1. (15 pts) Consider the system:

$$m\frac{d^2x}{dt^2} = \alpha(e^{\beta x} - 1),\ \alpha > 0, \beta > 0.$$

Linearize the system above around $x = 0$.

Method 1:

Let $f(x) = \alpha(e^{\beta x} - 1)$

Taylor expand around $x_E = 0$:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$

$$\approx 0 + \alpha\beta x + \frac{(\beta x)^2}{2!} + \cdots$$

$$\approx \alpha\beta x$$

Thus, $\underline{\frac{md^2x}{dt^2} = \alpha\beta x}$.

Method 2: Let $x(t) = \varepsilon x_1(t)$

Then, $\varepsilon m\frac{d^2x_1}{dt^2} = \alpha(e^{\beta\varepsilon x_1} - 1)$

$$= \alpha(1 + \varepsilon\beta x_1 + \cdots - 1)$$

$$= \varepsilon\alpha\beta x_1$$

Thus, $\underline{\frac{md^2x_1}{dt^2} = \alpha\beta x}$.

2. Given the following model:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

a. (12 pts) Solve the initial value problem with $x(0) = 1$ and $\frac{dx}{dt}(0) = 1$ when $c^2 > 4mk$.

roots: $r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

gen sol'n: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

init. cond.: $\left.\begin{aligned} x(0) &= c_1 + c_2 = 1 \\ \frac{dx}{dt}(0) &= r_1 c_1 + r_2 c_2 = 1 \end{aligned}\right\}$ solve for c_1 & c_2 in terms of r_1, r_2

$$\Rightarrow x(t) = \underbrace{\left(\frac{r_2 - 1}{r_2 - r_1}\right)}_{c_1} e^{r_1 t} + \underbrace{\left(\frac{1 - r_1}{r_2 - r_1}\right)}_{c_2} e^{r_2 t}$$

b. *(13 pts)* Take the limit of the solution in part *a* as $r_1 \to r_2$. Hint: the limiting solution is the same as for the case $c^2 = 4mk$.

Sol'n from part a:

$$x(t) = \frac{r_2-1}{r_2-r_1} e^{r_1 t} + \frac{1-r_1}{r_2-r_1} e^{r_2 t}$$

$$\lim_{r_1 \to r_2} x(t) = \lim_{r_1 \to r_2} \frac{(r_2-1)e^{r_1 t} + (1-r_1)e^{r_2 t}}{r_2-r_1} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{r_1 \to r_2} \frac{t(r_2-1)e^{r_1 t} - e^{r_2 t}}{-1}$$

$$= e^{r_2 t}\left((1-r_2)t + 1 \right)$$

Check: $r_1 \to r_2 \implies r = -\frac{c}{2m}$

Then, $x(t) = (At+B)e^{rt}$

$A = (1-r_2)$; $B = 1$ (in this case)

3. Consider the pendulum models:

(1) $$L\frac{d^2\theta}{dt^2} = -g\theta - k\frac{d\theta}{dt}$$

(2) $$L\frac{d^2\theta}{dt^2} = -g\sin\theta - k\frac{d\theta}{dt}.$$

a.(10 pts) What is the general solution of (1) if $k^2 < 4Lg$? *Make sure you define* ω.

roots : $r = \dfrac{-k \pm \sqrt{k^2-4Lg}}{2L}$

since $k^2 < 4Lg \iff$ 2 complex roots

then $r(t) = c_1 \cos\omega t + c_2 \sin\omega t$

where $\omega = \sqrt{\dfrac{g}{L} - \dfrac{k^2}{4L^2}}$

b. (10 pts) Suppose $k = 0$, **derive** the energy equation of (2)?

$$\frac{d\theta}{dt}\left(L\frac{d^2\theta}{dt^2}\right) = -g\sin\theta\frac{d\theta}{dt}$$

$$\int \frac{d}{dt}\left[\frac{L}{2}\left(\frac{d\theta}{dt}\right)^2\right] = \int -g\sin\theta\frac{d\theta}{dt}\,dt$$

$$\frac{L}{2}\left(\frac{d\theta}{dt}\right)^2\Big|_0^t = g\cos\theta\Big|_0^\theta$$

$$\Rightarrow \frac{L}{2}\left(\frac{d\theta}{dt}\right)^2 - g\cos\theta = \frac{L}{2}\left(\frac{d\theta_0}{dt}\right)^2 - g.$$

or

$$\Rightarrow \frac{L}{2}\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - 1) + E$$

c. *(10 pts)* For the model (2), assume $k = 0$. What is the potential energy? What are the equilibrium point(s)? State and justify the stability of the equilibrium point(s)

$$F(\theta) = \int_0^{\theta} g \sin\bar{\theta}\, d\bar{\theta} = g(1-\cos\theta)$$

$$\theta_E \text{ s.t. } F'(\theta_E) = 0 \iff g \sin\theta_E = 0 \Rightarrow \theta_E = n\pi,\ n \in \mathbb{Z}$$

2nd der. test: $F''(\theta_E) = \begin{cases} g\cos\theta_E > 0 & \text{if } n \text{ is even} \\ g\cos\theta_E < 0 & \text{if } n \text{ is odd} \end{cases}$

$\therefore$ $\theta_E = n\pi$, n even stable

$\theta_E = n\pi$, n odd unstable

d. *(5 pts)* Sketch some energy curves corresponding to the energy equation in part *b* for $0 \leq E \leq 2g$.

See pages 79-80 in textbook

4. Consider a simple spring-mass system with no forces other than a spring force and a linear damping force.

a. (5 pts) If mass is 1, the spring constant is 2, and the friction coefficient is 2, what is the governing equation?

$$m\frac{d^2x}{dt^2} + kx + c\frac{dx}{dt} = 0$$

b. (10 pts) What are the roots of the characteristic equation corresponding to the system above? Is the system overdamped, underdamped, or critically damped? What is the circular frequencey ω?

Ansatz : $x = e^{rt}$ $\Rightarrow r^2 + 2r + 2 = 0$

$\Rightarrow r = -1 \pm i$ 2 complex roots (underdamped)

$\Rightarrow \omega = 1$

c. (10 pts) With initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0) = 0$, what is the solution?

gen sol'n : $x(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$

$x(0) = c_1 = 1$

$\frac{dx}{dt}(0) = -(1) + c_2 = 0 \Rightarrow c_2 = 1$

(Note $\frac{dx}{dt}(t) = -e^{-t}(\cos t + c_2 \sin t) + e^{-t}(-\sin t + c_2 \cos t))$

$\Rightarrow x(t) = e^{-t}(\cos t + \sin t)$