## Assignment 7, due Friday, 19th November

**Theoretical:**

1. Find the stability function $R$ for the following Runge-Kutta method:

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ 1 & & 1 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Is it A-stable?

2. The following questions pertain to the extension of the linear stability analysis to inhomogeneous system.

a. Let $\Lambda$ be a nonsingular matrix. Prove that the solution of $\vec{y}' = \Lambda\vec{y} + \vec{a}$, $\vec{y}(t_0) = \vec{y}_0$ is

$$\vec{y}(t) = e^{\Lambda(t-t_0)}\vec{y_0} + \Lambda^{-1}[e^{\Lambda(t-t_0)} - I]\vec{a}, \quad t \geq t_0.$$

b. Prove that if $\Lambda$ has a full set of eigenvectors and all of its eigenvalues reside in $\mathbb{C}^-$, then $\lim_{t\to\infty} -\Lambda^{-1}\vec{a}$.

c. Show that after one step of the Runge-Kutta method applied to the simplest test problem,

$$\begin{aligned} y' &= \lambda y + a \\ y(t_0) &= y_0 \end{aligned}$$

results in

$$w_{n+1} = R(h\lambda)w_n + Q(h\lambda), \quad n = 0, 1, \ldots$$

where

$$R(z) = 1 + zb^T(I - zA)^{-1}\mathbf{1}$$

and

$$Q(z) = hab^T(I - zA)^{-1}\mathbf{1}$$

for $z \in \mathbb{C}$. Hence,

$$w_n = [R(h\lambda)]^n w_0 + \left(\frac{R(h\lambda)^n - 1}{R(h\lambda) - 1}\right) Q(h\lambda), \quad n = 0, 1, \ldots$$

3. Show that for all semi-implicit Runge-Kutta methods the denominator of the stability function $R(z)$ is a product of real linear factors.

**Computational:**

Consider the Curtiss-Hirschfelder equation

$$\frac{dy}{dt} = -50(y - \cos(t)), \quad y(0) = 1.$$

- Use the forward Euler and 4th order Runge-Kutta to solve the Curtiss-Hirschfelder equation for $t \in [0, 10]$.
- Determine a stepsize $h$ for each method where $w_n$ captures the correct qualitative behavior of the true solution. Plot $w_n$.
- Compare the two methods for solving the Curtiss-Hirschfeld equation.

*The exact solution is*

$$y(t) = \frac{2500}{2501}\cos(t) + \frac{50}{2501}\sin(t) + \frac{1}{2501}e^{-50t}, \quad y(0) = 1.$$