

The Lax-Friedrichs Sweeping Method for Optimal Control Problems in Continuous and Hybrid Dynamics

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Abstract. We implement the Lax-Friedrichs sweeping (LFS) method to approximate the solution to the Hamilton-Jacobi (HJ) equation arising in an infinite horizon optimal control problem. The fast sweeping method approximates the viscosity solution of a boundary value problem for HJ equations. The static HJ equation is discretized by using the Lax-Friedrichs numerical Hamiltonian and the solution is approximated iteratively by using a Gauss-Seidel like new updating process. Through the Dynamic Programming Principle and a convexity assumption on the control variable on Hamiltonian, we derive a pair of HJ equations for continuous-time dynamics. The LFS method is applied to solve the coupled PDEs by successive iteration of the optimal cost and control. The LFS scheme also applies to optimal control problems in hybrid systems in which it admits the viscosity solution for the HJ equation. We demonstrate the efficiency of the method through some numerical examples in both continuous and hybrid dynamics.

Keywords: Hamilton-Jacobi PDE, optimal control, hybrid system, Lax-Friedrichs sweeping method

1. INTRODUCTION

The aim of this paper is twofold: (1) to solve coupled HJ equations and (2) to approximate viscosity solutions of optimal control problems in hybrid dynamics. The Lax-Friedrichs sweeping (LFS) method is an effective tool for both problems.

These HJ-type equations appear in many problems in control theory. A pair of HJ-type equations arise in optimal control and H_∞ control problems and systems of HJ equations occur in mathematical finance.

Hybrid systems naturally arise in business and industry whenever there is an interaction between computer-embedded controllers with mechanical, chemical, or electrical processes. Hybrid systems occur in the interactions of both continuous and discrete-event dynamics. Here we consider hysteresis in smart materials as the hybrid system. Within their microstructures, smart materials, such as piezoelectric materials, electro-rheostatic and magneto-rheostatic materials, and shape memory alloys, have inherent features allowing them to have built-in sensors, actuators, and control mechanism. Because of these unique characteristics, smart materials have extensive applications, e.g. in aerospace, automotive industry, and manufacturing. However,

one debilitating factor of smart materials is hysteresis. Several papers have studied and classified hysteresis into phenomenological or physics-based models. The paper of Tan and Baras [15] explores the control methodology for smart actuators showing hysteresis based on the Jiles-Atherton model [8].

Since there is a such wide-range of applicability for solving pairs or systems of HJ equations in continuous dynamics and for approximating viscosity solutions in hybrid dynamics, it is imperative to study numerical techniques for these problems.

The paper is organized as follows. In next section, we describe the LFS method as a numerical technique for solving boundary value problems for HJ PDEs. In Section 3, the coupled equations are derived from the optimal control formulation for continuous dynamics. We also discuss several numerical examples. In the last section, we numerically solve for the optimal control of hysteresis. The derivation of the HJ equations corresponding to hybrid dynamics which appears in the ferromagnetic model is described.

2. LAX-FRIEDRICHS SWEEPING METHOD

The boundary value problem for the Hamilton-Jacobi equation that we solve here is

$$(2.1) \quad \begin{aligned} \lambda V(x) + H\left(x, \frac{\partial V}{\partial x}(x)\right) &= R(x), \quad x \in \Omega \\ V(x) &= V_0(x), \quad x \in \Gamma \end{aligned}$$

where $\Gamma \subset \Omega \subseteq \mathbb{R}^n$, $\lambda \geq 0$ represents a fixed discount factor, $R(x) > 0$ for all x and $V(x)$ is the unknown. The solutions to (2.1) are typically continuous but not differentiable even if $V_0(x)$ is smooth. The existence and uniqueness of the solution to (2.1) rely on the notions of viscosity solutions; see the classical paper of Crandall and Lions [6].

The fast sweeping method was motivated originally by Boué and Dupuis [3] and used in [18, 16, 10] with Godunov numerical Hamiltonian. Here we use the Lax-Friedrich sweeping method to solve HJ equations. The method is described in the one-dimensional case for simplicity of exposition. For the treatment of problems in higher dimensions, see [9]. The scheme is based on the Lax-Friedrichs monotone flux \hat{H}^{LF} :

$$(2.2) \quad \hat{H}^{LF}(p^+, p^-) = H\left(\frac{p^+ + p^-}{2}\right) - \frac{\sigma_x}{2}(p^+ - p^-)$$

where the artificial viscosity σ_x satisfies the monotonicity enforcing bound

$$\sigma_x \geq \max \left| \frac{\partial H}{\partial p} \right|.$$

Here we denote $p = \frac{\partial V}{\partial x}$ and p^\pm is the forward and backward difference approximations of $\frac{\partial V}{\partial x}$.

Assume $\Omega = [a, b]$ where $a = x_0 < x_1 < \dots < x_N < x_{N+1} = b$ and $h = x_{j+1} - x_j$. The function H in the boundary value problem is then discretized where $\frac{\partial V}{\partial x}$ is approximated according to (2.2); i.e.

$$\lambda V(x_j) + H\left(x_j, \frac{V(x_{j+1}) - V(x_{j-1}))}{2h}\right) - \frac{\sigma_x}{2} \left(\frac{V(x_{j+1}) - 2V(x_j) + V(x_{j-1}))}{h} \right) = R(x_j).$$

Other monotone schemes can be used, based on, for example, the Godunov numerical Hamiltonian [2]. However, LFS avoids a cumbersome optimization at each grid point. This feature keeps the LFS algorithm fast, simple and easily applicable to very complicated Hamiltonians.

Henceforth, we denote a function $f(x)$ evaluated at the mesh point x_j at the k iteration simply by $f_j^{(k)}$.

The discretized equation (2.3) is solved for $V_j^{(k)}$ iteratively and is updated by the current values of its nearest neighbors V_{j+1} and V_{j-1} . We require that the most current iterate value of its neighbors must be used to calculate $V_j^{(k)}$; this updating process is adapted from Gauss-Seidel type iterative schemes using a special sweeping strategy. In the 1-dimensional case, we have forward and backward sweeps. If the updating process is moving forward (left to right), then to find $V_j^{(k)}$ we utilize the values $V_{j-1}^{(k)}$ and $V_{j+1}^{(k-1)}$. The formula at the $k+1$ iteration in a forward sweep is:

$$V_j^{k+1} = \left(R_i - H \left(x_i, \frac{V_{j+1}^{(k)} - V_{j-1}^{(k+1)}}{2h} \right) + \sigma_x \frac{V_{j+1}^{(k)} + V_{j-1}^{(k+1)}}{2h} \right) / \left(\lambda + \frac{\sigma_x}{h} \right).$$

In one-dimension, two sweeps are counted as one iteration. The direction alternates at each sweep. In general, 2^n sweeps are required per iteration for n dimensional problems.

To update the value of points on the computational boundary $\Omega_C \subseteq \Omega$ requires numerical assignments on grid points lying outside the domain. The values on grid points outside Ω_C are selected in such a way that the inflow of spurious data along the characteristics into the domain is prohibited. For detailed discussion and formulas, see [9].

The HJ equation has been solved using the fast sweeping scheme for the value function $V(x)$. In this paper, solutions to a pair of HJ equations are approximated simultaneously generating a sequence of iterates $\{(V^{(k)}, u^{(k)})\}$. Most coupled toy problems reduce to solving a HJ equation for V and an equation where u is expressed in terms of $\frac{\partial V}{\partial x}$. We next describe the implementation of LFS in the following to these coupled systems.

3. OPTIMAL CONTROL IN CONTINUOUS-TIME

3.1. Hamilton-Jacobi-Bellman Equations. Hamilton-Jacobi (HJ) type equations arise in many control problems. In particular, the Hamilton-Jacobi-Bellman (HJB) partial differential equation occurs in the reformulation of the infinite horizon optimal control problem. The infinite horizon optimal control minimizes the cost

$$\int_t^\infty l(x, u) e^{\lambda(\tau-t)} d\tau$$

subject to the dynamics

$$(3.3) \quad \dot{x} = f(x, u)$$

and initial condition

$$(3.4) \quad x(t) = x_0.$$

The state vector $x \in \mathbb{R}^n$, the control $u \in \mathbb{R}^m$, and $\lambda \geq 0$ is a discount factor.

Define the value function as

$$V(x_t) = \min_u \int_t^\infty l(x(\tau), u(\tau)) e^{\lambda(\tau-t)} d\tau$$

where x_t is the trajectory satisfying (3.3,3.4) at time t . Let $t = 0$; then the optimality principle states:

$$V(x_0) = e^{-\lambda s} V(x_s) + \min_u \int_0^s l(x(\tau), u(\tau)) e^{-\lambda\tau} d\tau.$$

Through the dynamic programming formulation, if the maximum exists and $V(x_0)$ is a smooth function of the initial condition, it then satisfies the HJB PDE:

$$(3.5) \quad \lambda V(x) + \max_u \left\{ -\frac{\partial V}{\partial x}(x) f(x, u) - l(x, u) \right\} = 0$$

and the optimal control $u^*(x)$ satisfies

$$(3.6) \quad u^*(x) = \arg \max_u \left\{ -\frac{\partial V}{\partial x}(x) f(x, u) - l(x, u) \right\} = 0$$

We assume that $f(x, u)$ and $l(x, u)$ are sufficiently smooth and $V(x) \in \mathcal{C}^1$. Generally, the solution to nonlinear first order PDE is not in \mathcal{C}^1 . Therefore, the solution are interpreted in a weak or viscosity sense. Although the LFS method approximates viscosity solutions, our examples in the continuous case have value functions that have continuous derivatives. Our goal here is to show how LFS solves the coupling of these HJ equations.

3.2. Numerical Examples. For the following examples, the discount factor $\lambda = 0$ in (3.3). The optimal control problem can be conveniently expressed in terms of a Hamiltonian:

$$\tilde{H}(p, x) = \max_u \{-pf(x, u) - l(x, u)\}$$

where the argument $p = \frac{\partial V}{\partial x}$ is an n dimensional row vector. Then, we have equations (3.5,3.6)

$$\begin{aligned} 0 &= \tilde{H} \left(\frac{\partial V}{\partial x}(x), x \right) \\ u^*(x) &= \arg \max_u \left\{ -\frac{\partial V}{\partial x}(x) f(x, u) - l(x, u) \right\} \end{aligned}$$

If the Hamiltonian $\tilde{H}(p, x, u)$ is strictly convex in u for all p and x then (3.5, 3.6) become

$$(3.7) \quad \frac{\partial V}{\partial x}(x) f(x, u^*(x)) + l(x, u^*(x)) = 0$$

and

$$(3.8) \quad \frac{\partial V}{\partial x}(x) \frac{\partial f}{\partial u}(x, u^*) + \frac{\partial l}{\partial u}(x, u^*(x)) = 0$$

Observe that $\tilde{H} = H - R$ where H and R are taken from the boundary value problem. The first example is taken from the paper [12]. Let the cost functional to be minimized over u

$$\int_0^\infty \ln^2(x+1) + u^2 dt$$

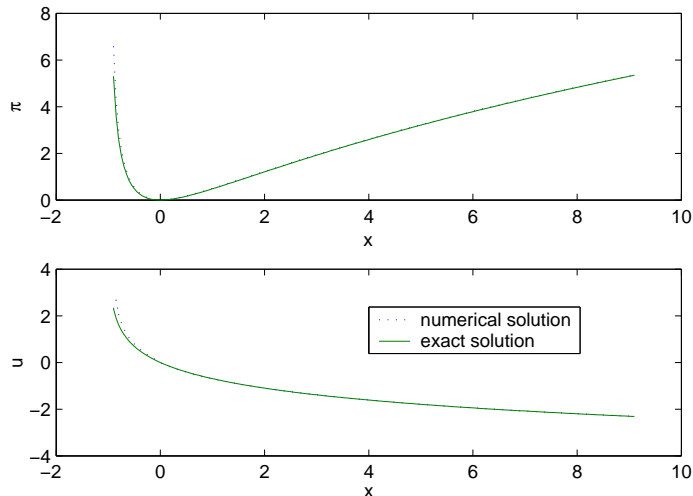


FIGURE 1. One-dimensional example: cost (above) and control (below).

subject to

$$\dot{x} = (x + 1)u, \quad x(0) = 0, \quad \text{for } x > -1.$$

One can easily verify that the true control law is $u^*(x) = -\ln(x + 1)$ and the corresponding value function is $V(x) = \ln^2(x + 1)$. The HJB equations are

$$\begin{cases} \frac{dV}{dx}(x + 1)u^* + \ln^2(x + 1) + u^{*2} = 0 \\ \frac{dV}{dx}(x + 1) + 2u^* = 0. \end{cases}$$

The above equations are reduced to

$$(3.9) \quad \left(\frac{dV}{dx}\right)^2 = \frac{4\ln^2(x + 1)}{(x + 1)^2}$$

$$(3.10) \quad u^* = -\frac{1}{2} \frac{dV}{dx}(x + 1).$$

The equation (3.9) is solved for iterates $V^{(k)}$ by the LFS scheme. Then $u^{*(k)}$ are updated according to (3.10). The optimal control u^* can be written as a function of $\frac{\partial V}{\partial x}$ when the dynamics (3.3) is affine in u .

The initial value $V^{(0)}$ is provided by the boundary condition $V(0) = 0$ and the assignment of arbitrary large values on the rest of the domain. After only 2 iteration, we see in Fig. 1 convergence is achieved in L_1 norm, i.e. $\|V^{(2)} - V^{(1)}\|_1 \leq 10^{-6}$.

We next consider another example in [12] where we minimize

$$\int_0^\infty (x_1^2 + x_2^{10/3} + x_1^{2/3} x_2^{4/3} (x_1^{2/3} + x_2^{2/3}) + u^{*2}) dt$$

subject to the dynamics

$$\begin{aligned} \dot{x}_1 &= x_1^{5/3} + 2x_1x_2^{4/3} - x_1 - 2x_1^{1/3}u \\ \dot{x}_2 &= -x_2^{7/3} - x_1^{2/3}x_2 \end{aligned}$$

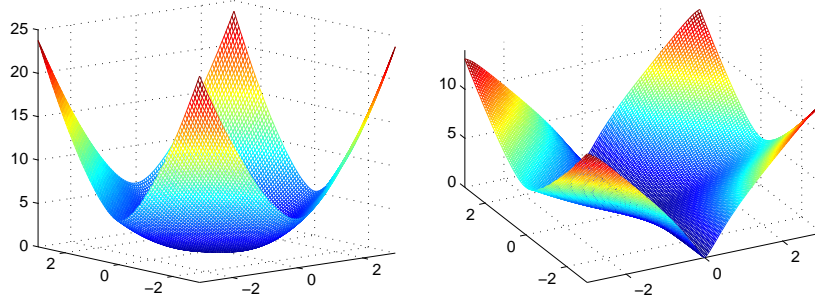


FIGURE 2. Optimal cost (left) and optimal control (right).

The HJB equations are

$$(3.11) \quad \frac{\partial V}{\partial x_1} x_1^{5/3} + 2x_1 x_2^{4/3} - x_1 - 2x_1^{1/3} u^* + \frac{\partial V}{\partial x_2} - x_2^{7/3} - x_1^{2/3} x_2 + x_1^2 + x_2^{10/3} + x_1^{2/3} x_2^{4/3} (x_1^{2/3} + x_2^{2/3}) + u^{*2} = 0$$

$$(3.12) \quad u^* = \frac{1}{2} \frac{\partial V}{\partial x_1} x_1^{1/3}$$

Similarly, the initial values $V^{(0)}$ and $u^{*(0)}$ are from the point source $V(0) = 0$ and $u(0) = 0$. In Fig. 2, after 115 iterations, the pair of approximations $\{(V^{(115)}), u^{*(115)}\}$ are within 10^{-6} of the L_1 norm.

Another example describes the motion of the inverted pendulum in [7]:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{\frac{g}{l} \sin(x_1) - \frac{1}{2} m_r x_2^2 \sin(2x_1) - \frac{m_r}{ml} \cos(x_1) u}{\frac{4}{3} - m_r \cos^2(x_1)} \end{aligned}$$

Here x_1 is the angle measured from the vertical up position, x_2 is the angular velocity, $m_r = (m + M)$ is the mass ratio of mass m of pendulum and mass M of the cart, and g is the gravitational constant. The cost function is quadratic and depends on the state variables x_1, x_2 , and the input u :

$$l(x_1, x_2, u) = \mu_1 x_1^2 + \mu_2 x_2^2 + \mu_u u^2.$$

We choose $m = 2\text{kg}$, $M = 8\text{kg}$, $l = 0.5\text{m}$, $g = 9.8\text{m/s}^2$, $\mu_1 = 0.1$, $\mu_2 = 0.05$, and $\mu_u = 0.01$. The contours of the optimal control and the associated optimal cost are in Fig. 3.

4. HJB ARISING IN HYBRID SYSTEMS

Viscosity solutions have been considered for optimal switching problem in [5]. Capuzzo-Dolcetta and Evans have studied controlling an ordinary differential equation that switches among m vector fields so that associated cost, the running and switching costs, is kept at a minimum. For optimal switching problem, the value functions satisfy the quasi-variational inequalities. If the switching cost is zero, then the value function satisfies the HJB equation in the viscosity sense. Yong [17] extended the results of Capuzzo-Dolcetta and Evans by adding continuous switching

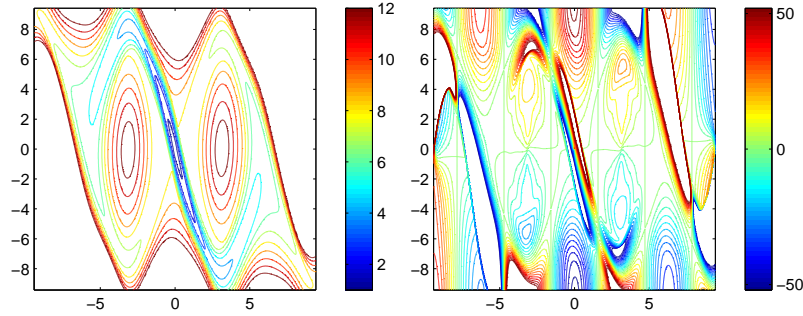


FIGURE 3. Inverted pendulum problem: cost (left) and control (right), 543 iterations on 400 by 400 grid

and impulse controls in the hybrid dynamics. In the paper of Tan and Baras [15], optimal control of hysteresis in smart actuators is specifically discussed. The dynamics of the low dimensional ferromagnetic bulk exhibit hysteresis along with continuous switching controls. Tan and Baras have shown that the corresponding HJB equation admits a unique value function in the viscosity sense.

In this section, we discuss our implementation of the LFS scheme as a numerical approach for approximating the optimal cost. But first, we give some preliminaries for the optimal control with hybrid dynamics.

4.1. Optimal Control for Hybrid Systems. Consider the problem of optimal control of an ordinary differential equation, whose dynamics with a control parameter can be changed into any of the m different settings at the price of switching and running costs.

Let us define the admissible control α as a pair of switching continuous control u_i and decisions d_i ;

$$\alpha = \{u_i, d_i\}_{i=0}^{\infty}$$

where $d_i = \{1, \dots, m\}$. For each $d = \{1, \dots, m\}$, we define an associated set \mathcal{A}^d where

$$\mathcal{A}^d = \{\alpha | \alpha \text{ is an admissible control, where } d_0 = d \text{ and } u_i(\cdot) \in \mathcal{U}\}$$

and

$$\mathcal{U} = \{u(\cdot) : [0, \infty) \rightarrow \mathbb{R}^m | u(\cdot) \text{ is measurable}\}.$$

\mathcal{A}^d is called the set of all admissible controls starting with decision d .

For any $d_0 \in \{1, 2, \dots, m\}$, $u_0(\cdot) \in \mathcal{U}$, the path $y_{x_0}(\alpha)$ satisfies the initial value problem

$$(4.13) \quad \dot{x} = f(x, u_i, d_i), \text{ for } i = \{1, 2, \dots, m\}$$

$$(4.14) \quad x(0) = x_0.$$

Define an associated cost

$$(4.15) \quad \sum_{i=1}^{\infty} \left[\int_{\theta_{i-1}}^{\theta_i} l(x, u_i, d_i) e^{-\lambda t} dt + k(d_{i-1}, d_i) e^{-\lambda \theta_i} \right].$$

where $l(x, u, d)$ is the running cost, $k(\tilde{d}, d)$ is the switching cost from \tilde{d} to d and $u_i(t), d_i(t)$ are functions defined for $t \in [\theta_{i-1}, \theta_i]$. The question is: what is the best

admissible control α^* or a sequence of $\{u_i^*, d_i^*\}_{i=0}^\infty$ so that the dynamics is continually adjusted that the cost incurred while traversing along the optimal path $y_{x_0}(\alpha^*)$ is always at the minimum.

Let an open set $\Omega \in \mathbb{R}^n$ and a curve $\xi(t) \in \Omega$. We shall describe the switching phenomena that are related to the ferromagnetic bulk model in [15]. One is a consequence of a hysteresis. This is when the dynamics switches among a finite number of vector fields in response to when the state variable $x = \xi(t)$ for some t . Another swapping event is when a decision d_i dictates a dynamics with a specific continuous control u_i .

Assume $k(\tilde{d}, d) = 0$ for $\tilde{d}, d \in \{1, 2, \dots, m\}$ as this is consistent with the modelling of the control of hysteresis. It follows that the minimum cost has no dependence on d_0 since the best setting can be chosen at no cost. In addition, we assume the running cost is independent of d_i as well. Then the associated cost in (4.15) reduces to

$$(4.16) \quad \sum_{i=1}^{\infty} \int_{\theta_{i-1}}^{\theta_i} l(x, u_i) e^{-\lambda t} dt.$$

and the admissible control $\alpha = \{u\}_{i=0}^\infty$. Thus we have an infinite horizon problem of minimizing a cost functional (4.16) subject to the dynamics (4.13). Moreover, formal calculations of the dynamic programming lead to the HJB equation

$$(4.17) \quad \lambda V(x) + \max_{i=1, \dots, m} \left\{ -\frac{\partial V}{\partial x} f(x, u_i, d_i) - l(x, u_i) \right\} = 0$$

as in Section 3.

4.2. Example 4. The bulk ferromagnetic hysteresis model in [15] is the following:

$$\dot{x} = \begin{cases} f_+(x)u, & u \in U_+, \text{ if } d = 1 \\ f_-(x)u, & u \in U_-, \text{ if } d = 2 \end{cases}$$

where

$$f_+(x) = \begin{cases} \begin{pmatrix} 1 \\ f_1(x) \end{pmatrix}, & \text{if } x \in \Omega_2 \\ \begin{pmatrix} 1 \\ f_3(x) \end{pmatrix}, & \text{if } x \in \Omega_1 \end{cases} \quad \text{and} \quad f_-(x) = \begin{cases} \begin{pmatrix} 1 \\ f_1(x) \end{pmatrix}, & \text{if } x \in \Omega_1 \\ \begin{pmatrix} 1 \\ f_2(x) \end{pmatrix}, & \text{if } x \in \Omega_2 \end{cases}$$

The domains are $\Omega_1 = \{x = (H, M) \in \mathbb{R}^2 \mid M < M_{an}\}$ and $\Omega_2 = \{x = (H, M) \in \mathbb{R}^2 \mid M \geq M_{an}\}$ and the control sets are $U_+ = \{u \mid u_c \geq u \geq 0\}$ and $U_- = \{u \mid -u_c \leq u \leq 0\}$. See Table 1. for f_1, f_2, f_3 and other parameters. The minimization of the cost functional

$$\int_0^\infty (100(H - H_0)^2 + .0M^2 + .01u^2) e^{-\lambda t} dt$$

is subject to the hysteresis model above. Then equation (4.17) becomes

$$(4.18) \quad \lambda V(x) + \max \left\{ \max_{u \in U_+} \left\{ -\frac{\partial V}{\partial x} f_+(x, u) - l(x, u) \right\}, \max_{u \in U_-} \left\{ -\frac{\partial V}{\partial x} f_-(x, u) - l(x, u) \right\} \right\} = 0.$$

Parameters	$\lambda = 1.58 \times 10^3$, $c = .3$, $M_s = 9.89 \times 10^3$, $\alpha = 190$, $k = 48$, $\mu_0 = 1$, $a = 1.9 \times 10^{-4}$, $T_1 = 189.8$, $T_2 = 8.40 \times 10^3$, $u_c = 1.22 \times 10^3$
Equations	$H_e = H + \alpha M$, $M_{an} = M_s L(z)$, $L(z) = \coth(z) - \frac{1}{z}$, $z = \frac{H_e}{a}$
Functions	$f_1(H, M) = \frac{cM_s L'}{a - \alpha c M_s L'}$, $f_2(H, M) = \frac{ckM_s L' - \mu_0 a(M_{an} - M)}{k(a - \alpha c M_s L') + \mu_0 \alpha a(M_{an} - M)}$, $f_3(H, M) = \frac{ckM_s L' + \mu_0 a(M_{an} - M)}{k(a - \alpha c M_s L') + \mu_0 \alpha a(M_{an} - M)}$
Cost	$l(H, M, u) = 100(H - H_0)^2 + .1M^2 + .01u^2$, $H_0 = 213.3$

TABLE 1. Parameters and functions for hysteresis model.

grid size	64	128	256	512	1024	2048
$\ r\ _1$	5.1786	2.6206	1.3187	0.6614	0.3313	0.1658
$\frac{\ V_{2h} - V_h\ _1}{\ V_h - V_{\frac{h}{2}}\ _1}$		2.0397	2.0202	2.0100	2.0048	

TABLE 2. Smart actuators

This is then solved by LFS for the value function $V(H, M)$. The control law is then found by solving

$$u^*(x) = \arg \max \left\{ \max_{u \in U_+} \left\{ -\frac{\partial V}{\partial x} f_+(x, u) - l(x, u) \right\}, \max_{u \in U_-} \left\{ -\frac{\partial V}{\partial x} f_-(x, u) - l(x, u) \right\} \right\}.$$

Given the approximation $V_{ij}^{(k)}$, the control $u_{ij}^{*(k)}$ is constructed such that

$$A(V_{ij}^{(k)}, u_{ij}^{*(k)}) = 0$$

where the operator A is the discrete version of the HJB operator in (4.18). Thus, the approximation of the optimal control is reduced to finding $u^{*(k)}$ such that $A(u^{*(k)}) = 0$. In Fig. 4, we see a smooth value function and an optimal control that has discontinuous derivatives. As in the Table. 2, the numerical approximations of $V(x)$ shows that the LHS method is order 1. Let the residual vector r be defined as

$$r^{(k)} = \lambda V^{(k)} + \left\{ -\frac{\partial V^{(k)}}{\partial x} f(x, u^{*(k)}) - l(x, u^{*(k)}) \right\}.$$

The L_1 norm of r associated with the pair of approximate solutions to (V, u^*) decreases to 0 as the grid is refined, as is shown in Table 2.

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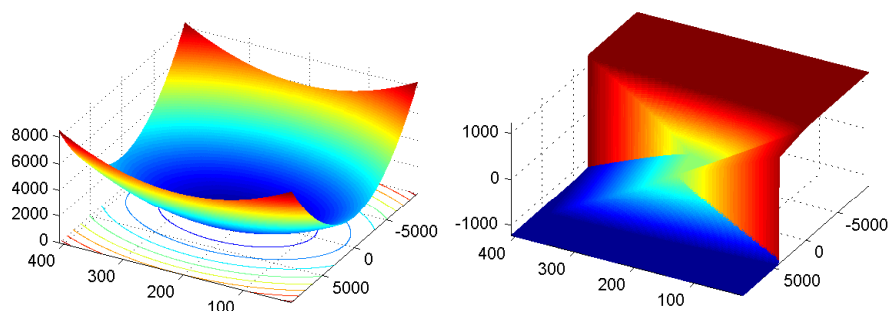


FIGURE 4. Smart actuators: optimal cost (left) and optimal control (right), 23 iterations on 512 by 512 grid

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