, we locally approximate the solution to the HJB PDE using polynomial estimation through the method of Al'brecht [1] based on solving linear equations and one Riccati equation. Like all polynomial approximation techniques, the solutions of Al'brecht are only valid for a small domain around zero. To overcome this weakness, the second-stage continues the construction of the power series solution to a larger region. From a boundary point on the domain of the Al'brecht's power series, an extremal trajectory is computed backward in time using the Pontryagin Maximum Principle. Then the ordinary differential equations are derived for the higher partial derivatives of the solution along the extremal. These equations are solved yielding a power series for the approximate solution in a neighborhood of the extremal.

The patchy method, the second approach [14], is a extension of the power series method of Albrecht [1], the Cauchy-Kovalevskaya technique [7], the fast marching method [18], [17] and the patchy technique of Ancona and Bressan [2]. It is also an extension of our previous paper [12]. Moreover, a one-dimensional example of this patchy method can be found in [13]. This approach also has two stages. Similarly to the first stage in [12], the technique of Al'brecht is used to generate the power series solution locally around the origin. The second-stage of this new approach starts by defining the zeroth patch. The zeroth patch is a sublevel set of the computed optimal cost function defined by how well it satisfies the HJB PDE on the level set that is its boundary. Then we pick a point on the boundary of the zeroth patch and assume the optimal cost and optimal feedback have a power series expansion around that point. This point on the boundary is what we call the patch point. Now the Cauchy-Kovalevskaya (CK) technique comes into play for finding power series solutions on neighborhoods around the patch point.

The disadvantage of our first approach [12] is the requirement of solving $2n$ boundary value problems for the optimal state and costate trajectories. In addition, an ODE system must be solved for the coefficients of the *moving* power series solutions along the extremals. The second approach does not have these problems; we simply calculate the partial derivatives algebraically. However, there are several aspects in which we would like to improve the method. For example, for each patch on the kth sub-level set requires a distinct set of algebraic equations to be solved. With the boundary conditions provided by the Cauchy data, the linear systems of equations from the Cauchy-Kovalevskaya method yield the coefficients of the power series. Moreover, the accuracy of our approximation heavily relies on the Cauchy data so that the polynomials are only as accurate as the estimated data. In addition, we compare the straightforward approach of the patchy method using the Cauchy data and the regularization technique for recovering power series coefficients for the CK technique.

2.1 Cauchy-Kovalevskaya's Method

The Cauchy problem is the following:

$$\frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) + l(x, \kappa(x)) = 0 \tag{9}$$

and

$$\frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) + \frac{\partial l}{\partial u}(x, \kappa(x)) = 0 \tag{10}$$

with the boundary conditions

$$\pi(x) = \pi^0(x) \text{ and } \kappa(x) = \kappa^0(x) \text{ for } x \in \Gamma \tag{11}$$

where Γ is a hyper-surface that is in the transversal direction to the closed loop dynamics. We assume that the Al'brecht closed loop dynamics is transverse to the boundary of the sublevel set and points inward; i.e.

$$\frac{\partial \pi^0}{\partial x}(z) f(z, \kappa^0(z)) < 0.$$

At the patch point $z = (\bar{x}, \bar{y})$ on the kth sublevel set, the Cauchy problem is solved through the following equations

$$\frac{\partial^k}{\partial x^k}\left(\frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) + l(x, \kappa(x))\right) = 0 \tag{12}$$

and

$$\frac{\partial^k}{\partial x^k}\left(\frac{\partial \pi}{\partial x}(z) \frac{\partial f}{\partial u}(z, \kappa(z)) + \frac{\partial l}{\partial u}(z, \kappa(z))\right) = 0 \tag{13}$$

with the side conditions

$$\frac{\partial^k \pi^0}{\partial x_{i_{\alpha_1}} \dots \partial x_{i_{\alpha_k}}}(z) = \frac{\partial^k \pi}{\partial x_{i_{\alpha_1}} \dots \partial x_{i_{\alpha_k}}}(z) \tag{14}$$

and

$$\frac{\partial^k \kappa^0}{\partial x_{i_{\alpha_1}} \dots \partial x_{i_{\alpha_k}}}(z) = \frac{\partial^k \kappa}{\partial x_{i_{\alpha_1}} \dots \partial x_{i_{\alpha_k}}}(z) \tag{15}$$

where $1 \le i_{\alpha_j} < l$ and $l < i_{\alpha_j} \le n$ and the jth index maximizes

$$|f_k(z) + g_k(z)\kappa^0(z)|.$$

The boundary conditions (11,14,15) are usually referred to as the Cauchy data. For simplicity, let $\dot{x} = f(x) + g(x)u(x)$ and $l(x,u) = q(x) + r(x)\frac{u(x)}{2}^2$. Here we look for power series solutions about $(\bar{x}, \bar{y})$:

$$\begin{aligned}
\pi(x,y) = {} & \pi(\bar{x},\bar{y}) + \frac{\partial \pi}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial \pi}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) \\
& + \frac{1}{2}\frac{\partial^2 \pi}{\partial x^2}(\bar{x},\bar{y})(x-\bar{x})^2 + \frac{\partial^2 \pi}{\partial x \partial y}(\bar{x},\bar{y})\frac{\partial^2 \pi}{\partial y}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y}) + \frac{1}{2}\frac{\partial^2 \pi}{\partial y}(\bar{x},\bar{y})(y-\bar{y})^2 \\
& + \frac{1}{6}\frac{\partial^3 \pi}{\partial x^3}(\bar{x},\bar{y})(x-\bar{x})^3 + \frac{1}{2}\frac{\partial^3 \pi}{\partial x^2 \partial y}(\bar{x},\bar{y})(x-\bar{x})^2(y-\bar{y}) + \frac{1}{2}\frac{\partial^3 \pi}{\partial x \partial y^2}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y})^2 \\
& + \frac{1}{6}\frac{\partial^3 \pi}{\partial y^3}(\bar{x},\bar{y})(y-\bar{y})^3 \\
& + \frac{1}{24}\frac{\partial^4 \pi}{\partial x^4}(\bar{x},\bar{y})(x-\bar{x})^4 + \frac{1}{6}\frac{\partial^4 \pi}{\partial x^3 \partial y}(\bar{x},\bar{y})(x-\bar{x})^3(y-\bar{y}) + \frac{1}{4}\frac{\partial^4 \pi}{\partial x^2 \partial y^2}(\bar{x},\bar{y})(x-\bar{x})^2(y-\bar{y})^2 \\
& + \frac{1}{6}\frac{\partial^4 \pi}{\partial x \partial y^3}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y})^3 + \frac{1}{24}\frac{\partial^4 \pi}{\partial y^4}(\bar{x},\bar{y})(y-\bar{y})^4 + \cdots
\end{aligned}$$

and

$$\begin{aligned}
\kappa(x,y) = {} & \kappa(\bar{x},\bar{y}) + \frac{\partial \kappa}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial \kappa}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) + \frac{1}{2}\frac{\partial^2 \kappa}{\partial x^2}(\bar{x},\bar{y})(x-\bar{x})^2 \\
& + \frac{\partial^2 \kappa}{\partial x \partial y}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y}) + \frac{1}{2}\frac{\partial^2 \kappa}{\partial y^2}(\bar{x},\bar{y})(y-\bar{y})^2 \\
& + \frac{1}{6}\frac{\partial^3 \kappa}{\partial x^3}(\bar{x},\bar{y})(x-\bar{x})^3 + \frac{1}{2}\frac{\partial^3 \kappa}{\partial x^2 \partial y}(\bar{x},\bar{y})(x-\bar{x})^2(y-\bar{y}) + \frac{1}{2}\frac{\partial^3 \kappa}{\partial x \partial y^2}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y})^2 \\
& + \frac{1}{6}\frac{\partial^3 \kappa}{\partial y^3}(\bar{x},\bar{y})(y-\bar{y})^3 + \cdots
\end{aligned}$$

The coefficients are solved in stages. The lowest partial derivatives are set to the partial derivatives of the Albrecht's solution; i.e.

$$\begin{aligned}
\frac{\partial \pi}{\partial x}(z) &= \frac{\partial \pi^0}{\partial x}(z) \\
\kappa(z) &= \kappa^0(z)
\end{aligned}$$

This is the Cauchy data in this level. Here we assume $l = 2$, that is, the y-direction which gives the maximum entry in the closed loop dynamics. We find $\frac{\partial \pi}{\partial y}$ from

$$\frac{\partial \pi}{\partial x}\left\{f_1(x,y)+g_1(x,y)u\right\}+\frac{\partial \pi}{\partial y}\left\{f_2(x,y)+g_2(x,y)u\right\}+l(x,y)+\frac{1}{2}r(x)u^2=0$$
$$\frac{\partial \pi}{\partial x}g_1(x,y)+\frac{\partial \pi}{\partial y}g_2(x,y)+r(x)u=0$$

which is equivalent to solving a quadratic equation in the unknown.

In the second level, the unknown partial derivatives are

$$\frac{\partial^2 \pi}{\partial x^2}(z), \frac{\partial^2 \pi}{\partial x \partial y}(z), \frac{\partial^2 \pi}{\partial y^2}(z), \frac{\partial \kappa}{\partial x}(z), \frac{\partial \kappa}{\partial y}(z). \tag{16}$$

From the Cauchy Problem, the following system

$$\begin{bmatrix} f_1+g_1u & f_2+g_2u & 0 & 0 & 0 \\ 0 & f_1+g_1u & f_2+g_2u & 0 & 0 \\ g_1 & g_2 & 0 & r & 0 \\ 0 & g_1 & g_2 & 0 & r \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \pi}{\partial x^2}(z) \\ \frac{\partial^2 \pi}{\partial x \partial y}(z) \\ \frac{\partial^2 \pi}{\partial y^2}(z) \\ \frac{\partial \kappa}{\partial x}(z) \\ \frac{\partial \kappa}{\partial y}(z) \end{bmatrix} = \begin{bmatrix} \xi_1^1(z) \\ \xi_2^1(z) \\ \zeta_1^1(z) \\ \zeta_2^1(z) \end{bmatrix} \tag{17}$$

where

$$\begin{aligned}
\xi_1^1(z) &= -\frac{\partial \pi}{\partial x}(z)\left(\frac{\partial f_1}{\partial x}(z)+\frac{\partial g_1}{\partial x}(z)u(z)\right)-\frac{\partial \pi}{\partial y}(z)\left(\frac{\partial f_2}{\partial x}(z)+\frac{\partial g_2}{\partial x}(z)u(z)\right)-\frac{\partial l}{\partial x}(z) \\
&\quad -\frac{1}{2}\frac{\partial r}{\partial x}(z)u^2(z) \\
\xi_2^1(z) &= -\frac{\partial \pi}{\partial x}(z)\left(\frac{\partial f_1}{\partial y}(z)+\frac{\partial g_1}{\partial y}(z)u(z)\right)-\frac{\partial \pi}{\partial y}(z)\left(\frac{\partial f_2}{\partial y}(z)+\frac{\partial g_2}{\partial y}(z)u(z)\right)-\frac{\partial l}{\partial y}(z) \\
&\quad -\frac{1}{2}\frac{\partial r}{\partial y}(z)u^2(z) \\
\zeta_1^1(z) &= 0 \\
\zeta_2^1(z) &= 0
\end{aligned}$$

is satisfied by the vector of the unknowns (17). Again we select the y-direction so that the partial derivatives $\frac{\partial^2 \pi}{\partial x^2}$ and $\frac{\partial \kappa}{\partial x}$ are known and the linear system

$$\begin{bmatrix} f_2+g_2u & 0 & 0 \\ f_1+g_1u & f_2+g_2u & 0 \\ g_1 & g_2 & r \end{bmatrix} \begin{bmatrix} \frac{\partial^2 \pi}{\partial x \partial y}(z) \\ \frac{\partial^2 \pi}{\partial y^2}(z) \\ \frac{\partial \kappa}{\partial y}(z) \end{bmatrix} = \begin{bmatrix} \xi_1^1(z)-(f_1+g_1u)\frac{\partial^2 \pi}{\partial x^2}(z) \\ \\ \xi_2^1(z) \\ \zeta_1^1(z)-g_1\frac{\partial^2 \pi}{\partial x^2}(z)-r\frac{\partial \kappa}{\partial x}(z) \end{bmatrix} \tag{18}$$

is now easily solved through matrix inversion as long as $f_2+g_2u(z) \neq 0$ and $r(z) \neq 0$.

The third level involves the following unknown partial derivatives:

$$\frac{\partial^3\pi}{\partial x^3}(z), \frac{\partial^3\pi}{\partial x^2\partial y}(z), \frac{\partial^3\pi}{\partial x\partial y^2}(z), \frac{\partial^3\pi}{\partial y^3}(z), \frac{\partial^2\kappa}{\partial x^2}(z), \frac{\partial^2\kappa}{\partial x\partial y}(z), \frac{\partial^2\kappa}{\partial y^2}(z).$$

Here is the linear system from the Cauchy Problem:

$$\begin{bmatrix} f_1+g_1u & f_2+g_2u & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1+g_1u & f_2+g_2u & 0 & 0 & 0 & 0 \\ 0 & 0 & f_1+g_1u & f_2+g_2u & 0 & 0 & 0 \\ g_1 & g_2 & 0 & 0 & r & 0 & 0 \\ 0 & g_1 & g_2 & 0 & 0 & r & 0 \\ 0 & 0 & g_1 & g_2 & 0 & 0 & r \end{bmatrix} \begin{bmatrix} \frac{\partial^3\pi}{\partial x^3}(z) \\ \frac{\partial^3\pi}{\partial x^2\partial y}(z) \\ \frac{\partial^3\pi}{\partial x\partial y^2}(z) \\ \frac{\partial^3\pi}{\partial y^3}(z) \\ \frac{\partial^2\kappa}{\partial x^2}(z) \\ \frac{\partial^2\kappa}{\partial x\partial y}(z) \\ \frac{\partial^2\kappa}{\partial y^2}(z) \end{bmatrix} = \begin{bmatrix} \xi_1^2(z) \\ \xi_2^2(z) \\ \xi_3^2(z) \\ \zeta_1^2(z) \\ \zeta_2^2(z) \\ \zeta_3^2(z) \end{bmatrix} \quad (19)$$

where

$$\begin{aligned}
\xi_1^2(z) = {} & -2\frac{\partial^2\pi}{\partial x^2}(z)\left(\frac{\partial f_1}{\partial x}(z) + \frac{\partial g_1}{\partial x}(z)u(z) + g_1(z)\frac{\partial\kappa}{\partial x}(z)\right) - \frac{\partial\pi}{\partial x}\left(\frac{\partial^2 f_1}{\partial x^2}(z)f_1(z) + \frac{\partial^2 g_1}{\partial x^2}(z)u(z)\right. \\
& \left. +2\frac{\partial g_1}{\partial x}(z)\frac{\partial\kappa}{\partial x}(z)\right) - 2\frac{\partial^2\pi}{\partial x\partial y}(z)\left(\frac{\partial f_2}{\partial x}(z) + \frac{\partial g_2}{\partial x}(z)u(z) + g_2(z)\frac{\partial\kappa}{\partial x}(z)\right) \\
& -\frac{\partial\pi}{\partial y}\left(\frac{\partial^2 f_2}{\partial x^2}(z)f_2(z) + \frac{\partial^2 g_2}{\partial x^2}(z)u(z) + 2\frac{\partial g_2}{\partial x}(z)\frac{\partial\kappa}{\partial x}(z)\right) \\
& -\frac{\partial^2 l}{\partial x^2}(z) - 2\frac{\partial r}{\partial x}(z)\kappa(z)\frac{\partial\kappa}{\partial x}(z) - r(z)(\frac{\partial\kappa}{\partial x}(z))^2 - r(z)\kappa(z)\frac{\partial^2\kappa}{\partial x^2}(z) - \frac{1}{2}\frac{\partial^2 r}{\partial x^2}\kappa^2(z) \\
\xi_2^2(z) = {} & -\frac{\partial^2\pi}{\partial x^2}(z)\left(\frac{\partial f_1}{\partial y}(z) + \frac{\partial g_1}{\partial y}(z)u(z) + g_1(z)\frac{\partial u}{\partial y}(z)\right) \\
& -\frac{\partial^2\pi}{\partial x\partial y}(z)\left(\frac{\partial f_1}{\partial y}(z) + \frac{\partial g_1}{\partial y}(z)u(z) + g_1(z)\frac{\partial u}{\partial x}(z)\right) \\
& -\frac{\partial\pi}{\partial x}\left(\frac{\partial^2 f_1}{\partial x\partial y}(z)f_1(z) + \frac{\partial^2 g_1}{\partial x\partial y}(z)u(z) + \frac{\partial g_1}{\partial x}(z)\frac{\partial u}{\partial y}(z) + \frac{\partial g_1}{\partial y}(z)\frac{\partial u}{\partial x}(z)\right) \\
& -\frac{\partial^2\pi}{\partial x\partial y}(z)\left(\frac{\partial f_2}{\partial y}(z) + \frac{\partial g_2}{\partial y}(z)u(z) + g_2(z)\frac{\partial\kappa}{\partial y}(z)\right) \\
& -\frac{\partial^2\pi}{\partial y^2}(z)\left(\frac{\partial f_2}{\partial x}(z) + \frac{\partial g_2}{\partial x}(z)u(z) + g_2(z)\frac{\partial\kappa}{\partial x}(z)\right) \\
& -\frac{\partial\pi}{\partial y}\left(\frac{\partial^2 f_2}{\partial x\partial y}(z)f_2(z) + \frac{\partial^2 g_2}{\partial x\partial y}(z)u(z) + \frac{\partial g_2}{\partial x}(z)\frac{\partial u}{\partial y}(z) + \frac{\partial g_2}{\partial y}(z)\frac{\partial u}{\partial x}(z)\right) \\
& -\frac{\partial^2 l}{\partial x\partial y}(z) - \frac{\partial r}{\partial y}(z)\kappa(z)\frac{\partial\kappa}{\partial x}(z) - \frac{\partial r}{\partial x}(z)\kappa(z)\frac{\partial\kappa}{\partial y}(z) - r(z)\frac{\partial\kappa}{\partial x}(z)\frac{\partial\kappa}{\partial y}(z) \\
& -r(z)\kappa(z)\frac{\partial^2\kappa}{\partial x\partial y}(z) - \frac{1}{2}\frac{\partial^2 r}{\partial x\partial y}\kappa^2(z) \\
\xi_3^2(z) = {} & -2\frac{\partial^2\pi}{\partial x\partial y}(z)\left(\frac{\partial f_1}{\partial y}(z) + \frac{\partial g_1}{\partial y}(z)u(z) + g_1(z)\frac{\partial\kappa}{\partial y}(z)\right) - \frac{\partial\pi}{\partial x}\left(\frac{\partial^2 f_1}{\partial x\partial y}(z)f_1(z) + \frac{\partial^2 g_1}{\partial x\partial y}(z)u(z)\right. \\
& \left. +2\frac{\partial g_1}{\partial y}(z)\frac{\partial\kappa}{\partial y}(z)\right) \\
& -2\frac{\partial^2\pi}{\partial y^2}(z)\left(\frac{\partial f_2}{\partial x}(z) + \frac{\partial g_2}{\partial x}(z)u(z) + g_2(z)\frac{\partial\kappa}{\partial x}(z)\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial \pi}{\partial y}\left(\frac{\partial^2 f_2}{\partial x \partial y}(z) f_2(z)+\frac{\partial^2 g_2}{\partial x \partial y}(z)\kappa(z)+\frac{\partial g_2}{\partial x}(z)\frac{\partial u}{\partial y}(z)+\frac{\partial g_2}{\partial y}(z)\frac{\partial u}{\partial x}(z)\right)\\
&-\frac{\partial^2 l}{\partial y^2}(z)-2\frac{\partial r}{\partial y}(z)\kappa(z)\frac{\partial \kappa}{\partial y}(z)-r(z)(\frac{\partial \kappa}{\partial y}(z))^2-r(z)\kappa(z)\frac{\partial^2 \kappa}{\partial y^2}(z)-\frac{1}{2}\frac{\partial^2 r}{\partial y^2}\kappa^2(z)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_1^2(z) &= 0\\
\zeta_2^2(z) &= 0\\
\zeta_3^2(z) &= 0.
\end{aligned}
$$

The Cauchy data on this level are the partial derivatives $\frac{\partial^3 \pi}{\partial x^3}$ and $\frac{\partial^2 \kappa}{\partial x^2}$. It follows that we obtain a new linear system that can be easily solved through matrix inversion. The higher oder partial derivatives can be obtained by following the same procedure.

2.2 Regularization technique

Boundary conditions are necessary to find solutions in patches to the HJB PDEs in patches of neighborhood. However, the HJB PDEs arising from the infinite horizon optimal control do not have natural boundary conditions except for $\pi(0) = 0$. To obviate the problem, the Albrecht's polynomial solutions are used as the boundary conditions defined on the zeroth patch. Since these solutions are only approximation, the ensuing power series solutions defined on the patches can only be at most as accurate as the Al'brecht polynomials. The regularization methods are often very useful for recovering the unknowns in the presence of noisy data, in our case, the noisy boundary conditions.

The standard regularization technique amounts to finding $\mathbf{x} \in \mathbb{R}^n$ where

$$
\min \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^{\mathbf{2}} + \alpha \|\mathbf{x}\|^{\mathbf{2}}
$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and α is some positive constant. This is the so-called Tikhonov regularization. We define $\|\cdot\|$ as the standard Euclidean norm. We will consider the case when $m < n$ since the linear systems (17,19) are of this form. The numerical implementation of the Tikhonov regularization have been well-developed; e.g. see the papers of Eldén [4] and Voevodin [19]. Similarly, the choice for the regularization parameter α have been well-studied; see the book of Engel, Hanke and Neubauer [5] and the references therein. For the error-free $\mathbf{b}$, several techniques may be applied to find α, namely, the L-curve criterion and the generalized cross-validation. In our case, we have a noisy $\mathbf{b}$ due to the error in the polynomial estimation on the patch points. There are two schemes for selecting the best α: one requires an a-priori knowledge of the error level in $\mathbf{b}$ and the other is an a-posteriori strategy based on the discrepancy principle of Morozov [11].

An advantage of this method is that the system of equations for solving the power series coefficient are obtained easily for each patch since the dependence is only the patch points and not in the direction in which the Cauchy data is defined; i.e. the linear system (17,19) holds for all patches.

3 Numerical Results

We solve problem:

$$\min_u \int_0^\infty \frac{1}{2}\sin^2 x + \frac{1}{2}\left(y - \frac{x^3}{3}\right)^2 + \frac{1}{2}u^2 \; dt$$

subject to

$$\dot{x} = \left(y - \frac{x^3}{3}\right)\sec x$$

$$\dot{y} = \left(x^2 y - \frac{x^5}{3}\right)\sec x + u$$

With the nonlinear transformation

$$z_1 = \sin x$$

$$z_2 = y - \frac{x^3}{3},$$

the optimal control above is equivalent to

$$\min_u \int_0^\infty \frac{1}{2}|z|^2 + \frac{1}{2}u^2 \; dt$$

subject to

$$\dot{z_1} = z_2$$

$$\dot{z_2} = u.$$

The true cost and control solutions are the following:

$$\pi(z) = \frac{1}{2}z' \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} z$$

$$\kappa(z) = -[1 \;\; \sqrt{3}]z.$$

In Figure 1, the Cauchy data are used to approximate the coefficients of the power series solutions. In Figure 2, the regularization technique is used for finding the coefficients of the power series. Notice that the lateral boundaries of the patches do not match up nicely. In addition, observe that the error graph dips the most exactly at the lateral boundaries of the patches. From these plots, the use of Cauchy data gives better numerical results on the patches and on the lateral boundaries. Although as the level of patching increases, maintaining the same accuracy on the patches requires smaller patch size. We are currently working on smoothing the lateral boundaries and experimenting on better patching schemes. Figure 3 gives a close look at the regularized patchy cost, its contour plots and the patch points represented as the red circles. Notice that the plot gives a crude approximation of the closed contours unlike the smooth contour plots of Al'brecht solutions where at some level set the contour breaks open.

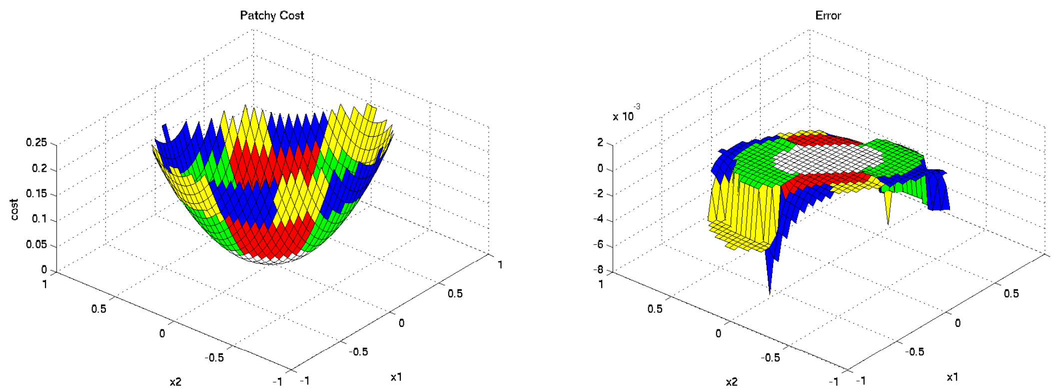

Figure 1. *Patchy Cost with the Cauchy Data*

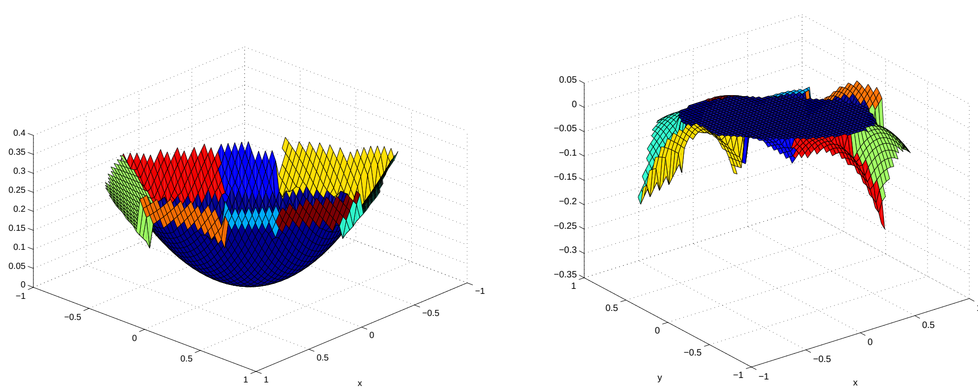

Figure 2. *Patchy Cost without the Cauchy Data*

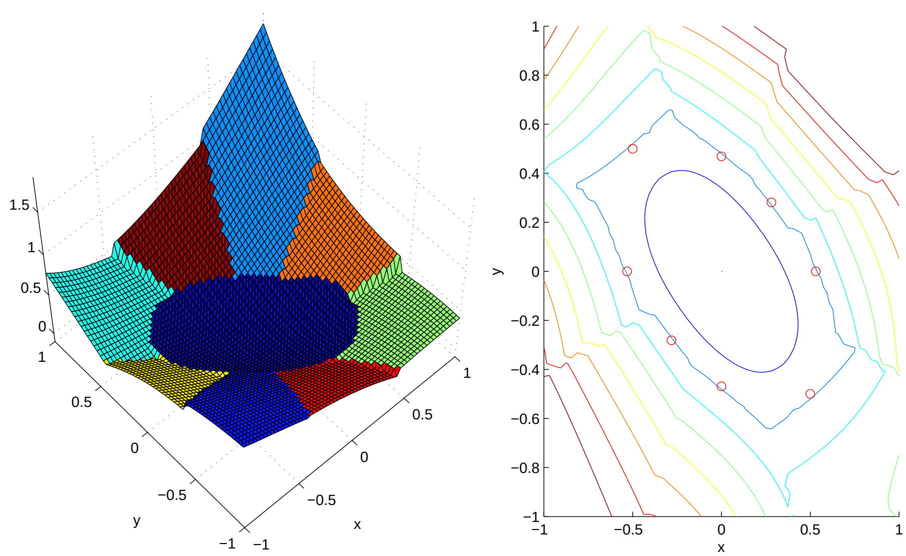

Figure 3. *The Regularized Patchy Cost, the Contours, and the Patch Points*

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