

Approximation of low rank solutions for linear quadratic control of partial differential equations

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Abstract

Algebraic Riccati equations (ARE) of large dimension arise when using approximations to design controllers for systems modelled by partial differential equations. We use a modified Newton method to solve the ARE that takes advantage of several special features of these problems. The modified Newton method leads to a right-hand side of rank equal to the number of inputs regardless of the weights. Thus, the resulting Lyapunov equation can be more efficiently solved. The Cholesky-ADI algorithm is used to solve the Lyapunov equation resulting at each step. The algorithm is straightforward to code. Performance is illustrated with a number of standard examples. An example on controlling the deflection of the Euler-Bernoulli beam indicates that for weakly damped problems a low rank solution to the ARE may not exist. Further analysis supports this point.

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Introduction

In linear quadratic controller design the objective is to find a control $u(t)$ so that the objective function

$$\int_0^{\infty} \langle Cz(t), Cz(t) \rangle + \langle u(t), Ru(t) \rangle dt \quad (1)$$

is minimized where R is a symmetric positive definite matrix weighting the control, C is a matrix weighting the state, and $z(t)$ is determined by the system of ordinary differential equations

$$\dot{z}(t) = Az(t) + Bu(t).$$

It is well-known that the solution to this problem is found by finding the positive semi-definite solution P to the algebraic Riccati equation (ARE) [28, e.g.]

$$A^*P + PA - PBR^{-1}B^*P = -C^*C. \quad (2)$$

The optimal control is then given by $u(t) = -Kx(t)$ where

$$K = -R^{-1}B^*P. \quad (3)$$

If the system is modelled by partial differential or delay differential equations, then the state $x(t)$ lies in an infinite-dimensional space. There is a fairly complete theory for linear quadratic control for infinite-dimensional systems; see [11, 15, 23, 24]. The optimal control is found through solution of an operator equation on the infinite-dimensional space that is similar in structure to the ARE (2). In practice, the control is calculated through approximation. The matrices A and B arise in a finite dimensional approximation of the infinite dimensional system. Letting n indicate the order of the approximation and m the number of control inputs; A is an $n \times n$ matrix and B is a $n \times m$ matrix. There have been many papers written describing conditions under which approximations lead to approximating controls that converge to the control for the original infinite-dimensional system, see for instance, [5, 16, 19, 23, 24]. In this paper we will assume that an approximation has been chosen so that a solution to the Riccati equation (2) exists for sufficiently large n and also that the approximating feedback operators converge.

In many cases, particularly in partial differential equation models with more than one space dimension, approximations lead to Riccati equations of very large order that must be solved. Standard direct methods are based on eigenvector calculation. The main methods are Potter's method [35] and the Schur vector-based technique [22]. These algorithms require computational effort proportional to n^3 where n is the order of the system; thus they are impractical for large order systems. A survey of currently used methods to solve large AREs can be found in [8]. Two common iterative methods are Chandrasekhar [4, 9] and Newton-Kleinman iterations [18, 21, 36]. In this paper, we use a modification of the Newton-Kleinman method first proposed by Banks and Ito [4] as a refinement for a partial solution to the Chandrasekhar equation. In [4] the Chandrasekhar equations are partially solved to obtain a stabilizing initial feedback K_o as a stabilizing initial guess to the modified Newton-Kleinman method. The modified Newton-Kleinman is subsequently used and compared to a Schur method.

Solution of the Lyapunov equation is a key step in implementing either modified or standard Newton-Kleinman. For most problems, the matrices involved are sparse. Also, in general, the number of controls, m , is considerably less than the number of states, n . We implement a method, Cholesky-ADI (CF-ADI) [25, 33], that takes advantage of these features. The use of this Lyapunov solver with a standard Newton-Kleinman method is described in [7, 8, 31, 34].

In the next section both standard and modified Newton-Kleinman are described. Then in Section 2 we compare both methods. We show that the modified Newton-Kleinman method is unconditionally convergent. We show that a modified Newton-Kleinman method is considerably faster than standard Newton-Kleinman when both are coupled with this Lyapunov solver. We then solve a number of standard control examples, including one with several space variables. Our results indicate that the modified Newton-Kleinman method achieves considerable savings in computation time over standard Newton-Kleinman.

Numerical results on a beam problem indicate that convergence is decreased with decreasing damping. In Section 3, we analyze the behavior of low rank approximations for damped systems. Numerical results and theoretical analysis indicate that in some cases a low rank approximation to the Riccati equation solution may not exist for a lightly damped system. Finally, in the last section, we make some concluding remarks.

1 Description of Algorithm

We say a matrix A_o is *Hurwitz* if $\sigma(A_o) \subset \mathbb{C}_-$. We will assume throughout this paper that

- (A, B) is stabilizable; that is there is K such that $A - BK$ is Hurwitz, and
- (A, C) is detectable; that is there is F such that $A - FC$ is Hurwitz.

The assumptions that (A, B) is stabilizable and (A, C) is detectable imply that there exists a unique $P \geq 0$ satisfying the Riccati equation equation (2) such that $A - B(R^{-1}B^*P)$ is Hurwitz [28, e.g.].

One approach to solving large Riccati equations is the Newton-Kleinman method [21]. The Riccati equation (2) can be rewritten as

$$(A - BK)^*P + P(A - BK) = -Q - K^*RK \quad (4)$$

where $Q = C^*C$ and $K = R^{-1}B^*P$.

Theorem 1.1 (Newton-Kleinman Method) [21] *Consider a stabilizable pair (A, B) with a feedback K_0 so that $A - BK_0$ is Hurwitz. Then the sequence of solutions P_i to the Lyapunov equation*

$$S_i^*P_i + P_iS_i = -Q - K_i^*RK_i \quad (5)$$

where $S_i = A - BK_i$ and $K_{i+1} = -R^{-1}B^*P_i$ has

$$\lim_{i \rightarrow \infty} P_i = P$$

with a quadratic convergence rate.

The key assumption in the above theorem is that there exists an initial estimate K_0 such that $A - BK_0$ is Hurwitz. This condition is not restrictive for Riccati equations arising in control of infinite-dimensional systems. Many of these systems are stable even when uncontrolled and so the initial iterate K_0 may be chosen as zero. If the original system is not stable, the initial stabilizing iterate can be chosen using a number of approaches. For example, if the approximation procedure is valid then convergence of the feedback gains is obtained with increasing model order. Thus, a gain obtained from a lower order approximation may be used as an ansatz for a higher order approximation. This technique was used successfully in [18, 31, 36].

A modification to the Newton-Kleinman method was proposed by Banks and Ito [4].

Theorem 1.2 (Modified Newton-Kleinman Method) *Consider equation (4) and choose K_0 and K_1 so that $A - BK_0$ and $A - BK_1$ are Hurwitz. Then the following Lyapunov equation and feedback update,*

$$\begin{aligned} (A - BK_i)^* X_i + X_i (A - BK_i) &= -[K_i - K_{i-1}]^* R [K_i - K_{i-1}], \\ K_{i+1} &= K_i - B^* X_i. \end{aligned} \quad (6)$$

has feedback gain K_i where

$$\lim_{i \rightarrow \infty} K_i = K$$

with quadratic convergence.

Proof: With $K_i = R^{-1} B^* P_{i-1}$, consider two iterations of the Newton-Kleinman method:

$$(A - BK_{i-1})^* P_{i-1} + P_{i-1} (A - BK_{i-1}) = -Q - K_{i-1}^* R K_{i-1}, \quad (7)$$

$$(A - BK_i)^* P_i + P_i (A - BK_i) = -Q - K_i^* R K_i. \quad (8)$$

Subtract (8) from (7), and rearrange to obtain

$$\begin{aligned} (-BK_{i-1} + BK_i)^* P_{i-1} + P_{i-1} (-BK_{i-1} + BK_i) + (A - BK_i)^* X_i \dots \\ + X_i (A - BK_i) = -K_{i-1}^* R K_{i-1} + K_i^* R K_i \end{aligned}$$

where $X_i = P_{i-1} - P_i$. The first term on the left hand side can be written $-(K_{i-1} - K_i)^* R K_i$, and similarly, the second term is $-K_i^* R (K_{i-1} - K_i)$. Move these two terms to the right-hand-side of equation, and simplify, to obtain (6). Since this is simply a reformulation of the Newton-Kleinman iterations, quadratic convergence of $\{K_i\}$ follows from the quadratic convergence of the Newton-Kleinman method. \square

The modified Newton-Kleinman method is mathematically equivalent to the original method, involving just a reformulation of the iterations. The advantage of the modified Newton-Kleinman method lies in the fact that the Lyapunov equation (6) to be solved at each step has a right-hand-side with rank limited by the number of controls, while the rank of the right-hand-side in the original formulation (5) has rank equal to that of the state weight $Q = C^* C$ which could be quite large.

The solution of a Lyapunov equation is a key step in both standard and modified Newton-Kleinman iterations. For $A_o \in \mathbb{R}^{n \times n}$ is Hurwitz and $D \in \mathbb{R}^{n \times r}$ consider the Lyapunov equation

$$XA_o + A_o^*X = -DD^* \quad (9)$$

If A_o is Hurwitz, then the Lyapunov equation has a symmetric positive semidefinite solution X .

As for the Riccati equation, direct methods such as Bartels-Stewart [6] are only appropriate for low model order and do not take advantage of sparsity in the matrices. The most basic iterative method to solve Lyapunov equation is Smith's method [38]. For $p < 0$, define $U = (A_o - pI)(A_o + pI)^{-1}$ and $V = -2p(A_o^* + pI)^{-1}DD^*(A_o + pI)^{-1}$. The Lyapunov equation (9) can be rewritten as

$$X = U^*XU + V.$$

In Smith's method, the solution X is found by using successive substitutions: $X = \lim_{i \rightarrow \infty} X_i$ where

$$X_i = U^*X_{i-1}U + V \quad (10)$$

with $X_0 = 0$. The convergence of the iterations can be improved by careful choice of the parameter p ; see [37, pg. 297]. This method of successive substitution is unconditionally convergent, but the convergence rate is linear convergence.

The ADI method [26, 40] improves Smith's method by using a different parameter p_i at each step. Two alternating linear systems,

$$(A_o^* + p_i I)X_{i-\frac{1}{2}} = -DD^* - X_{i-1}(A_o - p_i I) \quad (11)$$

$$(A_o^* + p_i I)X_i^* = -DD^* - X_{i-\frac{1}{2}}^*(A_o - p_i I) \quad (12)$$

are solved recursively starting with $X_0 = 0 \in \mathbb{R}^{n \times n}$ and parameters $p_i < 0$. If all parameters $p_i = p$ then equations (11,12) reduce to Smith's method. If the ADI parameters p_i are chosen appropriately, then convergence is obtained in J iterations where $J \ll n$. The parameter selection problem has been studied extensively [13, 26, 39]. For symmetric A the optimal parameters can be easily calculated. We estimate the complex ADI parameters as in [13] if the spectrum of A is complex. As the spectrum of A flattens to the real axis the ADI parameters are closer to optimal.

If A_o is sparse, then sparse arithmetic can be used in calculation of X_i . However, full calculation of the dense iterates X_i is required at each step. Setting $X_0 = 0$, it can be easily shown that X_i is symmetric and positive semidefinite for all i , and so we can write $X = ZZ^*$ where Z is a Cholesky factor of X [25, 33]. Substituting $Z_i Z_i^*$ for X_i and setting $X_0 = 0$, we obtain the following iterates for Z_i :

$$\begin{aligned} Z_1 &= \sqrt{-2p_1}(A_o^* + p_1 I)^{-1}D \\ Z_i &= [\sqrt{-2p_i}(A_o^* + p_i I)^{-1}D, (A_o^* + p_i I)^{-1}(A_o^* - p_i I)Z_{i-1}]. \end{aligned} \quad (13)$$

Note that $Z_1 \in \mathbb{R}^{n \times r}$, $Z_2 \in \mathbb{R}^{n \times 2r}$, and $Z_i \in \mathbb{R}^{n \times ir}$. At each iteration, an approximation to the solution of increasing rank is obtained. The algorithm is stopped when the Cholesky

Table 1: Complexity of CF-ADI and ADI [25]

	CF-ADI	ADI
Sparse A	$\mathcal{O}(Jrn)$	$\mathcal{O}(Jrn^2)$
Full A	$\mathcal{O}(Jrn^2)$	$\mathcal{O}(Jrn^3)$

Table 2: Cholesky-ADI Method

<p>Given A_o and D Choose ADI parameters $\{p_1, \dots, p_J\}$ with $\Re(p_i) < 0$ Define $z_1 = \sqrt{-2p_1}(A_o^* + p_1I)^{-1}D$ and $Z_1 = [z_1]$ For $i = 2, \dots, J$ Define $W_i = \left(\frac{\sqrt{-2p_{i+1}}}{\sqrt{-2p_i}}\right)[I - (p_{i+1} - p_i)(A_o^* + p_{i+1}I)^{-1}]$ (1) $z_i = W_i z_{i-1}$ (2) If $\ z\ > tol$ $Z_i = [Z_{i-1} \ z_i]$ Else, stop.</p>

iterations converge within some tolerance. In [25] these iterates are reformulated in a more efficient form using the observation that the order in which the ADI parameters are used is irrelevant. This leads to the algorithm shown in Table 2.

This solution method results in considerable savings in computation time and memory. The Cholesky factor Z is used to calculate the feedback K - the full solution X is never constructed. Also, the complexity of CF-ADI is also considerably less than that of standard ADI as shown in Table 1. For both standard and modified Newton-Kleinman iterations, $A_o = A - BK_i$. The factor D differs significantly. Let n_Q indicate the rank of $Q = C^*C$. In the case of standard Newton-Kleinman,

$$D = [C^* \ K_i^* R^{1/2}],$$

while for modified Newton-Kleinman,

$$D = (K_i - K_{i-1})^* R^{1/2}.$$

In the case of standard Newton-Kleinman, the rank r of the right-hand-side $r = m + n_Q$ while for modified Newton-Kleinman, $r = m$. Since n_Q is close to n in many cases, while m is generally much smaller than n , this means that the rank of the right-hand-side is typically much smaller for modified Newton-Kleinman than for the standard algorithm. The reduction in complexity of CF-ADI method over ADI is marked for the modified Newton-Kleinman method.

2 Comparison of Standard and Modified Newton-Kleinman

In this section we compare the standard and modified Newton-Kleinman methods on a number of standard examples. The computations were done in MatLab on a personal computer with two 2.5 GHz AMD Opteron processors with 1024KB cache memory and 2G RAM. The relative error of the feedback K is measured and used as a stopping criterion, with a tolerance of 10^{-9} and the stopping criterion for the Lyapunov solver is set to 10^{-9} for the heat equation and 10^{-12} for the beam problem. This criterion was used because it is cheap to calculate and also for most applications, an accurate calculation of K , not P is required. The residuals in the ARE were also calculated as a check on the algorithm and are shown in the tables. The total Lyapunov iterations includes the iterations from the required initializations.

One-dimensional Heat Equation

Consider the linear quadratic regulator problem of minimizing a cost functional (1) subject to [4, 10]

$$\begin{aligned}\frac{\partial z(t, x)}{\partial t} &= \frac{\partial^2 z(t, x)}{\partial x^2}, \quad x \in (0, 1), \\ z(0, x) &= \psi(x)\end{aligned}\tag{14}$$

with boundary conditions

$$\begin{aligned}\frac{\partial z(t, 0)}{\partial x} &= u(t) \\ \frac{\partial z(t, 1)}{\partial x} &= 0.\end{aligned}\tag{15}$$

We choose $R = 1$ and C is defined by

$$Cz(t) = \int_0^1 z(t, x) dx.\tag{16}$$

The solution to the infinite-dimensional Riccati equation is

$$Kz = \int_0^1 k(x)z(x) dx$$

where $k \equiv 1$ [5]. Thus, for this problem we have an exact solution to which we can compare the approximations. The equations (14-16) are discretized using the standard Galerkin approximation with linear spline finite element basis on an uniform partition of $[0, 1]$. This yields a sequence of solutions to the ARE that converges uniformly to the solution to the operator Riccati equation for the PDE [23].

In Tables 3 - 6 we compare the number of Newton-Kleinman and Lyapunov iterations as well as the CPU time per order n for C as in (16) and also $C = I$. We use the ansatz $k_0(x) = 100$ in all cases. Both algorithms yielded feedback operators that converged to the exact feedback. With the modified algorithm, there are fewer Riccati loops than with the original

Table 3: 1-d Heat Equation with single observation: Standard Newton-Kleinman

n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
25	12	348	.15	7.7×10^{-9}
50	12	406	.23	1.1×10^{-8}
100	12	464	.65	7.0×10^{-9}
200	12	522	4.55	8.9×10^{-9}

Table 4: 1-d Heat Equation with single observation: Modified Newton-Kleinman

n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
25	10	254	.11	7.7×10^{-9}
50	10	300	.17	1.1×10^{-8}
100	10	306	.43	7.0×10^{-9}
200	9	347	2.49	8.9×10^{-9}

Newton-Kleinman iteration. Also, the modified Newton-Kleinman method requires fewer Lyapunov iterations within the last few Newton-Kleinman loops. The computation time with the modified Newton-Kleinman algorithm is significantly less than that of the original algorithm. As predicted by the complexity analysis, the reduction in the number of iterations and computation time of the modified Newton-Kleinman method over that required in the standard method is greater with $C = I$ than for the case of a single observation.

Two-Dimensional Heating Problem

Define the rectangle $\Omega = [0, 1] \times [0, 1]$ with boundary $\partial\Omega$. Consider the two-dimensional partial differential equation [10]

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 20 \frac{\partial z}{\partial y} + 100z = f(x, y)u(t), & (x, y) \in \Omega \\ z(x, y, t) &= 0, & (x, y) \in \partial\Omega \end{aligned} \quad (17)$$

Table 5: 1-d Heat Equation with C=I: Standard Newton-Kleinman

n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
25	12	363	.22	7.7×10^{-9}
50	12	418	.83	1.1×10^{-8}
100	12	480	4.10	7.0×10^{-9}
200	12	535	28.35	8.9×10^{-9}

Table 6: 1-d Heat Equation with C=I: Modified Newton-Kleinman

n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
25	10	261	.12	7.7×10^{-9}
50	10	306	.21	1.1×10^{-8}
100	10	314	.68	7.0×10^{-9}
200	9	354	4.02	8.9×10^{-9}

Table 7: 2-d Heat Equation with single observation: Standard Newton-Kleinman

grid	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
12×12	144	13	431	1.7	7.9×10^{-7}
23×12	276	14	507	5.84	6.2×10^{-7}
23×23	529	14	557	29.18	9.5×10^{-8}

where z is a function of x, y and t and

$$f(x, y) = \begin{cases} 100, & \text{if } .1 < x < .3 \text{ and } .4 < y < .6 \\ 0, & \text{else} \end{cases} .$$

Central difference approximations are used to discretize (17) on a grid of $N \times M$ points. The resulting approximation has dimension $n = N \times M$ and thus, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$. The matrix A is sparse with at most 5 non-zero entries in any row. The matrix B is a sparse column vector. With a uniform grid, this approximation is equivalent to a Galerkin finite-element method and the results in [5, 23] imply uniform convergence of the solutions to the resulting ARE's.

We solved the Riccati equation on a number of grids, using both standard and modified Newton-Kleinman methods with $C = B^*$ and $R = 1$. The data is shown in Tables 7 and 8. We also investigated the performance of the algorithm with the same numerical approximation of the partial differential equation (17) but with control weight $R = 100$ and cost $C = I$. In all case, the ansatz $K_0 = 0$ was used. The results are shown in Tables 9 and 10. Fewer Lyapunov iterations are required for modified Newton-Kleinman than for standard Newton-Kleinman and there is also a significant reduction in computation time. The advantage of modified Newton-Kleinman over standard Newton-Kleinman is more pronounced for the case where the state weight has large rank.

Euler-Bernoulli Beam

Consider an Euler-Bernoulli beam clamped to a central hub at one end ($r = 0$) and free to vibrate at the other end ($r = 1$). A torque $u(t)$ at the hub causes a rotation of the entire beam about the hub. Let $w(r, t)$ denote the deflection of the beam from its rigid body motion at time t and position r . A detailed derivation of the partial differential equation can be

Table 8: 2-d Heat Equation with single observation: Modified Newton-Kleinman

grid	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
12×12	144	12	378	1.3	7.9×10^{-7}
23×12	276	13	447	4.17	6.2×10^{-7}
23×23	529	13	524	15.15	9.5×10^{-8}

Table 9: 2-d Heat Equation with $C = I$: Standard Newton-Kleinman

grid	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
12×12	144	6	96	3.3	8.5×10^{-5}
23×12	276	6	124	25.63	1.0×10^{-4}
23×23	529	7	166	237.21	4.0×10^{-7}

Table 10: 2-d Heat Equation with $C = I$: Modified Newton-Kleinman

grid	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
12×12	144	5	68	.59	8.4×10^{-5}
23×12	276	5	88	3.5	1.1×10^{-4}
23×23	529	6	103	27.4	4.0×10^{-7}

found in [29, e.g] We assume that the hub inertia I_h is much larger than the beam inertia I_b . In this situation, the partial differential equation model with Kelvin-Voigt and viscous damping is

$$\rho w_{tt}(r, t) + C_v w_t(r, t) + \frac{\partial^2}{\partial r^2} [C_d I_b w_{rrt}(x, t) + E I_b w_{rr}(r, t)] = \frac{\rho r}{I_h} u(t), \quad (18)$$

with boundary conditions

$$\begin{aligned} w(0, t) &= 0 \\ w_r(1, t) &= 0 \\ E I w_{rr}(1, t) + C_d I_b w_{rrt}(1, t) &= 0 \\ \frac{\partial}{\partial r} [E I(1) w_{rr}(r, t) + C_d I_b w_{rrt}(r, t)]_{r=1} &= 0. \end{aligned}$$

The values of the physical parameters in Table 2 are as in [2].

Define H be the closed linear subspace of the Sobolev space $H^2(0, 1)$,

$$H = \left\{ w \in H^2(0, 1) : w(0) = \frac{dw}{dr}(0) = 0 \right\}$$

and define the state-space to be $X = H \times L^2(0, 1)$ with state $z(t) = (w(\cdot, t), \frac{\partial}{\partial t} w(\cdot, t))$. A state-space formulation of the above partial differential equation problem is

$$\frac{d}{dt} x(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} 0 & I \\ -\frac{EI}{\rho} \frac{d^4}{dr^4} & -\frac{C_d I}{\rho} \frac{d^4}{dr^4} - \frac{C_v}{\rho} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{r}{I_h} \end{bmatrix},$$

with domain

$$\text{dom}(A) = \{(\phi, \psi) \in X : \psi \in H \text{ and}$$

$$M = EI \frac{d^2}{dr^2} \phi + C_d I \frac{d^2}{dr^2} \psi \in H^2(0, 1) \text{ with } M(L) = \frac{d}{dr} M(L) = 0\}.$$

Set $R = 1$ and define C by the tip position

$$w(1, t) = C[w(x, t) \quad \dot{w}(x, t)].$$

We also solved the control problem with $C = I$, and $R = 1$.

Let $H^{2N} \subset H$ be a sequence of finite-dimensional subspaces spanned by the standard cubic B-splines with a uniform partition of $[0, 1]$ into N subintervals. This yields an approximation in $H^{2N} \times H^{2N}$ [20, e.g.] of dimension $n = 4N$. This approximation method yields a sequence of solutions to the algebraic Riccati equation that converges strongly to the solution to the infinite-dimensional Riccati equation corresponding to the original partial differential equation description [5, 27].

The control problem was solved with several different choices of ansatz K_0 : 0 and B^* . The results are shown in Tables 12-15. In all cases, modified Newton-Kleinman converged with a significant reduction in iterations and computation time over standard Newton-Kleinman. As for the other examples, the reduction in the computation time is greater for full state observation than for a single observation.

E	$2.68 \times 10^{10} \text{ N/m}^2$
I_b	$1.64 \times 10^{-9} \text{ m}^4$
ρ	1.02087 kg/m
C_v	2 Ns/m^2
C_d	$5 \times 10^8 \text{ Ns/m}^2$
L	1 m
I_h	121.9748 kg m^2
d	$.041 \text{ kg}^{-1}$

Table 11: Beam physical parameters

Table 12: Beam with C=Ctip: Standard Newton-Kleinman

K_0	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
0	48	1	558	.3	1.3×10^{-11}
0	72	1	620	.7	1.3×10^{-11}
0	96	1	664	1.21	1.5×10^{-11}
0	120	1	698	1.99	3.4×10^{-11}
B'	48	1	558	.37	1.3×10^{-11}
B'	72	1	620	.72	1.3×10^{-11}
B'	96	1	664	1.2	1.4×10^{-11}
B'	120	1	698	2.0	3.5×10^{-11}

Table 13: Beam with C=Ctip: Modified Newton-Kleinman

K_0	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
0	48	1	280	.34	1.3×10^{-11}
0	72	1	311	.47	1.3×10^{-11}
0	96	1	333	.74	1.5×10^{-11}
0	120	1	350	1.14	3.4×10^{-11}
B'	48	2	547	.46	1.3×10^{-11}
B'	72	2	312	.60	1.3×10^{-11}
B'	96	1	333	.75	1.4×10^{-11}
B'	120	1	350	1.17	3.5×10^{-11}

Table 14: Beam with C=I: Standard Newton-Kleinman

K_0	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
0	48	1	566	3.49	3.7×10^{-10}
0	72	1	628	9.15	2.7×10^{-9}
0	96	1	672	18.41	3.0×10^{-9}
0	120	1	706	32.48	6.9×10^{-9}
B'	48	1	566	3.47	3.6×10^{-10}
B'	72	1	628	9.15	2.7×10^{-9}
B'	96	1	672	18.40	2.4×10^{-9}
B'	120	1	706	32.41	7.4×10^{-9}

Table 15: Beam with C=I: Modified Newton-Kleinman

K_0	n	N.K. Itn's	Lyapunov Itn's	time (s)	residual
0	48	1	284	1.87	3.6×10^{-10}
0	72	1	315	4.82	2.7×10^{-9}
0	96	1	337	9.59	2.8×10^{-9}
0	120	1	354	16.88	7.0×10^{-9}
B'	48	2	550	2.03	3.6×10^{-10}
B'	72	1	315	4.82	3.0×10^{-9}
B'	96	1	337	9.6	2.3×10^{-9}
B'	120	1	354	16.85	7.6×10^{-9}

3 Low rank approximations to Lyapunov function

For the parabolic equations in Examples 1 and 2 the number of Lyapunov iterations required in each Newton-Kleinman iteration was quite low- less than 50 in all cases. The convergence was much slower for the beam equation. Although for all n , only 1 or 2 Newton-Kleinman iterations were required, the number of Lyapunov iterates increases as n increases. Tables 16,17 show the effect of varying C_d on the number of iterations required for convergence. Larger values of C_d led to a decreasing number of iterations. Small values of C_d led to a large number of required iterations in each solution of a Lyapunov equation.

Recall that the CF-ADI algorithm used here starts with a rank 1 initial estimate of the Cholesky factor and the rank of the solution is increased at each step. The Cholesky-ADI method relies on its efficiency on the existence of a low-rank approximation to the solution X to the Lyapunov equation being solved. This is true of many other iterative algorithms to solve large Lyapunov equations.

Theorem 3.1 [17, e.g. Thm. 3.5.2]. *For any symmetric, positive semi-definite matrix X of rank n let $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n$ be its eigenvalues. For any rank $k < n$ matrix X_k ,*

$$\frac{\|X - X_k\|_2}{\|X\|_2} \geq \frac{\lambda_{k+1}}{\lambda_1}.$$

Thus, the relative size of the largest and smallest eigenvalues determines the existence of a low rank approximation X_k that is close to X , *regardless of how this approximation is obtained.*

A bound on the rank of the solution to a Lyapunov equation where A is symmetric is given in [32]. A tighter bound on the error in a low-rank approximation has been obtained [1] in terms of the eigenvalues and eigenvectors of A . This bound is applicable to all diagonalizable matrices A . The bound for the case where the right-hand-side D has rank one is as follows.

Theorem 3.2 [1, Thm 3.1] *Let V be the matrix of right eigenvectors of A , and denote its condition number by $\kappa(V)$. Denote the eigenvalues of A by μ_1, \dots, μ_n . There is a rank $k < n$ approximation to X satisfying*

$$\|X - X_k\|_2 \leq (n - k)^2 \delta_{k+1} (\kappa(V) \|D\|_2)^2 \tag{19}$$

where

$$\delta_{k+1} = \frac{-1}{2\text{Real}(\mu_{k+1})} \prod_{j=1}^k \left| \frac{\mu_{k+1} - \mu_j}{\mu_{k+1}^* + \mu_j} \right|^2.$$

The eigenvalues μ_i are ordered so that δ_i are decreasing.

If the condition number $\kappa(V)$ is not too large, for instance if A is normal, then δ_{k+1}/δ_1 gives a relative error estimate of the error λ_{k+1}/λ_1 in the approximation X_k . This estimate is observed in [1] to estimate the actual decay rate of the eigenvalues of X quite well, even in cases where the decay is very slow.

Consider a parabolic partial differential equation, where the A -matrix of the approximation is symmetric and negative definite and so all the eigenvalues μ_i of A are real. A routine calculation yields that

$$\frac{\delta_{k+1}}{\delta_1} \approx \frac{\mu_1}{\mu_k}$$

and so the rate of growth of the eigenvalues determines the accuracy of the low rank approximation. Since the eigenvalues typically increase in magnitude quite quickly, the Lyapunov function for such problems can be approximated by a matrix of low rank.

The accuracy of the low-rank approximant can be quite different if a matrix A is non-symmetric. The solution X to the Lyapunov equation could be, for instance, the identity matrix [1, 32] in which case the eigenvalues of X do not decay.

It is observed through several numerical examples in [1] that it is not just complex eigenvalues that cause the decay rate to slow down, but the dominance of the imaginary parts of the eigenvalues as compared to the real parts, together with absence of clustering in the spectrum of A .

Consider the bound (19) for the beam with $C_d = 0$, and an eigenfunction basis for the approximation. Defining $c_v = C_v/(\rho A)$, $c^2 = EI/(\rho A)$ and indicating the roots of $1 + \cos(a)\cosh(a)$ by a_k , the eigenvalues of A are [30]

$$\mu_k = -c_v/2 \pm i\omega_k$$

where

$$\omega_k = \sqrt{ca_k^2 - c_v^2/4}.$$

A simple calculation yields that for all k ,

$$\frac{\delta_k}{\delta_1} = \prod_{j=1}^{k-1} \frac{1}{1 + \frac{4c_v^2}{(w_k - w_j)^2}} \approx 1.$$

This indicates that the eigenvalues of the Lyapunov solution do not decay. This effect was observed in the beam equation. The ratio λ_{k+1}/λ_1 is plotted for several values of C_d in Figures 1 and 2. For larger values of C_d the solution X is closer to a low rank matrix than it is for smaller values of C_d .

These observations are supported by results in the theory of control of partial differential equations. If B and C are bounded finite-rank operators and A generates an exponentially stable semigroup then the solution to the ARE is nuclear:

$$\sum_{i=1}^{\infty} \lambda_i(P) < \infty.$$

[12, 14]. Uniform approximation by low order matrices is possible. This does not provide an estimate about the rate of decay of the eigenvalues. Figure 1 shows eigenvalue ratios for a beam example in which the underlying semigroup is stable, and C and B have rank 1. In all cases the eigenvalues decay, although more slowly for a lightly damped system.

The results quoted above only apply if B and C are compact operators. If the original semigroup is analytic, for instance in the case of Examples 1, 2 and the beam equation with

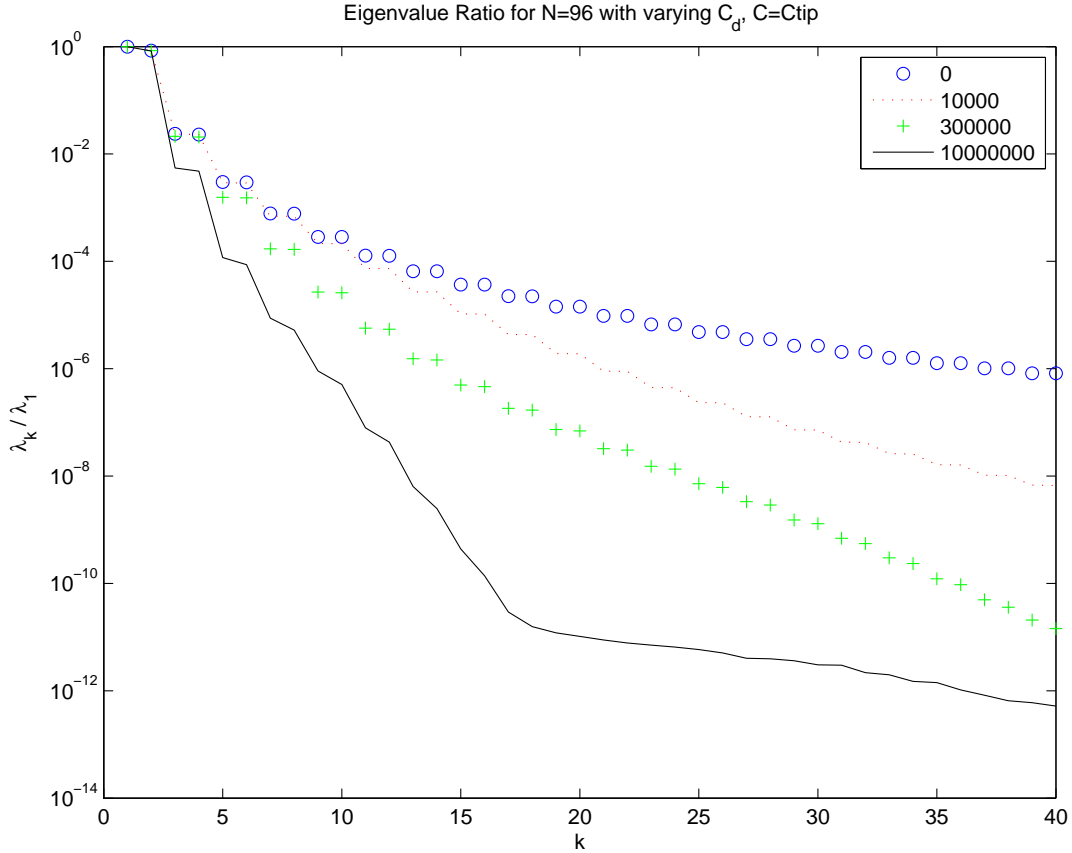


Figure 1: Eigenvalue ratio for solution to beam ARE for varying C_d ($N = 96, C = C_{tip}$)

$C_d > 0$, then the approximations converge uniformly in operator norm under fairly standard assumptions; see [23]. However, if the semigroup is not analytic, the solution P to the ARE may not be compact. For example, consider the class of systems where $A^* = -A$ (as in the undamped wave and beam equations). If $C = B^*$ then $P = I$ is a solution to the ARE [11]. Clearly, if P is not compact, it cannot be uniformly approximated by a finite rank operator. Convergence of the approximating solutions to the ARE for an non-analytic semigroup, in general requires that C must be compact; see [24]. In Figure 2 the observation C is not compact and with $C_d = 0$ the underlying semigroup is not analytic; note that no decay in the eigenvalues occurs in this case. For problems where the underlying semigroup is not analytic the state weighting matrix C must be compact. For lightly damped systems the decay of the eigenvalues in the Riccati solution may be very slow, even if the underlying semigroup is analytic, as in the case of the beam examined here. In such cases, the state weighting matrix C should be chosen compact. Even so, the decay of the solution eigenvalues may be quite slow.

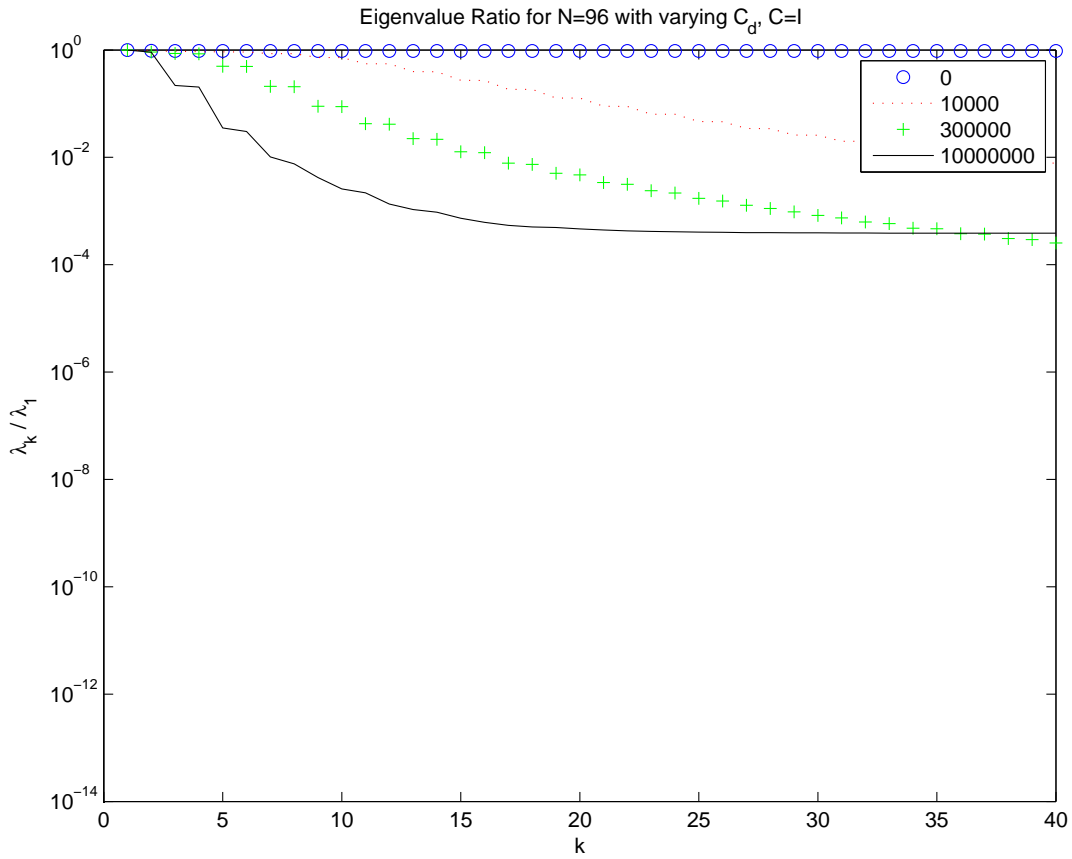


Figure 2: Eigenvalue ratio for solution to beam ARE for varying C_d ($N = 96, C = I$)

Table 16: Beam: Effect of Changing C_d (Standard Newton-Kleinman)

C_v	C_d	α	Newton-Kleinman Itn's	Lyapunov Itn's	CPU time
2	1×10^4	1.5699	—	—	—
	3×10^5	1.5661	—	—	—
	4×10^5	1.5654	3	1620;1620;1620	63.14
	1×10^7	1.5370	3	1316;1316;1316	42.91
	1×10^8	1.4852	3	744;744;744	18.01
	5×10^8	1.3102	3	301;301;301	5.32

Table 17: Beam: Effect of Changing C_d (Modified Newton-Kleinman)

C_v	C_d	α	Newton-Kleinman It's	Lyapunov It's	CPU time
2	1×10^4	1.5699	2	–	–
	3×10^5	1.5661	2	–	–
	4×10^5	1.5654	2	1620;1	24.83
	1×10^7	1.5370	2	1316;1	16.79
	1×10^8	1.4852	2	744;1	7.49
	5×10^8	1.3102	2	301;1	2.32

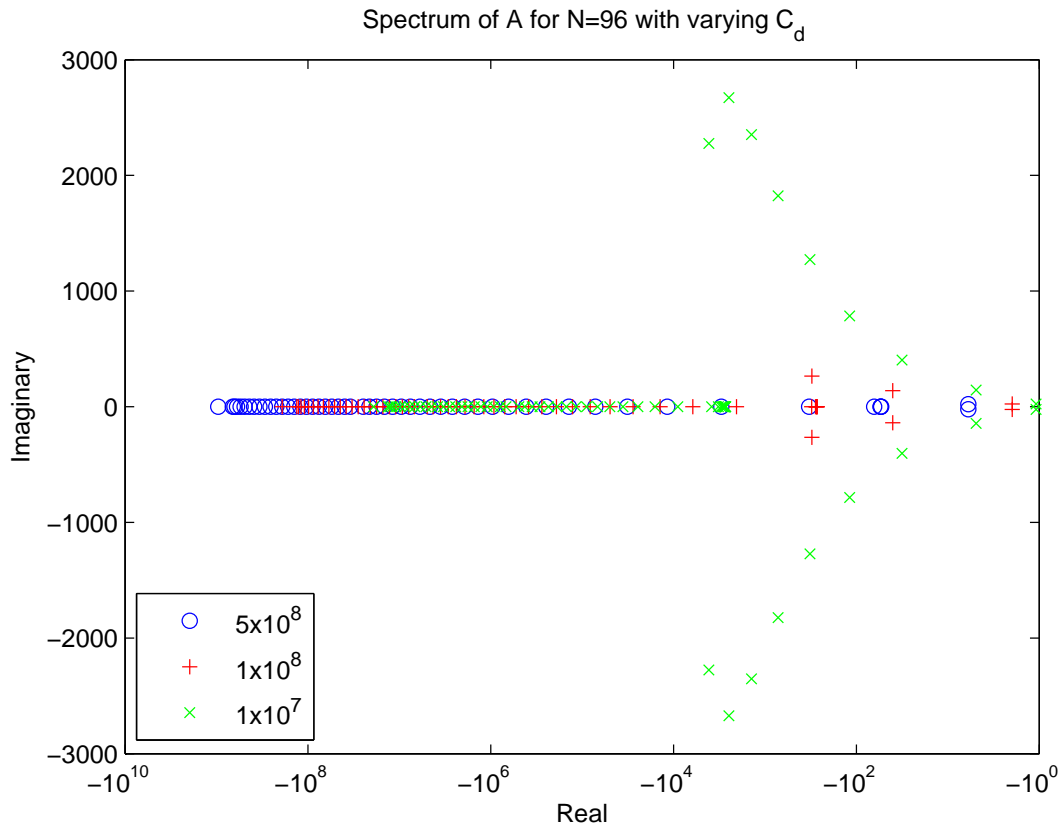


Figure 3: Beam: Spectrum of A for $C_d = 1 \times 10^4, 4 \times 10^5, 1 \times 10^8$.
 $\alpha = 1.5699, 1.5654, 1.4852$)

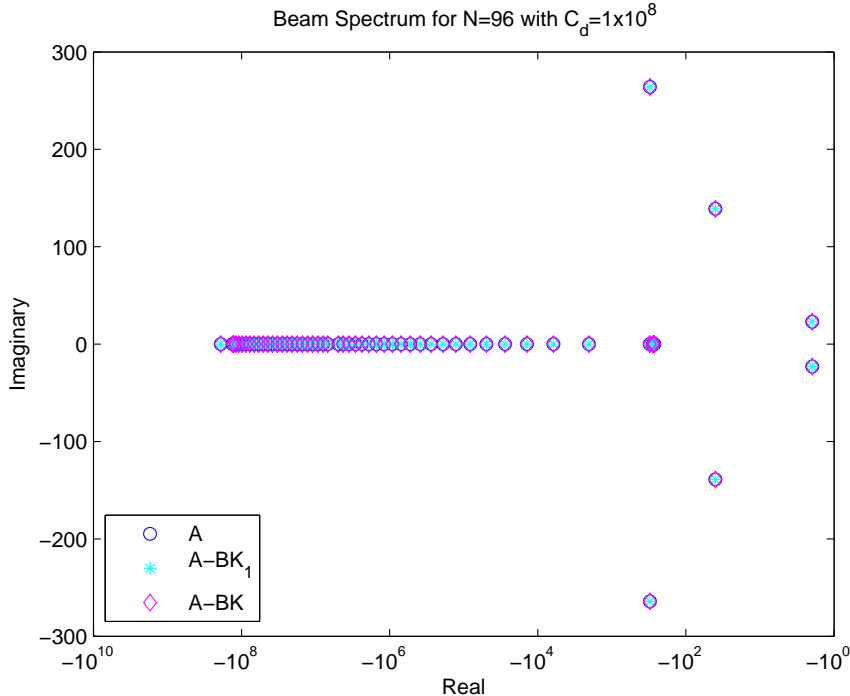


Figure 4: Spectra of operators A , $A - BK_1$, $A - BK$.

4 Conclusions

The analysis and results shown here indicate that a modified Newton-Kleinman method is considerably faster than standard Newton-Kleinman. The combination of the modified Newton-Kleinman method with the Cholesky-ADI method takes advantage of the fact that the number of controls is generally considerably less than the number of states. The reduction in number of iterations and computation time over standard Newton-Kleinman increases with the rank of the state-weighting matrix.

For the beam problem, increasing numbers of iterations were required as the damping was decreased. The convergence could probably be improved with better ADI parameter selection. However, as the damping is decreased, the ability to approximate the Riccati solution (or the solution to the Lyapunov equation) by a solution of low rank, is lost. Thus, one expects a greater number of iterations in the Lyapunov loop to be required as C_d is decreased.

Analysis indicates that the fundamental reason for the slow convergence with small C_d is the “hyperbolic-like” behaviour of the problem. This may have consequences for control of systems such as plates and coupled acoustic-structures where the spectra have large imaginary parts relative to the real parts. Special methods may be needed to calculate controllers for these systems.

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