

A CONDITIONAL PROOF OF THE NON-CONTRACTION PROPERTY FOR N FALLING BALLS

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ABSTRACT. Wojtkowski's system of N , $N \geq 2$, falling balls is a nonuniformly hyperbolic smooth dynamical system with singularities. It is still an open question whether this system is ergodic. We contribute toward an affirmative answer, by proving the non-contraction property, conditioned by the assumption of strict unboundedness. For a certain mass ratio the configuration space can be unfolded to a billiard table where the daunting proper alignment condition is satisfied. We prove, that the aforementioned unfolded system with three degrees of freedom is ergodic.

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1. INTRODUCTION

In [W90a, W90b], Maciej P. Wojtkowski introduced the system of N , $N \geq 2$, falling balls. It describes the motion of N point masses moving up and down a vertical line, colliding with each other elastically and the lowest point mass collides with a rigid floor placed at height zero. The system has N degrees of freedom, the positions q_1, \dots, q_N and the momenta p_1, \dots, p_N . The point masses are placed on top of each satisfying $0 \leq q_1 \leq \dots \leq q_N$. The overall standing assumption on the masses is $m_1 > \dots > m_N$. Movement occurs due to kinetic energy and a linear potential field on a compact energy surface E_c given by the Hamiltonian $H(q, p) = \sum_{i=1}^N p_i^2 / 2m_i + m_i q_i$. The dynamics are further reduced to the Poincaré

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section \mathcal{M} containing the states right after a collision of two point masses or a collision of the lowest point mass with the floor. Accordingly, the Poincaré map T describes the dynamics from one collision to the next. It preserves the smooth measure μ , obtained from the symplectic volume form on $\mathbb{R}^N \times \mathbb{R}^N$ via symplectic reduction. Out of historic convenience we will refer to the falling point masses as falling balls.

An intrinsic obstacle, which makes the treatment of this system challenging, is the presence of singular collisions. In physical terms, they occur in a triple collision or when the two lower balls hit the floor simultaneously. The singular collisions or singularities form codimension one submanifolds in phase space. The Poincaré map is not well defined on the singularities because it has two images.

The main question in Wojtkowski's original paper [W90a] revolved around the existence of non-zero Lyapunov exponents. Simányi settled the general case by proving that an arbitrary number of falling balls have non-zero Lyapunov exponents almost everywhere [S96]. For a family of potential fields $V(q)$, satisfying $\partial V(q)/\partial q > 0$, $\partial^2 V(q)/\partial q^2 < 0$, Wojtkowski proved the same result in [W90b]. The latter family of potentials does not include the linear potential field.

A new treatment, which ties in old and new ideas, can be found in Wojtkowski's latest work on falling balls [W98]. He starts off with N , $N \geq 2$, horizontally aligned balls falling to a moving floor, establishes complete hyperbolicity, and then carries the result over to a variety of falling ball systems by applying stacking rules on the balls. In the most extreme case he obtains his original system from [W90a]. The billiard system of each falling ball system corresponds to a particle falling in a particular wedge. The form of the wedge depends thereby on the masses and the physical model.

The main line of this work concerns the long time open conjecture whether three (or more) falling balls are ergodic. There are two results, confirming the ergodicity of two falling balls with mass configurations $m_1 > m_2$: One for the linear potential mentioned above [LW92] and one [Ch91] for the family of potentials considered above with the relaxed assumption $\partial^2 V(q)/\partial q^2 \leq 0$ and the additional restrictions $0 < C_1 \leq \partial V(q)/\partial q \leq C_2 < \infty$, $0 \leq |\partial^2 V(q)/\partial q^2| \leq C_3 < \infty$, for some constants $C_1, C_2, C_3 > 0$.

Since our system satisfies the mild conditions of Katok-Strelcyn [KS86], the theory of the latter implies that the phase space decomposes into at most countably many ergodic components.¹ From here, it is common to verify the Local Ergodic Theorem (LET) together with a transitivity argument to prove the existence of only one ergodic component of full measure and, thus, the ergodicity of the system. The LET dates back to Sinai's seminal proof of ergodicity for two discs moving uniformly in the two dimensional torus [S70] and was later generalized in the framework of semi-dispersing billiards [ChS87, KSSz90, BChSzT02]. In order to prove ergodicity we will use the LET, formulated for symplectic maps by Liverani and Wojtkowski [LW92].

The LET claims, that one can find an open neighbourhood of a hyperbolic point p with sufficient expansion, which lies (mod 0) in one ergodic component, if the following five conditions are satisfied:

- (C1) Regularity of singularity manifolds.

¹An ergodic component is a set of positive measure in phase space on which the conditional smooth measure is ergodic.

- (C2) Non-contraction property.
- (C3) Continuity of transversal Lagrangian subspaces.
- (C4) Chernov-Sinai ansatz.
- (C5) Proper alignment.

Assuming the validity of the LET and the abundance of sufficiently expanding points,² ensures that the neighbourhoods of the LET can be connected to one ergodic component of full measure.

The bulk of effort in this paper consists in giving a conditional proof of the non-contraction property (C2). We will use coordinate transformations (4.2), (4.3) for which the derivative of the flow equals the identity matrix. Hence, it is equivalent to verify the non-contraction property for the flow. The paramount advantage is that it is easier for us to express results in finite times rather than arbitrarily many derivative map compositions (cf. Section 7). There are two main ingredients for the proof of the non-contraction property: The first one requires that along every orbit and for every ball to ball collision there exists a subsequence of collision times, such that the pre-collisional velocity differences of the ball to ball collisions are uniformly bounded from below (cf. Theorem 6.1). The latter will imply that in every finite interval $[0, T]$, $T > 0$, the number of ball to ball collisions is uniformly bounded from above by a constant which depends only on the length T and the energy $c > 0$ of the system (cf. Lemma 7.4). The second ingredient requires the validity of the strict unboundedness property for every phase point (cf. Assumption SU, Section 2), i.e. the divergence of the quadratic form Q along every orbit and every vector of the closed expanding cone field (cf. Definition 4.4). For a constant $E_0 > 0$, the strict unboundedness property will help us to determine a time $T = T(E_0) > 0$, for which the Q -value of every vector from the expanding cone field has a uniform lower bound E_0 (cf. Lemma 7.1). This allows us to split the proof of the non-contraction property into two parts: First, we prove that the non-contraction property holds for every $t \leq T$ and, second, for every $t > T$. Note, that the uniform lower bound of the velocity differences and its implications is used for the first part only.

Conditioning the validity of the non-contraction property to the validity of strict unboundedness has the advantage that we free ourselves from having to find a Lyapunov semi-norm for this model, which is an inherently difficult task by itself. Additionally, as we discuss further below, the strict unboundedness property has already been verified for three falling balls with mass configuration given in (8.3) [HT20].³

It is already known, that the continuity of Lagrangian subspaces (C3) is true for an arbitrary number of balls [W90a, W91]. For the special restriction of masses (8.3) the configuration space (3.1) can be unfolded to a wide wedge [W98, Definition 6.1]. Wojtkowski's insight [W16] allowed to verify condition (C5), by showing that, due to the unfolding of the wedge, orbits hitting the unaligned triple collision singularity manifold can be uniquely continued [HT20, Subsection 7.3]. Except for the missing triple singularity manifold, this system is identical to the system of falling balls up to a Q -isometric coordinate transformation (8.1). In order to distinguish between

²The abundance of sufficiently expanding points is equivalent to saying that the set of sufficiently expanding points has measure one and is arcwise connected (cf. Subsection 5.2).

³In fact, the bold reader may verify that the result of Theorem 6.1 can be implemented into [HT20], which will yield strict unboundedness for every mass configuration $m_1 > m_2 > m_3$.

the two systems we follow Wojtkowski [W98] and call the former system a particle falling in a wide wedge system.

Since these systems relate to each other via a Q -isometric coordinate transformation it suffices to verify the conditions of the LET and the abundance of sufficiently expanding points in only one of the systems.

For the particular mass restrictions (8.3) we proved the strict unboundedness property, the Chernov-Sinai ansatz (C4) and the abundance of sufficiently expanding points [HT20]. Using [LW92, Lemma 7.7], it takes not much effort to check the regularity of singularity manifolds (C1) (cf. Section 8). Since the strict unboundedness assumption is valid, the new result of this work gives that the non-contraction property (C2) is valid as well. Therefore, we arrive at the conclusion that a particle falling in a three dimensional wide wedge is ergodic.

2. MAIN RESULTS

The phase space \mathcal{M} is partitioned (mod 0) into subsets \mathcal{M}_i , $i = 1, \dots, N$. \mathcal{M}_1 contains the states right after a collision with the floor and \mathcal{M}_i , $i = 2, \dots, N$, contains the states right after a collision of balls $i-1, i$. The Poincaré map $T : \mathcal{M} \circlearrowleft$ describes the movement from one collision to the next. After applying Wojtkowski's convenient coordinate transformation $(q, p) \rightarrow (h, v) \rightarrow (\xi, \eta)$ (4.2), (4.3), we obtain an expanding cone field $\{\mathcal{C}(x) : x \in \mathcal{M}\}$, explicitly given by

$$\begin{aligned} \mathcal{C}(x) &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : Q(\delta\xi, \delta\eta) > 0, \delta\xi_0 = 0, \delta\eta_0 = 0\} \cup \{\vec{0}\}, \\ \mathcal{C}'(x) &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : Q(\delta\xi, \delta\eta) < 0, \delta\xi_0 = 0, \delta\eta_0 = 0\} \cup \{\vec{0}\}. \end{aligned}$$

where $(\delta\xi, \delta\eta) = (\delta\xi_0, \dots, \delta\xi_{N-1}, \delta\eta_0, \dots, \delta\eta_{N-1})$ denote the coordinates in tangent space. The quadratic form Q is defined (cf. Definition 4.2) by a pair of constant, transversal Lagrangian subspaces (4.4) and the symplectic form ω . For this choice of Lagrangian subspaces Q becomes the Euclidean inner product

$$Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle = \sum_{i=1}^{N-1} \delta\xi_i \delta\eta_i.$$

Denote by $\overline{\mathcal{C}(x)}$ the closure of the cone $\mathcal{C}(x)$, let $d_x T^n = d_{T^n x} T \dots d_{T x} T d_x T$ and $(d_{T^n x} T)_{n \in \mathbb{N}} = (d_x T, d_{T x} T, d_{T^2 x} T, \dots)$. The sequence $(d_{T^n x} T)_{n \in \mathbb{N}}$ is called unbounded, if

$$\lim_{n \rightarrow +\infty} Q(d_x T^n v) = +\infty, \quad \forall v \in \mathcal{C}(x) \setminus \{\vec{0}\},$$

and strictly unbounded, if

$$\lim_{n \rightarrow +\infty} Q(d_x T^n v) = +\infty, \quad \forall v \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}.$$

For the proof of the non-contraction property (cf. Theorem A below), we have to assume that $(d_{T^n x} T)_{n \in \mathbb{N}}$ is strictly unbounded for every $x \in \mathcal{M}$.

Assumption (SU). *For every $x \in \mathcal{M}$, we have*

$$\lim_{n \rightarrow +\infty} Q(d_x T^n (\delta\xi, \delta\eta)) = +\infty,$$

for all $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}$.

Due to Proposition 6.2 and Theorem 6.8 of [LW92], Assumption (SU) also implies the strict unboundedness for the orbit in negative time $(d_{T^n x} T)_{n \in \mathbb{Z}^-}$, i.e.

$$\lim_{n \rightarrow -\infty} Q(d_x T^n v) = -\infty, \forall v \in \overline{\mathcal{C}'(x)} \setminus \{\vec{0}\}.$$

The singularity manifold on which T resp. T^{-1} is not well-defined is given by \mathcal{S}^+ resp. \mathcal{S}^- . Let $\mu_{\mathcal{S}^+}$ resp. $\mu_{\mathcal{S}^-}$ be the measures induced on the codimension one hypersurfaces \mathcal{S}^+ resp. \mathcal{S}^- , from the smooth T -invariant measure μ . We further abbreviate

$$\mathcal{S}_n^\pm = \mathcal{S}^\pm \cup T^{\mp 1} \mathcal{S}^\pm \cup \dots \cup T^{\mp(n-1)} \mathcal{S}^\pm.$$

Under assumption (SU), we prove the non-contraction property which is one of the five conditions (C1) - (C5) of the LET (cf. Section 5)

Theorem A (Non-contraction property). *Assume that assumption (SU) holds. Then, there exists $\zeta > 0$, such that*

(1) *for every $n \geq 1$, $x \in \mathcal{M} \setminus \mathcal{S}_n^+$, and $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}'(x)}$, we have*

$$\|d_x T^n(\delta\xi, \delta\eta)\| \geq \zeta \|(\delta\xi, \delta\eta)\|,$$

(2) *for every $n \geq 1$, $x \in \mathcal{M} \setminus \mathcal{S}_n^-$, and $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}'(x)}$, we have*

$$\|d_x T^{-n}(\delta\xi, \delta\eta)\| \geq \zeta \|(\delta\xi, \delta\eta)\|.$$

For three falling balls with the additional mass restriction (8.3), the configuration space can be unfolded to a billiard table where the ominous proper alignment condition (C5) is satisfied [W16, HT20]. The reason is, that the unfolded system misses the unaligned triple collision singularity manifold, since every orbit passing through it can be smoothly continued. Except for the missing triple singularity manifold, the system is identical to the system of falling balls up to a Q -isometric coordinate transformation (8.1). In order to distinguish between the two systems we follow Wojtkowski [W98] and call such a system a particle falling in a wide wedge. Assumption (SU) was proven for three falling balls with mass configurations (8.3) in [HT20]. Therefore, according to Theorem A, the non-contraction property holds for this system. Incorporating complementary previous results from [HT20] we will prove in Section 8

Theorem B (Ergodicity). *Consider the system of 3 falling balls with mass restrictions (8.3). Then, the unfolded system of a particle falling in a three dimensional wide wedge is ergodic.*

3. THE SYSTEM OF N FALLING BALLS

Let $q_i = q_i(t)$ be the position, $p_i = p_i(t)$ the momentum and $v_i = v_i(t)$ the velocity of the i -th ball. The balls are aligned on top of each other and are therefore confined to

$$(3.1) \quad \mathbf{N}_q = \{(q, p) \in \mathbb{R}^N \times \mathbb{R}^N : 0 \leq q_1 \leq \dots \leq q_N\}$$

where the subindex q in \mathbf{N}_q refers to the coordinates (q, p) . The momenta and the velocities are related by $p_i = m_i v_i$. We assume that the masses m_i decrease strictly as we go upwards $m_1 > \dots > m_N$. The movement of the balls are the result of a

linear potential field and their kinetic energies. The total energy of the system is given by the Hamiltonian function

$$H(q, p) = \sum_{i=1}^N \frac{p_i^2}{2m_i} + m_i q_i.$$

The Hamiltonian equations are

$$(3.2) \quad \begin{aligned} \dot{q}_i &= \frac{p_i}{m_i}, \\ \dot{p}_i &= -m_i. \end{aligned}$$

The dots indicate differentiation with respect to time t and the Hamiltonian vector field on the right hand side will be denoted as $X_H = X_H(q, p)$. For some energy value $c > 0$, the energy manifold E_c and its tangent space $\mathcal{T}E_c$ are given by

$$(3.3) \quad E_c = \{(q, p) \in \mathbb{R}_+^N \times \mathbb{R}^N : H(q, p) = \sum_{i=1}^N \frac{p_i^2}{2m_i} + m_i q_i = c\},$$

$$\mathcal{T}_{(q,p)}E_c = \{(\delta q, \delta p) \in \mathbb{R}^N \times \mathbb{R}^N : \nabla_{(q,p)}H(\delta q, \delta p) = \sum_{i=1}^N \frac{p_i \delta p_i}{m_i} + m_i \delta q_i = 0\}.$$

Including the restriction of the positions amounts to $E_c \cap \mathbf{N}_q$. The Hamiltonian vector field (3.2) gives rise to the Hamiltonian flow

$$\begin{aligned} \phi : \mathbb{R} \times (E_c \cap \mathbf{N}_q) &\rightarrow E_c \cap \mathbf{N}_q, \\ (t, (q, p)) &\mapsto \phi(t, (q, p)). \end{aligned}$$

For convenience, the image will also be written with the time variable as superscript, i.e. $\phi(t, (q, p)) = \phi^t(q, p)$.

The standard symplectic form $\omega = \sum_{i=1}^N dq_i \wedge dp_i$ induces the symplectic volume element $\Omega = \bigwedge_{i=1}^N \omega$. The volume element on the energy surface is obtained by contracting Ω , by a vector u , where u is a vector satisfying $dH(u) = 1$. Denoting the contraction operator by ι , the volume element on the energy surface is given by $\iota(u)\Omega$. Since the flow preserves the standard symplectic form, it preserves the volume element and, hence, the Liouville measure ν on $E_c \cap \mathbf{N}_q$ obtained from it.

We define the Poincaré section, which describes the states right after a collision as $\mathcal{M} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_N$, with

$$\begin{aligned} \mathcal{M}_1 &:= \{(q, p) \in E_c \cap \mathbf{N}_q : q_1 = 0, p_1/m_1 \geq 0\}, \\ \mathcal{M}_l &:= \{(q, p) \in E_c \cap \mathbf{N}_q : q_{l-1} = q_l, p_{l-1}/m_{l-1} \leq p_l/m_l\}, \quad l = 2, \dots, N. \end{aligned}$$

The set of states right before collision $\mathcal{M}^b = \mathcal{M}_1^b \cup \dots \cup \mathcal{M}_N^b$, are defined by

$$\begin{aligned} \mathcal{M}_1^b &:= \{(q, p) \in E_c \cap \mathbf{N}_q : q_1 = 0, p_1/m_1 < 0\}, \\ \mathcal{M}_l^b &:= \{(q, p) \in E_c \cap \mathbf{N}_q : q_{l-1} = q_l, p_{l-1}/m_{l-1} > p_l/m_l\}, \quad l = 2, \dots, N. \end{aligned}$$

The collision between balls i and $i+1$ is fully elastic, i.e. the total momentum and the kinetic energy are preserved. Therefore, the momenta resp. velocities change

according to

$$(3.4) \quad \begin{aligned} p_i^+ &= \gamma_i p_i^- + (1 + \gamma_i) p_{i+1}^-, \\ p_{i+1}^+ &= (1 - \gamma_i) p_i^- - \gamma_i p_{i+1}^-, \\ v_i^+ &= \gamma_i v_i^- + (1 - \gamma_i) v_{i+1}^-, \\ v_{i+1}^+ &= (1 + \gamma_i) v_i^- - \gamma_i v_{i+1}^-, \end{aligned}$$

where $\gamma_i = (m_i - m_{i+1}) / (m_i + m_{i+1})$, $i = 1, \dots, N - 1$. When the bottom particle collides with the floor the sign of its momentum resp. velocity is reversed

$$(3.5) \quad \begin{aligned} p_1^+ &= -p_1^-, \\ v_1^+ &= -v_1^-. \end{aligned}$$

This is derived from (3.4), by setting the floor velocity v_0 zero and letting the floor mass m_0 go to infinity. As a result, the floor collision does not preserve the total momentum.

These collision laws are described by the linear, symplectic, involutory collision map

$$\begin{aligned} \Phi_{i-1,i} : \mathcal{M}^b &\rightarrow \mathcal{M}, \\ (q, p^-) &\mapsto (q, p^+). \end{aligned}$$

We will write Φ if we do not want to refer to any specific collision. Let

$$(3.6) \quad \tau : E_c \cap \mathbf{N}_q \rightarrow \mathbb{R}_+,$$

be the first return time to \mathcal{M}^b . We define the Poincaré map as

$$\begin{aligned} T : \mathcal{M} &\rightarrow \mathcal{M}, \\ (q, p) &\mapsto \Phi \circ \phi^{\tau(q,p)}(q, p). \end{aligned}$$

T is the collision map, that maps the state from right after one collision to the next.

On \mathcal{M} , we obtain the volume element $\iota(X_H)\iota(u)\Omega$, by contracting the volume element $\iota(u)\Omega$ on the energy surface with respect to the direction of the flow X_H . This exterior form defines a smooth measure μ on \mathcal{M} , which is T -invariant.

Matching the present state with the next collision in the future resp. the past, we obtain two (mod 0) partitions of \mathcal{M}

$$\mathcal{M} = \mathcal{M}_{1,1}^+ \cup \bigcup_{i=1}^N \bigcup_{\substack{j=1 \\ j \neq i}}^N \mathcal{M}_{i,j}^+ = \mathcal{M}_{1,1}^- \cup \bigcup_{i=1}^N \bigcup_{\substack{j=1 \\ j \neq i}}^N \mathcal{M}_{i,j}^-,$$

where

$$\begin{aligned} \mathcal{M}_{1,1}^+ &= \{x \in \mathcal{M}_1 : Tx \in \mathcal{M}_1\}, \\ \mathcal{M}_{i,j}^+ &= \{x \in \mathcal{M}_i : Tx \in \mathcal{M}_j\}, \quad 1 \leq i, j \leq N, \quad j \neq i, \\ \mathcal{M}_{1,1}^- &= \{x \in \mathcal{M}_1 : T^{-1}x \in \mathcal{M}_1\}, \\ \mathcal{M}_{i,j}^- &= \{x \in \mathcal{M}_i : T^{-1}x \in \mathcal{M}_j\}, \quad 1 \leq i, j \leq N, \quad j \neq i. \end{aligned}$$

For some instances, it is useful to define the subset

$$\mathcal{M}_{1,1}^{m,+} := \mathcal{M}_{1,1}^+ \cap T^{-1}\mathcal{M}_{1,1}^+ \cap \dots \cap T^{-m}\mathcal{M}_{1,1}^+ \subset \mathcal{M}_{1,1}^+, \quad m \geq 1,$$

which contains the states returning $(m + 1)$ -times to the floor.

Each partition element $\mathcal{M}_{i,j}^\pm$ has a boundary $\partial\mathcal{M}_{i,j}^\pm$ and the intersection of two elements of the same partition is strictly contained in the intersection of their boundaries, i.e.

$$\mathcal{M}_{i,j}^\pm \cap \mathcal{M}_{k,l}^\pm \subset \partial\mathcal{M}_{i,j}^\pm \cap \partial\mathcal{M}_{k,l}^\pm, \quad (i,j) \neq (k,l).$$

The boundary of each partition consists of a regular part \mathcal{R}^\pm and a singular part \mathcal{S}^\pm , where we set $\partial\mathcal{M}^\pm = \mathcal{R}^\pm \cup \mathcal{S}^\pm$. The singular part comprises the following codimension one submanifolds

$$\begin{aligned} \mathcal{S}_{j,i}^+ &= \mathcal{M}_{j,i}^+ \cap \mathcal{M}_{j,i+1}^+, \quad \mathcal{S}_{i,j}^- = \mathcal{M}_{i,j}^- \cap \mathcal{M}_{i+1,j}^-, \\ i &= 2, \dots, N-1, \quad j = 1, \dots, N, \quad j \neq i, i+1, \\ \mathcal{S}_{k,1}^+ &= \mathcal{M}_{k,1}^+ \cap \mathcal{M}_{k,2}^+, \quad \mathcal{S}_{1,k}^- = \mathcal{M}_{1,k}^- \cap \mathcal{M}_{2,k}^-, \\ k &= 1, \dots, N, \quad k \neq 2. \end{aligned}$$

These sets are called singularity manifolds. The states in $\mathcal{S}_{j,i}^\pm$ face a triple collision next, while the states in $\mathcal{S}_{k,1}^+, \mathcal{S}_{1,k}^-$ experience a collision of the lower two balls with the floor next. The maps T resp. T^{-1} have two different images and are therefore not well-defined on the sets $\mathcal{S}_{j,i}^+, \mathcal{S}_{k,1}^+$ resp. $\mathcal{S}_{i,j}^-, \mathcal{S}_{1,k}^-$, because the compositions $\Phi_{i-1,i} \circ \Phi_{i,i+1}$ and $\Phi_{0,1} \circ \Phi_{1,2}$ do not commute. In this case, we follow the convention, that the orbit branches into two suborbits and we continue the system on each branch separately. We abbreviate, for $n \geq 1$,

$$\mathcal{S}^\pm = \bigcup_{i=2}^{N-1} \bigcup_{\substack{j=1 \\ j \neq i, i+1}}^N \mathcal{S}_{i,j}^\pm \cup \bigcup_{\substack{k=1 \\ k \neq 2}}^N \mathcal{S}_{k,1}^+ \cup \bigcup_{\substack{k=1 \\ k \neq 2}}^N \mathcal{S}_{1,k}^-, \quad \mathcal{S}_n^\pm = \mathcal{S}^\pm \cup T^{\mp 1} \mathcal{S}^\pm \cup \dots \cup T^{\mp(n-1)} \mathcal{S}^\pm.$$

In the upcoming sections we also want to refer to the singularity manifolds for the flow. We define them, informally, as \mathcal{S}_t^\pm , for every $t \in \mathbb{R}_+$.

Similarly to \mathcal{S}^\pm , the $T^{\pm 1}$ -image of all points in \mathcal{R}^\pm consists of two simultaneous collisions. The key difference to singular points is that the derivatives of the involved collision maps commute. This follows from the fact, that the two pairs of collisions do not share a common ball. Hence, for regular points our orbit does not split into two suborbits and can therefore be continued uniquely. Since the collision maps of the simultaneous collisions for points in \mathcal{R}^\pm commute and T is well-defined on $\mathcal{S}^- \setminus \mathcal{S}^+$, the regularity properties of the flow and the collision map imply that, for $n \geq 1$,

$$(3.7) \quad T^n : \mathcal{M} \setminus \mathcal{S}_n^+ \rightarrow \mathcal{M} \setminus \mathcal{S}_n^-$$

is a symplectomorphism, i.e. T extends diffeomorphically to \mathcal{R}^+ .

4. LYAPUNOV EXPONENTS

The study of Lyapunov exponents was carried out using a method developed by Wojtkowski in the string of papers [W85, W88, W91, LW92, W00]. This method has been successfully implemented to derive that an arbitrary number of falling balls has non-zero Lyapunov exponents almost everywhere [S96]. The basic tools of the Lyapunov exponent machinery were further advanced and are inevitable in the study of ergodicity of Hamiltonian systems [LW92]. We are therefore going to formulate the fundamentals of this method and how it applies to the system of

falling balls.

The standard symplectic form $\omega = \sum_{i=1}^N dq_i \wedge dp_i$ is given by

$$\omega(v_1, v_2) = \langle v_{1,1}, v_{2,2} \rangle - \langle v_{2,1}, v_{1,2} \rangle,$$

where $v_i = (v_{i,1}, v_{i,2}) \in \mathbb{R}^N \times \mathbb{R}^N$, $i = 1, 2$. A Lagrangian subspace V is a subspace of dimension N which is the ω -orthogonal complement to itself, i.e. the symplectic form is zero for every input from V [LM87, Definition 6.4]. It is equivalently the subspace of maximal dimension on which ω vanishes. Note, that for two transversal Lagrangian subspaces V_1, V_2 , every vector $v \in \mathbb{R}^N \times \mathbb{R}^N$ has a unique decomposition $v = v_1 + v_2$, $v_1 \in V_1, v_2 \in V_2$.

Definition 4.1. For two transversal Lagrangian subspaces V_1, V_2 we define the cone between V_1 and V_2 by

$$\mathcal{C}_{V_1, V_2} = \{v \in \mathbb{R}^N \times \mathbb{R}^N : \omega(v_1, v_2) > 0, v = v_1 + v_2, v_i \in V_i, i = 1, 2\} \cup \{\vec{0}\}.$$

Definition 4.2. The quadratic form Q_{V_1, V_2} , or Q_{V_1, V_2} -form, associated to a pair of transversal Lagrangian subspaces V_1, V_2 is given by

$$\begin{aligned} Q_{V_1, V_2} : \mathbb{R}^N \times \mathbb{R}^N &\rightarrow \mathbb{R}, \\ v &\mapsto \omega(v_1, v_2), \end{aligned}$$

where $v = v_1 + v_2$, $v_i \in V_i$, $i = 1, 2$.

Observe, that the quadratic Q_{V_1, V_2} -form is indefinite with signature (N, N) on $\mathbb{R}^N \times \mathbb{R}^N$. With the definitions above, the quadratic form can be used to define the cone

$$\mathcal{C}_{V_1, V_2} = \{v \in \mathbb{R}^N \times \mathbb{R}^N : Q_{V_1, V_2}(v) > 0\} \cup \{\vec{0}\}.$$

The complementary cone of \mathcal{C}_{V_1, V_2} is given by

$$\mathcal{C}'_{V_1, V_2} = \{v \in \mathbb{R}^N \times \mathbb{R}^N : Q_{V_1, V_2}(v) < 0\} \cup \{\vec{0}\}.$$

The arguably simplest expression of Q_{V_1, V_2} can be obtained by associating it to the standard Lagrangian subspaces given by

$$(4.1) \quad L_1 = \mathbb{R}^N \times \{\vec{0}\}, \quad L_2 = \{\vec{0}\} \times \mathbb{R}^N.$$

For this choice of transversal Lagrangian subspaces we will abbreviate $Q = Q_{L_1, L_2}$ and $\mathcal{C} = \mathcal{C}_{L_1, L_2}$. Further, for $v = v_1 + v_2$, the Q -form reads

$$Q(v) = \langle v_1, v_2 \rangle.$$

In [W90a], Wojtkowski introduced two coordinate transformations, $i = 1, \dots, N$,

$$(4.2) \quad h_i = \frac{p_i^2}{2m_i} + m_i q_i, \quad v_i = \frac{p_i}{m_i},$$

and

$$(4.3) \quad \begin{aligned} (\xi_0, \xi_1, \dots, \xi_{N-1})^T &= A^{-1}(h_1, h_2, \dots, h_N)^T \\ (\eta_0, \eta_1, \dots, \eta_{N-1})^T &= A^T(v_1, v_2, \dots, v_N)^T, \end{aligned}$$

where A is an invertible matrix depending only on the masses [W90a, p. 520]. In order to keep calculations concise and lucid, we will work exclusively in (ξ, η) -coordinates.

The energy manifold, its tangent space and the Hamiltonian vector field take the form

$$\begin{aligned} E_c &= \{(\xi, \eta) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} : H(\xi, \eta) = \xi_0 = c\}, \\ \mathcal{T}E_c &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} : \nabla_{(\xi, \eta)} H(\delta\xi, \delta\eta) = \delta\xi_0 = 0\}, \\ X_H(\xi, \eta) &= (0, \dots, 0, -1, 0, \dots, 0). \end{aligned}$$

Intersecting the standard Lagrangian subspaces (4.1) in $(\delta\xi, \delta\eta)$ -coordinates with the tangent space of the energy manifold and quotienting out the flow direction gives

$$(4.4) \quad \begin{aligned} \mathbf{L}_1 &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : \delta\xi_0 = 0, \delta\eta_i = 0, i = 0, \dots, N-1\} \simeq \mathbb{R}^{N-1}, \\ \mathbf{L}_2 &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : \delta\eta_0 = 0, \delta\xi_i = 0, i = 0, \dots, N-1\} \simeq \mathbb{R}^{N-1}. \end{aligned}$$

Thus, the Q -form given by $\mathbf{L}_1, \mathbf{L}_2$ reduces to $\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ and now amounts to

$$(4.5) \quad Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle = \sum_{i=1}^{N-1} \delta\xi_i \delta\eta_i,$$

with no further restrictions, when inserting a vector from $\mathbf{L}_1 \oplus \mathbf{L}_2$.

In these coordinates, the derivative of the flow $d\phi^t$ equals the identity map. Thus, only the derivatives of the collision maps $d\Phi_{i,i+1}$ are relevant to the dynamics in tangent space. Since $\delta\xi_0 = 0, \delta\eta_0 = 0$ we can reduce the derivatives of the collision maps to $(2N-2 \times 2N-2)$ -matrices. In these coordinates they are given by

$$(4.6) \quad d\Phi_{0,1} = \begin{pmatrix} \text{id}_{N-1} & 0 \\ B & \text{id}_{N-1} \end{pmatrix}, \quad d\Phi_{i,i+1} = \begin{pmatrix} D_i & F_i \\ 0 & D_i^T \end{pmatrix}, \quad i = 1, \dots, N-1,$$

where $B = (b_{m,n})_{m,n=1}^{N-1}$, $F_i = (f_{m,n})_{m,n=1}^{N-1}$ have the structure of the zero matrix, except for the entries $b_{1,1} = \beta$, $f_{i,i} = -\alpha_i$ and $D_i = (d_{m,n})_{m,n=1}^N$ has the structure of the identity matrix, except for the following entries in the i -th row

$$d_{i,i-1} = 1 - \gamma_i, \quad d_{i,i} = -1, \quad d_{i,i+1} = 1 + \gamma_i.$$

The terms $\alpha_1, \dots, \alpha_N$ and β in the matrices are non-negative and given by

$$(4.7) \quad \beta = -\frac{2}{m_1 v_1^-}, \quad \alpha_i = \frac{2m_i m_{i+1} (m_i - m_{i+1}) (v_i^- - v_{i+1}^-)}{(m_i + m_{i+1})^2}.$$

Observe, that the strict inequality $m_1 > \dots > m_N$ of the mass configurations implies, that $\alpha_i > 0$, since $v_i^- - v_{i+1}^- > 0$.

Using the quadratic form Q , we define the open cone \mathcal{C} and the complementary cone \mathcal{C}' associated to the Lagrangian subspaces $\mathbf{L}_1, \mathbf{L}_2$ by

$$\begin{aligned} \mathcal{C} &= \{(\delta\xi, \delta\eta) \in \mathbf{L}_1 \oplus \mathbf{L}_2 : Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle > 0\} \cup \{\vec{0}\}, \\ \mathcal{C}' &= \{(\delta\xi, \delta\eta) \in \mathbf{L}_1 \oplus \mathbf{L}_2 : Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle < 0\} \cup \{\vec{0}\}. \end{aligned}$$

The cone field $\{\mathcal{C}(x) : x \in \mathcal{M}\}$ is constant and therefore continuous in \mathcal{M} . Denote by $\bar{\mathcal{C}}$ the closure of the cone \mathcal{C} .

Definition 4.3. 1. The cone \mathcal{C} is called *invariant* at $x \in \mathcal{M}$, if

$$d_x T\bar{\mathcal{C}} \subseteq \bar{\mathcal{C}},$$

2. The cone \mathcal{C} is called *strictly invariant* at $x \in \mathcal{M}$, if

$$d_x T\bar{\mathcal{C}} \subseteq \mathcal{C},$$

3. The cone \mathcal{C} is called *eventually strictly invariant* at $x \in \mathcal{M}$, if there exists a positive integer $k = k(x) \geq 1$, such that

$$d_x T^k \bar{\mathcal{C}} \subseteq \mathcal{C},$$

4. The map $d_x T$ is called *Q-monotone*, if

$$Q(d_x T v) \geq Q(v),$$

for all $v \in \mathbf{L}_1 \oplus \mathbf{L}_2$.

5. The map $d_x T$ is called *strictly Q-monotone*, if

$$Q(d_x T v) > Q(v),$$

for all $v \in \mathbf{L}_1 \oplus \mathbf{L}_2 \setminus \{\vec{0}\}$.

6. The map $d_x T$ is called *eventually strictly Q-monotone*, if there exists a positive integer $k = k(x) \geq 1$, such that

$$Q(d_x T^k v) > Q(v),$$

for all $v \in \mathbf{L}_1 \oplus \mathbf{L}_2 \setminus \{\vec{0}\}$.

In the definition above, statements 1, 2, 3 are equivalent to statements 4, 5, 6 [LW92, Section 4]. In order to obtain non-zero Lyapunov exponents we repeat Wojtkowski's criterion [W85], which links eventual strict Q -monotonicity to non-zero Lyapunov exponents

Q-Criterion (Theorem 5.1, [W85]). *If $d_x T$ is eventually strictly Q -monotone for μ -a.e. x , then all Lyapunov exponents are non-zero almost everywhere.*

The derivative $d_x T$ is Q -monotone for every $x \in \mathcal{M}$ and any number of falling balls [W90a]. Simányi established that N , $N \geq 2$, falling balls have non-zero Lyapunov exponents for μ -a.e. $x \in \mathcal{M}$, by verifying the Q -criterion [S96].

Observe, that the coordinate transformation (4.3) is Q -isometric, i.e.

$$Q(\delta\xi, \delta\eta) = Q(A^{-1}\delta h, A^T\delta v) = Q(\delta h, \delta v),$$

which represents a change of basis inside of both Lagrangian subspaces. Therefore, it does not make a difference in terms of the Q -form's value whether we operate in $(\delta h, \delta v)$ or $(\delta\xi, \delta\eta)$ -coordinates.

We close this subsection by formulating the (strict) unboundedness property and the least expansion coefficients, which will be used to establish criteria for ergodicity.

The *least expansion coefficients* σ , $\sigma_{\mathcal{C}'}$, for $n \geq 1$, are defined as

$$\sigma(d_x T^n) = \inf_{0 \neq v \in \mathcal{C}(x)} \sqrt{\frac{Q(d_x T^n v)}{Q(v)}}, \quad \sigma_{\mathcal{C}'}(d_x T^{-n}) = \inf_{0 \neq v \in \mathcal{C}'(x)} \sqrt{\frac{Q(d_x T^{-n} v)}{Q(v)}}.$$

Definition 4.4. 1. The sequence $(d_{T^n x} T)_{n \in \mathbb{N}}$ is called *unbounded*, if

$$\lim_{n \rightarrow +\infty} Q(d_x T^n v) = +\infty, \quad \forall v \in \mathcal{C}(x) \setminus \{\vec{0}\}.$$

2. The sequence $(d_{T^n x} T)_{n \in \mathbb{N}}$ is called *strictly unbounded*, if

$$\lim_{n \rightarrow +\infty} Q(d_x T^n v) = +\infty, \quad \forall v \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}.$$

The least expansion coefficient and the property of strict unboundedness relate to each other in the following way

Theorem 4.1 (Theorem 6.8, [LW92]). *The following assertions are equivalent:*

- (1) *The sequence $(d_{T^n x} T)_{n \in \mathbb{N}}$ is strictly unbounded.*
- (2) $\lim_{n \rightarrow \infty} \sigma(d_x T^n) = \infty$.

Remark 4.1. The strict unboundedness property can also be stated in negative time, i.e.

$$\lim_{n \rightarrow -\infty} Q(d_x T^n v) = -\infty, \forall v \in \overline{\mathcal{C}'(x)} \setminus \{\vec{0}\}.$$

Following the proof of [LW92, Theorem 6.8], Theorem 4.1 also extends to this case, i.e.

- (1) The sequence $(d_{T^n x} T)_{n \in \mathbb{Z}_-}$ is strictly unbounded.
- (2) $\lim_{n \rightarrow \infty} \sigma_{\mathcal{C}'}(d_x T^{-n}) = \infty$.

5. ERGODICITY

Due to the theory of Katok-Strelcyn [KS86] we know that our phase space decomposes into at most countably many components on which the conditional smooth measure is ergodic. The strategy to prove ergodicity involves two steps:

- (1) Proving local ergodicity (or the Local Ergodic Theorem), which implies that every ergodic component is a (mod 0) open set.
- (2) Proving that the set of sufficiently expanding points (Definition 5.1) is arcwise connected and of full measure, which implies that any two (mod 0) open ergodic components can be connected with each other, such that their intersection is of positive μ -measure.

The validity of both points above proves the existence of only one ergodic component of full measure.

5.1. Local Ergodicity. We use the Local Ergodic Theorem (LET) of [LW92] and begin with the definition of a sufficiently expanding point.

Definition 5.1. A point $p \in \mathcal{M}$ is called *sufficient* (or *sufficiently expanding*) if there exists a neighbourhood $\mathcal{U} = \mathcal{U}(p)$ and an integer $N = N(p) > 0$ such that either

- (3) $\mathcal{U} \cap \mathcal{S}_N^- = \emptyset$ and $\sigma(d_y T^N) > 3$, for all $y \in T^{-N}\mathcal{U}$, or
- (4) $\mathcal{U} \cap \mathcal{S}_N^+ = \emptyset$ and $\sigma_{\mathcal{C}'}(d_y T^{-N}) > 3$, for all $y \in T^N\mathcal{U}$.

Note, that in the sufficiency definition the requirements $\mathcal{U} \cap \mathcal{S}_N^- = \emptyset$ in (3) and $\mathcal{U} \cap \mathcal{S}_N^+ = \emptyset$ in (4) additionally demand, that the orbit meets no singular manifold in the first $N(p) - 1$ iterates.

The LET amounts to showing that around a sufficient point, it is possible to find an open neighbourhood, which lies (mod 0) in one ergodic component.

Local Ergodic Theorem. *Let $p \in \mathcal{M}$ be a sufficient point and let $\mathcal{U} = \mathcal{U}(p)$ be the neighbourhood from Definition 5.1. Suppose conditions (C1) - (C5) below are satisfied.*

- (C1) (*Regularity of singularity manifolds*): *The sets \mathcal{S}_n^+ and \mathcal{S}_n^- , $n \geq 1$, are regular subsets.*⁴
- (C2) (*Non - contraction property*): *There exists $\zeta > 0$, such that*

⁴For the definition of a regular subset refer to [LW92, Definition 7.1]

(a) for every $n \geq 1$, $x \in \mathcal{M} \setminus \mathcal{S}_n^+$, and $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)}$, we have

$$\|d_x T^n(\delta\xi, \delta\eta)\| \geq \zeta \|(\delta\xi, \delta\eta)\|,$$

(b) for every $n \geq 1$, $x \in \mathcal{M} \setminus \mathcal{S}_n^-$, and $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}'(x)}$, we have

$$\|d_x T^{-n}(\delta\xi, \delta\eta)\| \geq \zeta \|(\delta\xi, \delta\eta)\|.$$

(C3) (Continuity of Lagrangian subspaces) : The ordered pair of transversal Lagrangian subspaces $(L_1(x), L_2(x))$ varies continuously in $\text{int } \mathcal{M}$.

(C4) (Chernov – Sinai ansatz) : For $\mu_{\mathcal{S}^\mp}$ -a.e. $x \in \mathcal{S}^\mp$,

$$\lim_{n \rightarrow \pm\infty} Q(d_x T^n v) = \pm\infty,$$

for every $v \in \overline{\mathcal{C}(x)} \setminus \{0\}$, if $x \in \mathcal{S}^-$ and for every $v \in \overline{\mathcal{C}'(x)} \setminus \{0\}$, if $x \in \mathcal{S}^+$.

(C5) (Proper alignment) : There exists $M \geq 0$, such that for every $x \in \mathcal{S}^+$ resp. \mathcal{S}^- , we have $d_x T^{-M} v_x^+$ resp. $d_x T^M v_x^-$ belong to $\overline{\mathcal{C}'(T^{-M}x)}$ resp. $\overline{\mathcal{C}(T^Mx)}$, where v_x^+ resp. v_x^- are the characteristic lines⁵ of $\mathcal{T}_x \mathcal{S}^+$ resp. $\mathcal{T}_x \mathcal{S}^-$.

Then, the open neighbourhood $\mathcal{U}(p)$ is contained (mod 0) in one ergodic component.

5.2. Abundance of sufficiently expanding points. The notion of a sufficiently expanding point was given in Definition 5.1. Once local ergodicity is established we deduce that every ergodic component is (mod 0) open. One possibility to obtain a single ergodic component is

Theorem 5.1 (Abundance of sufficiently expanding points). *The set of sufficiently expanding points has full measure and is arcwise connected.*

The abundance of sufficiently expanding points can be proven at once by requiring the strict unboundedness assumption (SU), the proper alignment property (C5) and the explicit construction of the neighbourhood lying in one ergodic component from the LET in the beginning of Section 8 in [LW92].

Proof of Theorem 5.1. Recall that a point $x \in \mathcal{M}$ is sufficient if there exists a positive integer $N = N(x) > 0$, such that either (3) or (4) from Definition 5.1 are satisfied. Due to the strict unboundedness (SU), Theorem 4.1 and Remark 4.1, $\sigma(d_x T^n)$ and $\sigma_{\mathcal{C}'}(d_x T^{-n})$ diverge to infinity for every $x \in \mathcal{M}$. Therefore, every orbit which experiences at most one singular collision satisfies either

$$(5) \quad \sigma(d_x T^{N(x)}) > 3, T^k \mathcal{U} \cap \mathcal{S}^+ = \emptyset, 0 \leq k \leq N(x), \text{ or}$$

$$(6) \quad \sigma_{\mathcal{C}'}(d_x T^{-N(x)}) > 3, T^{-k} \mathcal{U} \cap \mathcal{S}^- = \emptyset, 0 \leq k \leq N(x).$$

It follows that the only non-sufficient orbits lie in a subset of double singular collisions. Due to the proper alignment property (C5), \mathcal{S}^+ and \mathcal{S}^- are transversal for every point, thus, the points of double singular collisions form a set of (at least) codimension two. Hence, there is an arcwise connected set of measure one such that the least expansion coefficient is larger than three. For the last part, the proof follows the beginning of Section 8 in [LW92]:

Without loss of generality assume that $\sigma_{\mathcal{C}'}(d_x T^N) > 3$. We can choose a small enough neighbourhood \mathcal{U} around the point x such that $T^N : T^{-N}\mathcal{U} \rightarrow \mathcal{U}$ is a diffeomorphism. This implies that $\mathcal{U} \cap \mathcal{S}_N^- = \emptyset$ and $T^{-N}\mathcal{U} \cap \mathcal{S}_N^+ = \emptyset$. Further, the

⁵The characteristic line v_x^\pm is a vector of $\mathcal{T}_x \mathcal{S}^\pm$ that has the property of annihilating every other vector $w \in \mathcal{T}_x \mathcal{S}^\pm$ with respect to the symplectic form ω , i.e. $\omega(v_x^\pm, w) = 0, \forall w \in \mathcal{T}_x \mathcal{S}^\pm$. Alternatively stated, it is the ω -orthogonal complement of $\mathcal{T}_x \mathcal{S}^\pm$. Note, that in symplectic geometry the ω -orthogonal complement of a codimension one subspace is one dimensional.

functional $y \mapsto \sigma(d_y T^{\mathbb{N}})$ is continuous on \mathcal{U} and by making \mathcal{U} smaller, if necessary, we obtain $\sigma(d_y T^{\mathbb{N}}) > 3$, for every $y \in T^{-\mathbb{N}}\mathcal{U}$. \square

6. UNIFORM LOWER BOUND OF VELOCITY DIFFERENCES

The investigation regarding a uniform lower bound of velocity differences $v_i^- - v_{i+1}^-$, for any $i \in \{1, \dots, N-1\}$, is of main interest for the non-contraction property. Denote by $(i, i+1)$ the collision between ball i and ball $i+1$, i.e. when $q_i = q_{i+1}$.

Let $x = x(t) \in \mathcal{M}_{i+1}$, $i \in \{1, \dots, N-1\}$. The velocity difference $v_i^-(t) - v_{i+1}^-(t)$ is non-negative and due to the collision laws (3.4), changes sign after the collision, i.e.

$$(6.1) \quad 0 \leq v_i^-(t) - v_{i+1}^-(t) = -(v_i^+(t) - v_{i+1}^+(t)).$$

The Hamiltonian equations imply, that during free flight this quantity remains preserved (3.2). Using (3.2), (3.4), we see that the term $v_i^-(t) - v_{i+1}^-(t)$ is only affected by a $(i-1, i)$ resp. $(i+1, i+2)$ collision when being expanded backwards, i.e.

$$(6.2) \quad \begin{aligned} 0 \leq v_i^-(t) - v_{i+1}^-(t) &= (1 + \gamma_{i-1})(v_{i-1}^-(t_c) - v_i^-(t_c)) \\ &\quad + (v_i^-(t_c) - v_{i+1}^-(t_c)), \text{ resp.} \\ 0 \leq v_i^-(t) - v_{i+1}^-(t) &= (1 - \gamma_{i+1})(v_{i+1}^-(t_c) - v_{i+2}^-(t_c)) \\ &\quad + (v_i^-(t_c) - v_{i+1}^-(t_c)), \end{aligned}$$

where $t_c < t$ is the collision time of the $(i-1, i)$ resp. $(i+1, i+2)$ collision. Since we stopped our expansion right before a $(i-1, i)$ resp. $(i+1, i+2)$ collision, we have

$$(6.3) \quad v_{i-1}^-(t_c) - v_i^-(t_c) \geq 0, \quad v_{i+1}^-(t_c) - v_{i+2}^-(t_c) \geq 0.$$

Formula (6.2) can be generalized in the following way. Let $t_1 < t_2$ be collision times of two successful $(i, i+1)$ collisions and $m, n \in \mathbb{N}$. Assume that in between those two $(i, i+1)$ collisions we have m $(i-1, i)$ collisions and n $(i+1, i+2)$ collisions, with collision times r_1, \dots, r_m and u_1, \dots, u_n . Expanding only the $(i, i+1)$ velocity difference backwards without changing the appearing $(i-1, i)$ and $(i+1, i+2)$ velocity differences, we obtain for $i \geq 2$

$$(6.4) \quad \begin{aligned} 0 &\leq v_i^-(t_2) - v_{i+1}^-(t_2) \\ &= (1 + \gamma_{i-1}) \sum_{j=1}^m (v_{i-1}^-(r_j) - v_i^-(r_j)) \\ &\quad + (1 - \gamma_{i+1}) \sum_{l=1}^n (v_{i+1}^-(u_l) - v_{i+2}^-(u_l)) \\ &\quad + v_i^+(t_1) - v_{i+1}^+(t_1). \end{aligned}$$

In between two $(1, 2)$ collisions, we assume to have one floor collision, m full returns to the floor of the lowest ball and n $(2, 3)$ collisions, again with collision times

r_1, \dots, r_m and u_1, \dots, u_n . Expanding (1, 2) at t_2 backwards yields

$$\begin{aligned}
(6.5) \quad 0 &\leq v_1^-(t_2) - v_2^-(t_2) \\
&= 2 \sum_{j=1}^m 2j v_1^+(r_j) \\
&+ (1 - \gamma_2) \sum_{i=1}^n (v_2^-(u_i) - v_3^-(u_i)) \\
&+ 2\sqrt{(v_1^+(t_1))^2 + 2q_1(t_1)} + v_1^+(t_1) - v_2^+(t_1).
\end{aligned}$$

If there is at least one floor collision between two (1, 2) collisions, then the square root term $2\sqrt{(v_1^+(t_1))^2 + 2q_1(t_1)}$ appears in (6.5). The latter is part of the time the lowest ball needs to fall to the floor after a (1, 2) collision.^{6 7}

Remember, that

$$(6.6) \quad v_1^+(r_j) \geq 0, \quad \forall j \in \{1, \dots, m\},$$

since this is the velocity of the first ball right after taking off from the floor.

At the heart of this work lies the following

Theorem 6.1. *There exists a constant $C > 0$, such that for every $x \in \mathcal{M}$ and every $i \in \{1, \dots, N - 1\}$, there exists a divergent sequence of collision times $(\mathbf{t}_n)_{n \in \mathbb{N}} = (\mathbf{t}_n(x, i))_{n \in \mathbb{N}} : v_i^-(\mathbf{t}_n) - v_{i+1}^-(\mathbf{t}_n) \geq C$.*

Outline of the proof: We start describing a certain collision pattern. Since every collision happens infinitely often, this collision pattern can be found (non-uniquely) infinitely often in every orbit. The time interval of this pattern in the proof below is given by $[t_{-(1,2)}, t_{(1,2)}]$. At $t_{-(1,2)}, t_{(1,2)}$ we have a (1, 2) collision and somewhere in between is at least one (0, 1) collision.

We start a proof by contradiction assuming that every pre-collisional velocity difference of (1, 2) collisions in $[t_{-(1,2)}, t_{(1,2)}]$ is arbitrarily small. Using the above formulas (6.2) - (6.5) this will amount to having every ball arbitrarily close to the floor with velocities being arbitrarily close to each other at time $t_{-(1,2)}$. Since there is at least one (0, 1) collision between the two (1, 2) collisions at $t_{-(1,2)}, t_{(1,2)}$ the square root term in (6.5) exists. This implies that at $t_{-(1,2)}$ all the balls will have arbitrarily small velocities, which results in a contradiction since the energy of the system would be arbitrarily small. Hence, the velocity difference $v_1^- - v_2^-$ of at least one (1, 2) collision in $[t_{-(1,2)}, t_{(1,2)}]$ is bounded from below.

Repeatedly using the above formulas, we obtain lower bounds for at least one velocity difference $v_i^- - v_{i+1}^-$, for every $i \in \{1, \dots, N - 1\}$. Since this collision pattern appears infinitely often along every orbit we can extend these considerations obtaining the result from Theorem 6.1.

Proof. Pick an arbitrary (1, 2) collision and mark the time as $t_{-(1,2)}$. Then, pick the next (2, 3) collision in the future and mark the time as $t_{-(2,3)}$. Continuing this procedure for the next (3, 4), \dots $(N - 1, N)$ collisions, gives us collision times $t_{-(3,4)}, \dots, t_{-(N-1,N)}$. After that we pick the first (0, 1) collision and mark its collision time with t_0 . We now reverse the order of collisions after t_0 and mark the future collision times of the first consecutively appearing $(N - 1, N), \dots, (1, 2)$

⁶The exact time the lowest ball needs to fall to the floor is $v_1^+(t_1) + \sqrt{(v_1^+(t_1))^2 + 2q_1(t_1)}$.

⁷Note, that we can have a floor collision between two (1, 2) collision without a full return of the lowest ball to the floor, i.e. the square root term is present in (6.5) but $m = 0$ in the first sum.

collisions as $t_{(N-1,N)}, \dots, t_{(1,2)}$. Note, that in the intervals $[t_{-(i,i+1)}, t_{-(i+1,i+2)}]$, $i \in \{1, \dots, N-2\}$, exactly one $(i+1, i+2)$ collision occurs, while in the interval $[t_0, t_{(N-1,N)}]$ resp. $[t_{(i,i+1)}, t_{(i-1,i)}]$, $i \in \{2, \dots, N-1\}$, exactly one $(N-1, N)$ resp. $(i-1, i)$ collision occurs, but there is no restriction on other collisions happening.

The collision times of each $(i, i+1)$ collision, including floor collisions, induce a partition \mathcal{P}_i of the time interval $[t_{-(1,2)}, t_{(1,2)}]$: For every $i \in \{2, \dots, N-1\}$, there exists a positive integer $n = n(i) \geq 2$, such that the collision times of all the $(i, i+1)$ collisions in the interval $[t_{-(i,i+1)}, t_{(i,i+1)}]$ are given by $s_{i,1}, \dots, s_{i,n}$, with $s_{i,1} := t_{-(i,i+1)}$ and $s_{i,n} := t_{(i,i+1)}$. For $i = 1$, $n = n(1) \geq 0$, and by default $s_{1,0} = t_{-(1,2)}$, $s_{1,n+1} = t_{(1,2)}$. For $i = 0$, $s_{0,1}, \dots, s_{0,n}$, $n = n(0) \geq 1$, are simply the collision times of the lowest ball with the floor in the open interval $(t_{-(1,2)}, t_{(1,2)})$. We augment the collision time sequences by a first element $s_{i,0} := t_{-(1,2)}$ and a last element $s_{i,n+1} := t_{(1,2)}$, which yields the partitions $\mathcal{P}_i = \bigcup_{k=0}^n [s_{i,k}, s_{i,k+1}]$, for every $i \in \{0, \dots, N-1\}$.

If at time $s_{i,k}$ of our partition, we face a singular collision, we might have to repartition. The details of this procedure are described in the last four paragraphs of the proof.

Assume that for every $\varepsilon > 0$ and every $k \in \{0, \dots, n+1\}$, where $n = n(1)$, we have

$$(6.7) \quad v_1^-(s_{1,k}) - v_2^-(s_{1,k}) < \varepsilon,$$

that is, the velocity differences right before every $(1, 2)$ collision in $[t_{-(1,2)}, t_{(1,2)}]$ are arbitrarily small.

In order to apply (6.5) we need to quantify how many $(2, 3)$ collisions and floor returns of the lowest ball are in between two successful $(1, 2)$ collisions. We introduce, for $i \in \{0, \dots, N-1\}$, $j \in \{1, \dots, N-1\}$, $k \in \{0, \dots, n\}$, where $n = n(j)$, the functional

$$\begin{aligned} \mathbf{c}_i : \mathcal{P}_j &\rightarrow \mathbb{N} \\ [s_{j,k}, s_{j,k+1}] &\mapsto \mathbf{c}_i([s_{j,k}, s_{j,k+1}]) =: \mathbf{c}_{i,j,k}. \end{aligned}$$

The term $\mathbf{c}_{i,j,k}$ counts how many $(i, i+1)$ collisions appear in the interval $[s_{j,k}, s_{j,k+1}]$ of the partition \mathcal{P}_j , i.e. in between two successful $(j, j+1)$ collisions happening at time $s_{j,k}$ and $s_{j,k+1}$. Applying this notation, we expand the velocity differences in (6.7) backwards and according to (6.5) obtain for every $k \in \{1, \dots, n(1)+1\}$

$$\begin{aligned} 0 &\leq v_1^-(s_{1,k}) - v_2^-(s_{1,k}) \\ &= 2 \sum_{j=1}^{c_{0,1,k}} 2j v_1^+(s_{0,g_0(j)}) \\ (6.8) \quad &+ (1 - \gamma_2) \sum_{l=1}^{c_{2,1,k}} (v_2^-(s_{2,g_2(l)}) - v_3^-(s_{2,g_2(l)})) \\ &+ 2\sqrt{(v_1^+(s_{1,k-1}))^2 + 2q_1(s_{1,k-1})} + v_1^+(s_{1,k-1}) - v_2^+(s_{1,k-1}), \end{aligned}$$

where the functions $g_0(j) \in \{1, \dots, n(0)\}$ and $g_2(l) \in \{1, \dots, n(2)\}$ enumerate the collision times subindices. Using (6.8) together with (6.3), (6.6) and our assumption

(6.7), implies for every $\varepsilon > 0$,

$$(6.9a) \quad v_2^-(s_{2,k}) - v_3^-(s_{2,k}) < \varepsilon, \quad \forall k \in \{1, \dots, n(2)\},$$

$$(6.9b) \quad v_1^+(s_{0,k}) < \varepsilon, \quad \forall k \in \{1, \dots, n(0)\},$$

$$(6.9c) \quad v_1^+(s_{1,k-1}) < \varepsilon, \quad q_1(s_{1,k-1}) < \varepsilon, \quad \forall k \in \{1, \dots, n(1) + 1\}.$$

We repeat step (6.8), by expanding the remaining velocity differences $v_i^-(s_{i,k}) - v_{i+1}^-(s_{i,k})$, for all $i \in \{2, \dots, N-1\}$, $k \in \{2, \dots, n(i)\}$ backwards. Using (6.3), (6.4), (6.9a), this leads to

$$(6.10) \quad v_i^-(s_{i,k}) - v_{i+1}^-(s_{i,k}) < \varepsilon,$$

for all $\varepsilon > 0$, $i \in \{2, \dots, N-1\}$, $k \in \{2, \dots, n(i)\}$. Every pre-collisional velocity difference $v_i^- - v_{i+1}^-$ occurring in $[t_{-(1,2)}, s_{i,1}]$ resp. $(s_{i,n}, t_{(1,2)})$ can be expanded forward resp. backward and by using (6.10) will be arbitrarily small as well. Therefore, (6.7) implies that every ball to ball pre-collisional velocity difference in $[t_{-(1,2)}, t_{(1,2)}]$ is arbitrarily small.

If the next ball to ball collision is $(i, i+1)$, $i \in \{1, \dots, N-1\}$, the collision time is given by

$$\frac{q_{i+1} - q_i}{v_i - v_{i+1}}.$$

If the denominator $v_i - v_{i+1}$ is arbitrarily small, q_{i+1}, q_i , has to be arbitrarily small as well, otherwise the collision time would be arbitrarily large, which would result in arbitrarily large velocities and contradict the finite energy assumption. Since every velocity difference in $[t_{-(1,2)}, t_{(1,2)}]$ is arbitrarily small, at time $t_{-(1,2)}$, all the balls are lying arbitrarily close to the floor with velocities being arbitrarily equal. Due to our construction, there is at least one $(0, 1)$ collision in $[t_{-(1,2)}, t_{(1,2)}]$. Hence, the square root term in (6.8) is present, which further implies (6.9c). Thus, at time $t_{-(1,2)}$, every ball lies arbitrarily close to the floor with arbitrarily small velocity. In this way, $H(q(t_0), p(t_0)) < \varepsilon$, for every $\varepsilon > 0$, which means that our orbit would break through the constant energy surface. Since this is impossible, we obtain a contradiction to our beginning assumption (6.7), hence, there exists a constant $C_1 > 0$ and at least one $k \in \{0, \dots, n(1) + 1\}$, such that

$$(6.11) \quad v_1^-(s_{1,k}) - v_2^-(s_{1,k}) \geq C_1.$$

In order to obtain the existence of a constant $C > 0$ and at least one $(i, i+1)$ collision, such that $v_i^- - v_{i+1}^- \geq C$, for all $i \in \{1, \dots, N-1\}$, we first pick the previous resp. next $(2, 3)$ collision before resp. after the $(1, 2)$ collision in (6.11). Let the past resp. future $(2, 3)$ collision happen at t_p resp. t_f . Using (6.4) we expand $v_2^-(t_f) - v_3^-(t_f)$ backwards and obtain

$$\begin{aligned} 0 &< v_2^-(t_f) - v_3^-(t_f) \\ &= (1 + \gamma_1) \sum_{j=1}^m (v_1^-(r_j) - v_2^-(r_j)) \\ &+ (1 - \gamma_3) \sum_{l=1}^n (v_3^-(u_l) - v_4^-(u_l)) \\ &+ v_2^+(t_p) - v_3^+(t_p), \end{aligned}$$

where r_1, \dots, r_m resp. u_1, \dots, u_n are the collision times of the $(1, 2)$ resp. $(3, 4)$ collisions in between the two $(2, 3)$ collisions occurring at times t_p, t_f . Note, that

the reason we denoted these collision times as r_j resp. u_l (and not $s_{1,j}$ resp. $s_{3,l}$) is because one of the (2, 3) collisions may lie outside of $[t_{-(1,2)}, t_{(1,2)}]$. This depends on the position of the (1, 2) collision at time $s_{1,k}$ from (6.11).

Assuming that both (2, 3) velocity differences in the past and future are arbitrarily small yields a contradiction since $v_1^-(s_{1,k}) - v_2^-(s_{1,k}) \geq C_1$. Hence, there exists a constant $C_2 > 0$, such that either $v_2^-(t_f) - v_3^-(t_f) \geq C_2$ or $v_2^-(t_p) - v_3^-(t_p) \geq C_2$. Successfully continuing this procedure we find positive constants $C_1, \dots, C_{N-1} > 0$ and at least one $(i, i+1)$ collision, for all $i \in \{1, \dots, N-1\}$, such that

$$(6.12) \quad v_i^- - v_{i+1}^- \geq \min\{C_1, \dots, C_{N-1}\}.$$

It follows from the way we obtained (6.12), that the collision times of all $(i, i+1)$ collisions satisfying (6.12) do not necessarily belong to $[t_{-(1,2)}, t_{(1,2)}]$.

The above steps can be repeated, thus, creating infinitely many compact intervals with a sequence of constant positive lower bounds for at least one $v_1^- - v_2^-$ per compact interval. This holds along every orbit. Those lower bounds have a global minimum, i.e.

$$\min_{x \in \mathcal{M}} \min_{n \in \mathbb{N}} v_1^-(\mathbf{t}_n(x, 1)) - v_2^-(\mathbf{t}_n(x, 1))$$

exists. Otherwise this would imply (6.10) and, hence, a contradiction. For this global lower bound we can repeat the steps from the last paragraph to obtain a global lower bound, say $C > 0$, for every pre-collisional velocity difference.

In the event of a singular collision between balls $i-1, i, i+1, i \geq 1$, which happens at $s_{i-1,k} = s_{i,k}$, for some $k \in \{2, \dots, n-1\}$, our orbit branches into two suborbits and the procedure above works for both branches, because there are further $(i-1, i), (i+1, i)$ collisions flanking the singular collisions in the past and the future.

If the singularity occurs at the last possible collision time $s_{i-1,1} = s_{i,1}$ or $s_{i-1,n} = s_{i,n}$, we have to repartition the collision times for one of the suborbits. We only outline $s_{i-1,1} = s_{i,1}$ since $s_{i-1,n} = s_{i,n}$ works in a similar way: If, for the first suborbit, the collision order is $(i-1, i) \rightarrow (i, i+1)$, nothing changes. If, $(i, i+1) \rightarrow (i-1, i)$, then we do not consider the $(i, i+1)$ collision but rather set $s_{i-1,1} = s_{i,1}$ to be the collision time of $(i-1, i)$. Then, continue as described in the beginning of the proof by picking the next collisions $(i, i+1), \dots, (N-1, N)$ with corresponding (and possibly new) collision times $s_{i,1}, \dots, s_{N-1,1}$.

Note, that if $i = 1$ in the last paragraph we face no problem with either collision order $(0, 1) \rightarrow (1, 2), (1, 2) \rightarrow (0, 1)$. In both cases we associate the collision time $s_{1,0} = t_{-(1,2)}$ with the (1, 2) collision (and in exactly the same manner, we associate $s_{1,n+1} = t_{(1,2)}$ either (1, 2) in the future).

The same procedure as in the last three paragraphs is initiated if the orbit experiences a singularity involving more than three balls. \square

Remark 6.1. We want to bring to the readers attention, that it may be possible (depending on the dynamics), for some $x \in \mathcal{M}, i \in \{1, \dots, N-1\}$, to have diverging collision time subsequences $(\mathbf{u}_n)_{n \in \mathbb{N}}$, which satisfy, for instance

$$\lim_{n \rightarrow \infty} v_i^-(\mathbf{u}_n(x, i)) - v_{i+1}^-(\mathbf{u}_n(x, i)) = 0.$$

The importance is, that such a behaviour may only happen along a collision time subsequence, since there must be enough space left for $(\mathbf{t}_n)_{n \in \mathbb{N}}$ from Theorem 6.1 to exist.

7. THE NON-CONTRACTION PROPERTY

We begin this section by pointing out, that it is sufficient for the non-contraction property to hold if we only prove it for every $v \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(0, 1)$, where $\partial B_{\|\cdot\|}(0, 1)$ is the compact ball of unit radius, with respect to the norm $\|\cdot\|$, in tangent space.

Since the flow derivative between collisions is equal to the identity matrix, it is equivalent to formulate the non-contraction property in terms of the flow, i.e.

$$\exists \zeta > 0, \forall t > 0, \forall x \in \mathcal{M} \setminus \mathcal{S}_t^+, \forall v \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(0, 1) : \|d_x \phi^t v\| \geq \zeta.$$

We know [HT19, Remark 10.3], that arbitrarily many $(0, 1)$ collisions can occur in finite time. This is why we prefer to formulate the non-contraction property in terms of the flow, because we rather deal with finite times than arbitrarily many derivative map compositions.

Assume now that the strict unboundedness property (SU) holds for every point. We fix $E_0 > 0$ and define the function

$$\begin{aligned} \tau_{E_0}^+ : \mathcal{M} &\rightarrow \mathbb{R}_+, \\ x &\mapsto \tau_{E_0}^+(x), \end{aligned}$$

where

$$(7.1) \quad \tau_{E_0}^+(x) = \min\{t > 0 : Q(d_x \phi^t v) > E_0, \forall v \in \overline{\mathcal{C}(x)}\}.$$

The assumption of strict unboundedness (SU) together with the compactness of \mathcal{M} and $\overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(0, 1)$ will help us to assert that $\tau_{E_0}^+$ is uniformly bounded from above, i.e.

$$(7.2) \quad \exists T > 0, \forall x \in \mathcal{M} : \tau_{E_0}^+(x) \leq T.$$

This information is then utilized to split up the proof of the non-contraction property into two parts: First, we prove the non-contraction property for every collision of every feasible orbit in the fixed time interval $[0, T]$ and, second, for every $t > T$.

We begin with the proof of the uniform upper bound for $\tau_{E_0}^+$.

Lemma 7.1. *The function $\tau_{E_0}^+$ is uniformly bounded from above.*

Proof. The assertion of strict unboundedness (SU) is equivalent to

$$\forall K \geq 0, \forall x \in \mathcal{M}, \forall v \in \overline{\mathcal{C}(x)}, \exists s_0 = s_0(K, x, v) : Q(d_x \phi^t v) > K, \forall t \geq s_0.$$

Since the Q -form is homogeneous (of degree two), the previous statement does not lose its general validity if we only assume it for $v \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(0, 1)$.

We want to prove

$$\exists T > 0, \forall x \in \mathcal{M}, \forall v \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(0, 1) : s_0(E_0, x, v) \leq T.$$

Assume on the contrary, that

$$\forall T > 0, \exists x = x(T) \in \mathcal{M}, \exists v = v(T) \in \overline{\mathcal{C}(x(T))} \cap \partial B_{\|\cdot\|}(0, 1) : s_0(E_0, x(T), v(T)) > T.$$

Due to compactness, the limits $\lim_{T \rightarrow \infty} x(T) = x_*$ resp. $\lim_{T \rightarrow \infty} v(T) = v_*$ lie in \mathcal{M} resp. $\overline{\mathcal{C}(x_*)} \cap \partial B_{\|\cdot\|}(0, 1)$. In view of the role of s_0 in the strict unboundedness statement, our assumption implies

$$\exists x_* \in \mathcal{M}, \exists v_* \in \overline{\mathcal{C}(x_*)} \cap \partial B_{\|\cdot\|}(0, 1) : \lim_{t \rightarrow \infty} Q(d_{x_*} \phi^t v_*) \leq E_0,$$

which clearly yields a contradiction to the strict unboundedness property. \square

We introduce the norm

$$\|(\delta\xi, \delta\eta)\|_{HT} := \|(\delta\xi, \delta\eta)\|_2 + \|\delta\eta\|_{CW},$$

where

$$\|\delta\eta\|_{CW}^2 = \sum_{i=1}^{N-2} \frac{(\delta\eta_{i+1} - \delta\eta_i)^2}{m_i},$$

is a norm on \mathbb{R}^{N-1} introduced by Cheng and Wojtkowski in [ChW91, (11)]. It is invariant with respect to the submatrices D_i, D_i^T of the ball to ball collision map derivatives given in (4.6). The norm $\|\cdot\|_2$ refers to the Euclidean norm.

We start with the first part of the proof by investigating how the fixed length of a vector changes, when it is subjected to floor or ball to ball collisions.

Lemma 7.2. *There exists a constant $E_1 > 0$, such that for all $i \in \{1, \dots, N\}$, $x \in \mathcal{M}_{i,1}^+$, $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$ and $n \geq 1$, we have*

$$\|d_x \Phi_{0,1}^n(\delta\xi, \delta\eta)\|_{HT} \geq E_1.$$

Proof. Using the definition of the floor derivative $d\Phi_{0,1}$ (4.6), we estimate

$$\|d_x \Phi_{0,1}^n(\delta\xi, \delta\eta)\|_{HT} \geq \|(\delta\xi, nB\delta\xi + \delta\eta)\|_2 \geq \max\{\|\delta\xi\|_2, \|nB\delta\xi + \delta\eta\|_2\}.$$

We will be proving the following statement: There exists a constant $E_1 > 0$, such that for all $i \in \{1, \dots, N\}$, $x \in \mathcal{M}_{i,1}^+$, $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$ and $n \geq 1$, we have

$$(7.3) \quad \|\delta\xi\|_2 \geq E_1 \quad \vee \quad \|nB\delta\xi + \delta\eta\|_2 \geq E_1.$$

Assume on the contrary that the previous statement does not hold, i.e. for every $E_1 > 0$, there exists an $i \in \{1, \dots, N\}$, $x \in \mathcal{M}_{i,1}^+$, $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$ and $n \geq 1$, such that

$$(7.4) \quad \|\delta\xi\|_2 < E_1 \quad \wedge \quad \|nB\delta\xi + \delta\eta\|_2 < E_1.$$

For E_1 sufficiently small, conditions (7.4) imply

$$(7.5) \quad |\delta\xi_1|, |\delta\xi_2|, \dots, |\delta\xi_{N-1}| < E_1, \quad |n\beta\delta\xi_1 + \delta\eta_1|, |\delta\eta_2|, \dots, |\delta\eta_{N-1}| < E_1.$$

Since $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$, (7.5) implies that the length of the $\delta\eta$ component of the vector $(\delta\xi, \delta\eta)$ is concentrated on the first entry $\delta\eta_1$, i.e. there exists a constant $E_3 = E_3(E_1) > 0$, such that

$$(7.6) \quad |\delta\eta_1| \geq E_3.$$

If

$$\delta\xi_1, \delta\eta_1 > 0, \quad \delta\xi_1, \delta\eta_1 < 0 \quad \text{or} \quad \delta\xi_1 = 0,$$

then

$$|n\beta\delta\xi_1 + \delta\eta_1| = n\beta|\delta\xi_1| + |\delta\eta_1| \geq |\delta\eta_1| \geq E_3,$$

which contradicts (7.5). Assume therefore that $\delta\xi_1\delta\eta_1 < 0$.

Since the vector lies in the cone $\overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$, $\sum_{i=1}^{N-1} \delta\xi_i\delta\eta_i \geq 0$ must be satisfied. Combining this with the above yields

$$(7.7) \quad 0 > \delta\xi_1\delta\eta_1 \geq -(\delta\xi_2\delta\eta_2 + \dots + \delta\xi_{N-1}\delta\eta_{N-1}).$$

Due to (7.5), (7.6) the second inequality in (7.7) is violated, because the right hand side is of quadratic order $\mathcal{O}(E_1^2)$, while the $\delta\xi_1\delta\eta_1$ term is of linear order $\mathcal{O}(E_1)$.

Hence, for sufficiently small E_1 , this implies $(\delta\xi, \delta\eta) \notin \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$, which contradicts assumption (7.4) and yields our claim (7.3). \square

A uniform lower bound for multiple ball to ball collisions can only be established for a fixed number of ball to ball collisions. Let $v_{\max} > 0$ be the largest possible velocity a ball can reach within the compact energy surface.

Lemma 7.3. *For a fixed $\mathbf{n}_0 \geq 1$, let $d_x T^{\mathbf{n}_0}$ be a product of \mathbf{n}_0 ball to ball collision derivatives. Then, there exists a constant $E_2 > 0$, such that for all $m \geq 1$, $x \in \mathcal{M} \setminus (\bigcup_{i=2}^N \mathcal{M}_{i,1}^+ \cup \mathcal{M}_{1,1}^{m,+})$, $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$, $n \leq \mathbf{n}_0$, we have*

$$\|d_x T^n(\delta\xi, \delta\eta)\|_{HT} \geq E_2.$$

Proof. Using the ball to ball collision map derivatives (4.6), a first estimate gives,

$$\|d_x T^n(\delta\xi, \delta\eta)\|_{HT} \geq \max\{\|D_n \delta\xi + U_n \delta\eta\|_2, \|D_n^T \delta\eta\|_{CW}\}.$$

If $E_4 > 0$ and $\|\delta\eta\|_{CW} \geq E_4$, then the invariance with respect to D_n^T of the norm $\|\cdot\|_{CW}$ immediately yields, that the vector is bounded from below.

Assume therefore that $\|\delta\eta\|_{CW} < E_4$, for a value E_4 , which will be chosen sufficiently small. Since $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1)$, there exists a constant $E_5 = E_5(E_4) > 0$, such that $\|\delta\xi\|_{CW} \geq E_5$. It is clear, that if E_4 decreases, E_5 increases.

The matrix product U_n is recursively defined by

$$U_1 = F_{i_1}, \quad U_n = D_{i_n} U_{n-1} + F_{i_n} D_{n-1}^T,$$

for some $i_1, \dots, i_n \in \{1, \dots, N\}$ depending on x and $D_n = D_{i_n} \dots D_{i_1}$. Repeatedly using the triangle inequality and the D_i -invariance of the $\|\cdot\|_{CW}$ -norm, we estimate

$$\begin{aligned} \|U_n\|_{CW} &\leq \|D_{i_n}\|_{CW} \|U_{n-1}\|_{CW} + \|F_{i_n}\|_{CW} \|D_{n-1}^T\|_{CW} \\ &= \|U_{n-1}\|_{CW} + \|F_{i_n}\|_{CW} \\ &\leq \|F_{i_1}\|_{CW} + \dots + \|F_{i_n}\|_{CW}. \end{aligned}$$

Remembering the definition of F_{i_k} (4.6) and α_k (4.7), we obtain the upper bound

$$\|U_n\|_{CW} < n 2v_{\max} \max_{i \in \{1, \dots, N-1\}} \frac{2m_i m_{i+1} (m_i - m_{i+1})}{(m_i + m_{i+1})^2}.$$

We abbreviate

$$E_6 := \max_{i \in \{1, \dots, N-1\}} \frac{2m_i m_{i+1} (m_i - m_{i+1})}{(m_i + m_{i+1})^2},$$

and estimate

$$\begin{aligned} \|D_n \delta\xi + U_n \delta\eta\| &\geq \|\delta\xi\|_2 - \|U_n\| \|\delta\eta\|_2 \\ &\geq c_1 \|\delta\xi\|_{CW} - c_2 \|U_n\|_{CW} \|\delta\eta\|_{CW} \\ &\geq c_1 E_5 - c_2 n 2v_{\max} E_6 E_4 \\ &\geq c_1 E_5 - c_2 \mathbf{n}_0 2v_{\max} E_6 E_4, \end{aligned}$$

where $c_1, c_2 > 0$ are constants derived from the equivalence of norms.

For the original statement to hold, we choose E_4 sufficiently small and obtain a lower bound $E_2 > 0$ for the last inequality. \square

To conclude the first step of the non-contraction property it remains to prove

Lemma 7.4. *Let $T > 0$. The number of ball to ball collisions in $[0, T]$ is bounded from above by a constant, which depends only on the length of the interval and the energy of the system.*

Proof. We know from [G78, G81, V79], that the number of ball to ball collisions in $[0, T]$, without any floor interaction, is bounded from above. Assume therefore that we have arbitrarily many ball to ball collisions and floor interactions in $[0, T]$. Let $i_k \in \{0, \dots, N-1\}$, $k \in \mathbb{N}$, and $(i_k, i_k + 1)$ be the aforementioned diverging collision sequence in decreasing order, i.e. $(i_k, i_k + 1)$ happens prior to $(i_{k-1}, i_{k-1} + 1)$. Additionally, we let each $(i_k, i_k + 1)$ happen at time $t_{(i_k, i_k + 1)}$. Due to the energy restriction it is clear that $v_{\max} > |v_{i_1}^+(t_{(i_1, i_1 + 1)})|$. Using the formulas of the velocity time evolution (3.2) and elastic collisions (3.4), (3.5), we expand $v_{i_1}^+(t_{(i_1, i_1 + 1)})$ n -times backward and obtain

$$(7.8) \quad \begin{aligned} v_{\max} &> |v_{i_1}^+(t_{(i_1, i_1 + 1)})| \\ &= |-(t_{(i_1, i_1 + 1)} - t_{(i_n, i_n + 1)}) + v_{i_n}^+(t_{(i_n, i_n + 1)}) + \sum_{k=1}^{c(n)} \gamma_{i_k} (v_{i_k}^- - v_{i_k + 1}^-)|. \end{aligned}$$

The positive integer function $c(n)$ counts how many ball to ball collisions happened within the n -backward iterations. Since we also consider floor collisions $c(n) < n$, for n large enough. Observe that each ball to ball collision adds a velocity difference term to the positive sum in (7.8). Hence, since we assume to have arbitrarily many ball to ball collisions, $\lim_{n \rightarrow \infty} c(n) = \infty$. Letting n go to infinity in (7.8), we first obtain that $\lim_{n \rightarrow \infty} -(t_{(i_1, i_1 + 1)} - t_{(i_n, i_n + 1)})$ is bounded since $[0, T]$ is. Second, due to Theorem 6.1, the sequence $(\gamma_{i_k} (v_{i_k}^- - v_{i_k + 1}^-))_{k \in \mathbb{N}}$ does not converge to zero and, thus, $\lim_{n \rightarrow \infty} \sum_{k=1}^{c(n)} \gamma_{i_k} (v_{i_k}^- - v_{i_k + 1}^-) = \infty$, which results in the contradiction $v_{\max} > \infty$.

We want to supplement the details for the reader, that the sum will be large enough for (7.8) to be violated after a uniform number of summations. The proof of this fact is similar to the proof of Lemma 7.1. Abbreviate

$$a_{c(n)}(x) := \sum_{k=1}^{c(n)} \gamma_{i_k} (v_{i_k}^-(t_{(i_k, i_k + 1)}) - v_{i_k + 1}^-(t_{(i_k, i_k + 1)})).$$

The divergence of $a_{c(n)}(x)$ is equivalent to

$$(7.9) \quad \forall K > 0, \forall x \in \mathcal{M}, \exists M = M(K, x) \in \mathbb{N} : a_{c(n)}(x) > K, \forall n \geq M(K, x).$$

Due to the energy $c > 0$ there is a value $K_0 = K_0(c) > 0$ of $a_{c(n)}$ for which inequality (7.8) is violated. Observe that $K_0(c)$ additionally depends on the term of opposite sign $-(t_{(i_1, i_1 + 1)} - t_{(i_n, i_n + 1)})$ and, thus, on the length of $[0, T]$, hence, we have the dependence $K_0 = K_0(c, T)$.

We want to prove

$$\forall c > 0, \forall T > 0, \exists M_0 = M_0(c, T) \in \mathbb{N}, \forall x \in \mathcal{M} : M(K_0, x) < M_0(c, T).$$

Assume on the contrary, that this does not hold, i.e.

$$\exists c > 0, \exists T > 0, \forall M_0 \in \mathbb{N}, \exists x = x(M_0) : M(K_0, x(M_0)) \geq M_0.$$

Since \mathcal{M} is compact, the limit $\lim_{M_0 \rightarrow \infty} x(M_0) = x_*$ lies in \mathcal{M} . For this x_* , we obtain from (7.9)

$$\lim_{n \rightarrow \infty} a_{c(n)}(x_*) \leq K_0,$$

which contradicts the divergence of $a_{c(n)}(x_*)$. Therefore, the number of ball to ball collisions in $[0, T]$ are bounded by a constant $M_0(c, T)$, which depends only on the length T of $[0, T]$ and the energy $c > 0$ of the system. \square

Combining the last three lemmas yields

Corollary 7.1. *Let $T > 0$ and $c > 0$ be the energy of the system. Then, there exists a constant $\zeta_1 = \zeta_1(T, c) > 0$, such that the non-contraction property holds for every finite time interval $[0, T]$, i.e.*

$$\exists \zeta_1 > 0, \forall t \leq T, \forall x \in \mathcal{M} \setminus \mathcal{S}_t^+, \forall v \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|_{HT}}(0, 1) : \|d_x \phi^t v\| \geq \zeta_1.$$

The corollary applies directly to the interval $[0, T]$, where T is the uniform upper bound of $\tau_{E_0}^+$ (7.2). We will now conclude the proof of the non-contraction property by proving it for all $t > T$.

Due to $\langle \delta\xi - \delta\eta, \delta\xi - \delta\eta \rangle \geq 0$, the Euclidean norm $\|\cdot\|_2$ and the Q -form can be related via

$$\|(\delta\xi, \delta\eta)\|_2 \geq \sqrt{2} \sqrt{Q(\delta\xi, \delta\eta)}.$$

Using the Q -monotonicity of the derivative and (7.1), (7.2), we obtain

$$\begin{aligned} \|d_x \phi^t(\delta\xi, \delta\eta)\|_{HT} &\geq \|d_x \phi^t(\delta\xi, \delta\eta)\|_2 \geq \sqrt{2} \sqrt{Q(d_x \phi^t(\delta\xi, \delta\eta))} \\ &\geq \sqrt{2} \sqrt{Q(d_x \phi^T(\delta\xi, \delta\eta))} \\ &\geq \sqrt{2E_0}, \forall t > T. \end{aligned}$$

This immediately proves the non-contraction property

Corollary 7.2. *The non-contraction property formulated in terms of the norm $\|\cdot\|_{HT}$ holds with constant $\zeta = \min\{\zeta_1, \sqrt{2E_0}\}$.*

8. ERGODICITY OF A PARTICLE FALLING IN A THREE DIMENSIONAL WIDE WEDGE

In [W98], Wojtkowski investigated the hyperbolicity of dynamical systems, which describe the motion of a particle subjected to constant acceleration in a variety of wedges. We start by recapitulating the necessary prerequisites from [W98, HT20] to prove our results. For a thorough introduction to the subject we recommend reading [W98].

The unrestricted configuration space \mathbf{N}_q (3.1) of N falling balls has the form of a wedge. Abbreviating $\mathbf{q} = (q_1, \dots, q_N)$, we can alternatively formulate it as

$$W(b_1, \dots, b_N) = \left\{ \mathbf{q} \in \mathbb{R}^N : \mathbf{q} = \sum_{i=1}^N d_i b_i, d_i \in \mathbb{R}_+^N, i \in \{1, \dots, N\} \right\},$$

where the set of linearly independent vectors $\{b_1, \dots, b_N\}$, called *generators*, are given by $b_i = (b_{i,k})_{k=1}^N$ with $b_{i,1} = \dots = b_{i,i-1} = 0$, $b_{i,i} = \dots = b_{i,N} = 1$. Observe, that for every mass configuration (m_1, \dots, m_N) , the system of falling balls has the same unrestricted configuration space. As before, we obtain the dynamics by intersecting the wedge with the energy surface E_c (3.3).

We introduce the Q -isometric coordinate transformation

$$(8.1) \quad x_i = \sqrt{m_i}q_i, \quad v_i = \frac{p_i}{\sqrt{m_i}}.$$

The newly obtained unit generators $\{e_1, \dots, e_N\}$ in these coordinates become $\sqrt{M_i}e_i = (0, \dots, 0, \sqrt{m_i}, \dots, \sqrt{m_N})$, where $M_i = m_i + \dots + m_N$. We observe that, up to a scalar multiple, every mass configuration defines a different wedge $W(e_1, \dots, e_N)$, since the angles between the generators now depend on the masses. The inner product given by the kinetic energy in these coordinates is the standard scalar product $\langle \cdot, \cdot \rangle$ in \mathbb{R}^N . Since $\langle e_i, e_j \rangle = \sqrt{M_j}/\sqrt{M_i}$, it is easy to verify that

$$(8.2) \quad \begin{aligned} \langle e_i, e_{i+1} \rangle &> 0, \quad \forall i \in \{1, \dots, N-1\}, \\ \langle e_i, e_j \rangle &= \prod_{l=i}^{j-1} \langle e_l, e_{l+1} \rangle, \quad \forall i, j \in \{1, \dots, N\}, \quad i \neq j. \end{aligned}$$

Wedges satisfying (8.2) are called *simple* [W98, Proposition 2.3].

In the three dimensional case, the hitting of the face $W(e_1, e_3)$, resp. $W(e_1, e_2)$ corresponds in the physical model to a (1, 2) resp. (2, 3) collision. The triple collision states are given by the intersection of the former faces, which amounts to the first generator, i.e. $W(e_1, e_3) \cap W(e_1, e_2) = e_1$.

It was shown in [HT20] that for the mass restriction

$$(8.3) \quad 2\sqrt{m_1 m_3} = \sqrt{m_1 + m_2 + m_3},$$

the configuration wedge can be unfolded, by continuously reflecting it in the faces $W(e_1, e_2)$ resp. $W(e_2, e_3)$, into a *wide* wedge [W98, Definition 6.1]. This wide wedge consists exactly of six simple wedges [HT20, Figure 1]. This idea is due to Wojtkowski and can be generalized to N dimensions [W16].

The triple collision states in the configuration space, which are represented by the first generator e_1 , disappear in the wide wedge. More precisely, each trajectory which passes through the spot where e_1 was has a smooth continuation. Since the triple collision singularity manifold is the only obstacle to proper alignment [LW92, HT20], the system of a particle falling in the wide wedge, obtained for the special mass configuration (8.3), satisfies the proper alignment condition (C5).

For the Chernov-Sinai ansatz (C4) to be valid, we need that the orbit for $\mu_{\mathcal{S}^\pm}$ -a.e. $x \in \mathcal{S}^\pm$ emerging from the singularity manifold is strictly unbounded. This certainly holds since strict unboundedness is established for every orbit in \mathcal{M} [HT20, Main Theorem].

As was proven earlier in this work, the non-contraction property (C2) follows directly from the validity of the strict unboundedness for every point.

The Lagrangian subspaces $\mathbf{L}_1, \mathbf{L}_2$ (4.4) of the eventually strictly invariant cone field \mathcal{C} are both constant in \mathcal{M} and therefore continuous. This verifies condition (C3).

For the regularity of singularity manifolds (C1), we employ [LW92, Lemma 7.7]. The aforementioned lemma states that if $T : \mathcal{M} \setminus \mathcal{S}^+ \rightarrow \mathcal{M} \setminus \mathcal{S}^-$ is a diffeomorphism, the proper alignment condition (C5) holds and $d_x T$ is Q -monotone for every $x \in \mathcal{M}$, then the regularity of singularity subsets follows. The last two conditions have already been affirmatively addressed. For the first one, observe that we outlined at the end of Section 3, that T is a symplectomorphism up to and including the regular boundary \mathcal{R}^+ (3.7).

Since the proper alignment condition (C5) holds and every point is strictly unbounded [HT20, Main Theorem] it follows from Subsection 5.2 that the set of

sufficient points has measure one and is arcwise connected. Thus, the model of a particle falling in a three dimensional wedge is ergodic.

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