THE CHERNOV-SINAI ANSATZ HOLDS FOR N FALLING BALLS

MICHAEL HOFBAUER-TSIFLAKOS

Abstract. Wojtkowski's system of N, $N \geq 2$, falling balls is a nonuniformly hyperbolic smooth dynamical system with singularities. It is still an open question whether this system is ergodic. We contribute toward an affirmative answer, by providing proofs for the strict unboundedness property and the Chernov-Sinai ansatz. This is supplemented by conditional proofs of Chernov's transversality condition and the abundance of sufficiently expanding points. The condition put in place for the latter to hold, is that double singular points form a set of codimension two.

Contents

1. Introduction

In [W90a, W90b], Maciej P. Wojtkowski introduced the system of N, $N \geq 2$, falling balls. It describes the motion of N point masses moving up and down a vertical line, colliding with each other elastically and the lowest point mass collides with a rigid floor placed at height zero. The system has N degrees of freedom, the positions $q_1, \ldots, q_N$ and the momenta $p_1, \ldots, p_N$. The point masses are placed

Date: January 11, 2021.

2010 *Mathematics Subject Classification.* Primary 37D50; Secondary 37J10.

Key words and phrases. Ergodic Theory, Hyperbolic dynamical systems with singularities.

on top of each satisfying $0 \le q_1 \le \ldots \le q_N$. The overall standing assumption on the masses is $m_1 > \ldots > m_N$. Movement occurs due to kinetic energy and a linear potential field on a compact energy surface E_c given by the Hamiltonian $H(q,p) = \sum_{i=1}^{N} p_i^2/2m_i + m_i q_i$. The dynamics are further reduced to the Poincaré section $\mathcal{M}$ containing the states right after a collision of two point masses or a collision of the lowest point mass with the floor. Accordingly, the Poincaré map T describes the dynamics from one collision to the next. It preserves the smooth measure μ, obtained from the symplectic volume form on $\mathbb{R}^N \times \mathbb{R}^N$ via symplectic reduction. Out of historic convenience we will refer to the falling point masses as falling balls.

An intrinsic obstacle, which makes the treatment of this system challenging, is the presence of singular collisions. In physical terms, they occur in a triple collision or when the two lower balls hit the floor simultaneously. The singular collisions or singularities form codimension one submanifolds $\mathcal{S}^{\pm}$ in phase space. The Poincaré map T resp. T^{-1} is not well defined on the singularities $\mathcal{S}^{+}$ resp. $\mathcal{S}^{-}$ because it has two images.

The main question in Wojtkowski's original paper [W90a] revolved around the existence of non-zero Lyapunov exponents. Simányi settled the general case by proving that an arbitrary number of falling balls have non-zero Lyapunov exponents almost everywhere [S96]. For a family of potential fields $V(q)$, satisfying $\partial V(q)/\partial q > 0$, $\partial^2 V(q)/\partial q^2 < 0$, Wojtkowski proved the same result in [W90b]. The latter family of potentials does not include the linear potential field.

The main line of this work concerns the long time open conjecture whether three (or more) falling balls are ergodic. There are two results, confirming the ergodicity of two falling balls with mass configurations $m_1 > m_2$: One for the linear potential mentioned above [LW92] and one [Ch91] for the family of potentials considered above with the relaxed assumption $\partial^2 V(q)/\partial q^2 \le 0$ and the additional restrictions $0 < C_1 \le \partial V(q)/\partial q \le C_2 < \infty$, $0 \le \left|\partial^2 V(q)/\partial q^2\right| \le C_3 < \infty$, for some constants $C_1, C_2, C_3 > 0$.

Since our system satisfies the mild conditions of Katok-Strelcyn [KS86], the theory of the latter implies that the phase space decomposes into at most countably many ergodic components.[1] From here, it is common to verify the Local Ergodic Theorem (LET) together with a transitivity argument to prove the existence of only one ergodic component of full measure and, thus, the ergodicity of the system. The LET dates back to Sinai's seminal proof of ergodicity for two discs moving uniformly in the two dimensional torus [S70] and was later generalized in the framework of semi-dispersing billiards [ChS87, KSSz90, BChSzT02]. In order to prove ergodicity we will use the LET, formulated for symplectic maps by Liverani and Wojtkowski [LW92].

The LET claims, that one can find an open neighbourhood of a hyperbolic point p with sufficient expansion, which lies (mod 0) in one ergodic component, if the following five conditions are satisfied:

- (C1) Regularity of singularity manifolds.
- (C2) Non-contraction property.
- (C3) Continuity of transversal Lagrangian subspaces.
- (C4) Chernov-Sinai ansatz.

[1] An ergodic component is a set of positive measure in phase space on which the conditional smooth measure is ergodic.

(C5) Proper alignment.

In [Ch93], Chernov introduced condition

(C5') Transversality.

This condition is weaker than (C5) and the LET still holds with (C5') substituted in place of (C5). The big advantage of (C5') over (C5) is that the transversality condition (C5') has to hold only for $\mu_{\mathcal{S}^\pm}$-almost every point, while the proper alignment condition (C5) is formulated for every point of the singularity manifolds $\mathcal{S}^\pm$. Once the LET is established, it remains to prove, that the obtained ergodic neighbourhoods reach into every part of the phase space. The transitivity condition which ensures this is called

(A) Abundance of sufficiently expanding points.[2]

The validity of (A) guarantees that the neighbourhoods of the LET can be connected to one ergodic component of full measure.

It is already known, that the continuity of Lagrangian subspaces (C3) is true for an arbitrary number of balls [W90a, W91]. In this paper we provide proofs for the strict unboundedness property (cf. Theorem 2.1), condition (C4) and conditional proofs for conditions (A) and (C5').

The derivation of non-zero Lyapunov exponents and the formulation of the LET heavily relies on a quadratic form Q, which is defined using the canonical symplectic form ω and a pair of transversal, point-independent Lagrangian subspaces $\mathbf{L}_1, \mathbf{L}_2$ contained in the energy surface tangent space $\mathcal{T}_x E_c$ for every $x \in \mathcal{M}$ (cf. Section 4).

Strict unboundedness asserts the divergence of the quadratic form Q along every non-double singular orbit and every vector of the closed expanding cone field (cf. Definition 4.4). The Chernov-Sinai ansatz (C4) follows immediately from this fact as a simple corollary (cf. Corollary 2.1).

The validity of conditions (C5'), (A) is conditioned by the transversality of singularity manifolds $T^m\mathcal{S}^-$, $\mathcal{S}^+$ for every $m \in \mathbb{N}$. The latter immediately implies (a) the regularity of singularity manifolds (cf. Section 6) and that (b) the set of double singular points has (at least) codimension two. Both points (a), (b) are needed to derive condition (C5') by using strict unboundedness within a compactness argument (cf. Section 9). Condition (A) only utilizes point (b), such that the strict unboundedness property can be applied to every non-double singular orbit in order to derive sufficiency (cf. Subsection 5.2).

The verification of strict unboundedness itself uses two main ingredients: The first one requires that along every non-double singular orbit and for every ball to ball collision there exists a subsequence of collision times, such that the pre-collisional velocity differences of the ball to ball collisions are uniformly bounded from below (cf. Theorem 7.1). The second one considers the previously derived result from [S96, Sublemma 3.8], which claims that each vector from the neutral space has constant length along its entire orbit. A priori, the neutral space contains vectors $v \in \mathbf{L}_1$, for which $Q(d_x T^n v) = 0$, for every $n \in \mathbb{N}$. Part of the strict unboundedness proof, will establish that the neutral space is in fact empty for every non-double singular point. The proof of strict unboundedness is carried out in Section 8.

---

[2] The abundance of sufficiently expanding points is equivalent to saying that the set of sufficiently expanding points has measure one and is arcwise connected (cf. Subsection 5.2).

2. Main results

The phase space $\mathcal{M}$ is partitioned (mod 0) into subsets $\mathcal{M}_i$, $i = 1, \ldots, N$. $\mathcal{M}_1$ contains the states right after a collision with the floor and $\mathcal{M}_i$, $i = 2, \ldots, N$, contains the states right after a collision of balls $i-1, i$. The Poincaré map $T : \mathcal{M} \circlearrowleft$ describes the movement from one collision to the next. After applying Wojtkowski's convenient coordinate transformation $(q, p) \to (h, v) \to (\xi, \eta)$ (4.2), (4.3), we obtain an expanding cone field $\{\mathcal{C}(x) : x \in \mathcal{M}\}$, explicitly given by

$$\mathcal{C}(x) = \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : Q(\delta\xi, \delta\eta) > 0, \ \delta\xi_0 = 0, \ \delta\eta_0 = 0\} \cup \{\vec{0}\},$$
$$\mathcal{C}'(x) = \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : Q(\delta\xi, \delta\eta) < 0, \ \delta\xi_0 = 0, \ \delta\eta_0 = 0\} \cup \{\vec{0}\}.$$

where $(\delta\xi, \delta\eta) = (\delta\xi_0, \ldots, \delta\xi_{N-1}, \delta\eta_0, \ldots, \delta\eta_{N-1})$ denote the coordinates in tangent space. The quadratic form Q is defined (cf. Definition 4.2) by a pair of constant, transversal Lagrangian subspaces (4.4) and the symplectic form ω. For this choice of Lagrangian subspaces Q becomes the Euclidean inner product

$$Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle = \sum_{i=1}^{N-1} \delta\xi_i \delta\eta_i.$$

The singularity manifold on which T resp. T^{-1} is not well-defined is given by $\mathcal{S}^+$ resp. $\mathcal{S}^-$. Let $\mu_{\mathcal{S}^+}$ resp. $\mu_{\mathcal{S}^-}$ be the measures induced on the codimension one hypersurfaces $\mathcal{S}^+$ resp. $\mathcal{S}^-$, from the smooth T-invariant measure μ. We further abbreviate

$$\mathcal{S}_n^{\pm} = \mathcal{S}^{\pm} \cup T^{\mp 1}\mathcal{S}^{\pm} \cup \ldots \cup T^{\mp(n-1)}\mathcal{S}^{\pm}.$$

Points which hit a singularity in the past and the future are called double singular. We define the set of all double singular points as $\mathcal{D} \subset \mathcal{M}$.
Denote by $\overline{\mathcal{C}(x)}$ the closure of the cone $\mathcal{C}(x)$, let $d_xT^n = d_{T^nx}T \ldots d_{Tx}T d_xT$ and $(d_{T^nx}T)_{n\in\mathbb{N}} = (d_xT, d_{Tx}T, d_{T^2x}T, \ldots)$. The sequence $(d_{T^nx}T)_{n\in\mathbb{N}}$ is called unbounded, if

$$\lim_{n\to+\infty} Q(d_xT^n v) = +\infty, \ \forall \ v \in \mathcal{C}(x) \setminus \{\vec{0}\},$$

and strictly unbounded, if

$$\lim_{n\to+\infty} Q(d_xT^n v) = +\infty, \ \forall \ v \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}.$$

With this in mind, we formulate our first result.

Theorem 2.1 (Strict unboundedness)**.** *For every $x \in \mathcal{M} \setminus \mathcal{D}$, we have*

$$\lim_{n\to+\infty} Q(d_xT^n(\delta\xi, \delta\eta)) = +\infty,$$

for all $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}$.

Due to Proposition 6.2 and Theorem 6.8 of [LW92], Theorem A also implies the strict unboundedness for the orbit in negative time $(d_{T^nx}T)_{n\in\mathbb{Z}^-}$, i.e.

$$\lim_{n\to-\infty} Q(d_xT^n v) = -\infty, \ \forall \ v \in \overline{\mathcal{C}'(x)} \setminus \{\vec{0}\}.$$

Theorem 2.1 immediately yields the Chernov-Sinai ansatz (C4), which is one of three conditions of the LET we are going to prove (cf. Section 5).

Corollary 2.1 (Chernov-Sinai ansatz)**.** *For $\mu_{\mathcal{S}^{\mp}}$-a.e. $x \in \mathcal{S}^{\mp}$,*

$$\lim_{n\to\pm\infty} Q(d_x T^n v) = \pm\infty,$$

for every $v \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}$, if $x \in \mathcal{S}^-$ and for every $v \in \overline{\mathcal{C}'(x)} \setminus \{\vec{0}\}$, if $x \in \mathcal{S}^+$.

We further give two conditional proofs of conditions (C5') and (A). Both utilize the transversality of singularity manifolds $T^m\mathcal{S}^-$, $\mathcal{S}^+$ for every $m \in \mathbb{N}$, which implies

(a) the regularity of singularity manifolds (cf. Section 6).
(b) the codimension of double singular points $\mathcal{D}$ is (at least) two.

Denote by $W^u(x)$ resp. $W^s(x)$ the unstable resp. stable manifold at point x. Their existence are guaranteed from the existence of non-zero Lyapunov exponents [S96] and Katok-Strelcyn theory [KS86].

Theorem 2.2 (Transversality)**.** *Assume points (a), (b) hold. Then, for $\mu_{\mathcal{S}^{\pm}}$-a.e. x, the stable manifold $W^s(x)$ resp. unstable manifold $W^u(x)$ is transversal to $\mathcal{S}^-$ resp. $\mathcal{S}^+$.*

Our last result is used to connect ergodic neighbourhoods derived from the LET into one ergodic component of full measure (cf. Subsection 5.2). As with Theorem 2.2, it necessitates that the set of double singular points $\mathcal{D}$ has codimension two.

Theorem 2.3 (Abundance of sufficient points)**.** *Assume point (b) holds. Then, the set of sufficiently expanding points has full measure and is arcwise connected.*[3]

The strict unboundedness property, as stated in Theorem 2.1, is used in both proofs of Theorem 2.2 and Theorem 2.3.
Given the results in this work, the only obstacles left to prove the ergodicity of N falling balls are

(I) the non-contraction property (C2),
(II) the transversality of singularity manifolds $T^m\mathcal{S}^-$, $\mathcal{S}^+$ for every $m \in \mathbb{N}$.

Both points (I), (II) are subject of future work [HT21, S21].

3. The system of N falling balls

Let $q_i = q_i(t)$ be the position, $p_i = p_i(t)$ the momentum and $v_i = v_i(t)$ the velocity of the i-th ball. The balls are aligned on top of each other and are therefore confined to

$$\mathbf{N}_q = \{(q,p) \in \mathbb{R}^N \times \mathbb{R}^N : \ 0 \le q_1 \le \ldots \le q_N\} \tag{3.1}$$

where the subindex q in $\mathbf{N}_q$ refers to the coordinates (q,p). The momenta and the velocities are related by $p_i = m_i v_i$. We assume that the masses m_i decrease strictly as we go upwards $m_1 > \ldots > m_N$. The movement of the balls are the result of a linear potential field and their kinetic energies. The total energy of the system is given by the Hamiltonian function

$$H(q,p) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + m_i q_i.$$

[3]The definition of a sufficiently expanding point can be found in Definition 5.1.

The Hamiltonian equations are

$$\begin{aligned} \dot{q}_i &= \frac{p_i}{m_i}, \\ \dot{p}_i &= -m_i. \end{aligned} \tag{3.2}$$

The dots indicate differentiation with respect to time t and the Hamiltonian vector field on the right hand side will be denoted as $X_H = X_H(q, p)$. For some energy value $c > 0$, the energy manifold E_c and its tangent space $\mathcal{T}E_c$ are given by

$$\begin{aligned} E_c &= \{(q,p) \in \mathbb{R}_+^N \times \mathbb{R}^N : \ H(q,p) = \sum_{i=1}^N \frac{p_i^2}{2m_i} + m_i q_i = c\}, \\ \mathcal{T}_{(q,p)} E_c &= \{(\delta q, \delta p) \in \mathbb{R}^N \times \mathbb{R}^N : \ \nabla_{(q,p)} H(\delta q, \delta p) = \sum_{i=1}^N \frac{p_i \delta p_i}{m_i} + m_i \delta q_i = 0\}. \end{aligned} \tag{3.3}$$

Including the restriction of the positions amounts to $E_c \cap \mathbf{N}_q$. The Hamiltonian vector field (3.2) gives rise to the Hamiltonian flow

$$\begin{aligned} \phi : \mathbb{R} \times (E_c \cap \mathbf{N}_q) &\to E_c \cap \mathbf{N}_q, \\ (t,(q,p)) &\mapsto \phi(t,(q,p)). \end{aligned}$$

For convenience, the image will also be written with the time variable as superscript, i.e. $\phi(t,(q,p)) = \phi^t(q,p)$.
The standard symplectic form $\omega = \sum_{i=1}^N dq_i \wedge dp_i$ induces the symplectic volume element $\Omega = \bigwedge_{i=1}^N \omega$. The volume element on the energy surface is obtained by contracting Ω, by a vector u, where u is a vector satisfying $dH(u) = 1$. Denoting the contraction operator by ι, the volume element on the energy surface is given by $\iota(u)\Omega$. Since the flow preserves the standard symplectic form, it preserves the volume element and, hence, the Liouville measure ν on $E_c \cap \mathbf{N}_q$ obtained from it.
We define the Poincaré section, which describes the states right after a collision as $\mathcal{M} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_N$, with

$$\begin{aligned} \mathcal{M}_1 &:= \{(q,p) \in E_c \cap \mathbf{N}_q : q_1 = 0, \ p_1/m_1 > 0\}, \\ \mathcal{M}_l &:= \{(q,p) \in E_c \cap \mathbf{N}_q : q_{l-1} = q_l, \ p_{l-1}/m_{l-1} < p_l/m_l\}, \ l = 2, \ldots, N. \end{aligned}$$

The set of states right before collision $\mathcal{M}^b = \mathcal{M}_1^b \cup \ldots \cup \mathcal{M}_N^b$, are defined by

$$\begin{aligned} \mathcal{M}_1^b &:= \{(q,p) \in E_c \cap \mathbf{N}_q : q_1 = 0, \ p_1/m_1 < 0\}, \\ \mathcal{M}_l^b &:= \{(q,p) \in E_c \cap \mathbf{N}_q : q_{l-1} = q_l, \ p_{l-1}/m_{l-1} > p_l/m_l\}, \ l = 2, \ldots, N. \end{aligned}$$

The collision between balls i and $i+1$ is fully elastic, i.e. the total momentum and the kinetic energy are preserved. Therefore, the momenta resp. velocities change according to

$$\begin{aligned} p_i^+ &= \gamma_i p_i^- + (1+\gamma_i) p_{i+1}^-, \\ p_{i+1}^+ &= (1-\gamma_i) p_i^- - \gamma_i p_{i+1}^-, \\ v_i^+ &= \gamma_i v_i^- + (1-\gamma_i) v_{i+1}^-, \\ v_{i+1}^+ &= (1+\gamma_i) v_i^- - \gamma_i v_{i+1}^-, \end{aligned} \tag{3.4}$$

where $\gamma_i = (m_i - m_{i+1})/(m_i + m_{i+1})$, $i = 1, \ldots, N-1$. When the bottom particle collides with the floor the sign of its momentum resp. velocity is reversed

$$\begin{aligned} p_1^+ &= -p_1^-, \\ v_1^+ &= -v_1^-. \end{aligned} \tag{3.5}$$

This is derived from (3.4), by setting the floor velocity v_0 zero and letting the floor mass m_0 go to infinity. As a result, the floor collision does not preserve the total momentum.
These collision laws are described by the linear, symplectic, involutory collision map

$$\begin{aligned} \Phi_{i-1,i}: \quad & \mathcal{M}^b \to \mathcal{M}, \\ & (q, p^-) \mapsto (q, p^+). \end{aligned}$$

We will write Φ if we do not want to refer to any specific collision. Let

$$\tau : E_c \cap \mathbf{N}_q \to \mathbb{R}_+, \tag{3.6}$$

be the first return time to $\mathcal{M}^b$. We define the Poincaré map as

$$\begin{aligned} T : \mathcal{M} &\to \mathcal{M}, \\ (q, p) &\mapsto \Phi \circ \phi^{\tau(q,p)}(q, p). \end{aligned}$$

T is the collision map, that maps the state from right after one collision to the next. On $\mathcal{M}$, we obtain the volume element $\iota(X_H)\iota(u)\Omega$, by contracting the volume element $\iota(u)\Omega$ on the energy surface with respect to the direction of the flow X_H. This exterior form defines a smooth measure μ on $\mathcal{M}$, which is T-invariant.
Matching the present state with the next collision in the future resp. the past, we obtain two (mod 0) partitions of $\mathcal{M}$

$$\mathcal{M} = \mathcal{M}_{1,1}^+ \cup \bigcup_{i=1}^{N} \bigcup_{\substack{j=1 \\ j \neq i}}^{N} \mathcal{M}_{i,j}^+ = \mathcal{M}_{1,1}^- \cup \bigcup_{i=1}^{N} \bigcup_{\substack{j=1 \\ j \neq i}}^{N} \mathcal{M}_{i,j}^-,$$

where

$$\begin{aligned} \mathcal{M}_{1,1}^+ &= \{x \in \mathcal{M}_1 : \ Tx \in \mathcal{M}_1\}, \\ \mathcal{M}_{i,j}^+ &= \{x \in \mathcal{M}_i : \ Tx \in \mathcal{M}_j\}, \ 1 \leq i, j \leq N, \ j \neq i, \\ \mathcal{M}_{1,1}^- &= \{x \in \mathcal{M}_1 : \ T^{-1}x \in \mathcal{M}_1\}, \\ \mathcal{M}_{i,j}^- &= \{x \in \mathcal{M}_i : \ T^{-1}x \in \mathcal{M}_j\}, \ 1 \leq i, j \leq N, \ j \neq i. \end{aligned}$$

For some instances, it is useful to define the subset

$$\mathcal{M}_{1,1}^{m,+} := \mathcal{M}_{1,1}^+ \cap T^{-1}\mathcal{M}_{1,1}^+ \cap \ldots \cap T^{-m}\mathcal{M}_{1,1}^+ \subset \mathcal{M}_{1,1}^+, \ m \geq 1,$$

which contains the states returning $(m+1)$-times to the floor.
Each partition element $\mathcal{M}_{i,j}^\pm$ has a boundary $\partial\mathcal{M}_{i,j}^\pm$ and the intersection of two elements of the same partition is strictly contained in the intersection of their boundaries, i.e.

$$\mathcal{M}_{i,j}^\pm \cap \mathcal{M}_{k,l}^\pm \subset \partial\mathcal{M}_{i,j}^\pm \cap \partial\mathcal{M}_{k,l}^\pm, \ (i,j) \neq (k,l).$$

The boundary of each partition consists of a regular part $\mathcal{R}^\pm$ and a singular part $\mathcal{S}^\pm$, where we set $\partial\mathcal{M}^\pm = \mathcal{R}^\pm \cup \mathcal{S}^\pm$. The singular part comprises the following codimension one submanifolds

$$\begin{aligned} &\mathcal{S}^+_{j,i} = \mathcal{M}^+_{j,i} \cap \mathcal{M}^+_{j,i+1},\ \mathcal{S}^-_{i,j} = \mathcal{M}^-_{i,j} \cap \mathcal{M}^-_{i+1,j}, \\ &i = 2, \dots, N-1,\ j = 1, \dots, N,\ j \neq i, i+1, \\ &\mathcal{S}^+_{k,1} = \mathcal{M}^+_{k,1} \cap \mathcal{M}^+_{k,2},\ \mathcal{S}^-_{1,k} = \mathcal{M}^-_{1,k} \cap \mathcal{M}^-_{2,k}, \\ &k = 1, \dots, N,\ k \neq 2. \end{aligned}$$

These sets are called singularity manifolds. The states in $\mathcal{S}^\pm_{j,i}$ face a triple collision next, while the states in $\mathcal{S}^+_{k,1}$, $\mathcal{S}^-_{1,k}$ experience a collision of the lower two balls with the floor next. The maps T resp. T^{-1} have two different images and are therefore not well-defined on the sets $\mathcal{S}^+_{j,i}$, $\mathcal{S}^+_{k,1}$ resp. $\mathcal{S}^-_{i,j}$, $\mathcal{S}^-_{1,k}$, because the compositions $\Phi_{i-1,i} \circ \Phi_{i,i+1}$ and $\Phi_{0,1} \circ \Phi_{1,2}$ do not commute. In this case, we follow the convention, that the orbit branches into two suborbits and we continue the system on each branch separately. We abbreviate, for $n \geq 1$,

$$\mathcal{S}^\pm = \bigcup_{i=2}^{N-1} \bigcup_{\substack{j=1 \\ j \neq i, i+1}}^{N} \mathcal{S}^\pm_{i,j} \cup \bigcup_{\substack{k=1 \\ k \neq 2}}^{N} \mathcal{S}^+_{k,1} \cup \bigcup_{\substack{k=1 \\ k \neq 2}}^{N} \mathcal{S}^-_{1,k},\ \mathcal{S}^\pm_n = \mathcal{S}^\pm \cup T^{\mp 1}\mathcal{S}^\pm \cup \ldots \cup T^{\mp(n-1)}\mathcal{S}^\pm.$$

Finally, we define the set of double singular points $\mathcal{D}$. These are points x for which there exist $m, n \in \mathbb{N}$, such that $x \in T^{-m}\mathcal{S}^+ \cap T^n \mathcal{S}^-$.
Similarly to $\mathcal{S}^\pm$, the $T^{\pm 1}$-image of all points in $\mathcal{R}^\pm$ consists of two simultaneous collisions. The key difference to singular points is that the derivatives of the involved collision maps commute. This follows from the fact, that the two pairs of collisions do not share a common ball. Hence, for regular points our orbit does not split into two suborbits and can therefore be continued uniquely. Since the collision maps of the simultaneous collisions for points in $\mathcal{R}^\pm$ commute and T is well-defined on $\mathcal{S}^- \setminus \mathcal{S}^+$, the regularity properties of the flow and the collision map imply that, for $n \geq 1$,

$$T^n :\ \mathcal{M} \setminus \mathcal{S}^+_n \to \mathcal{M} \setminus \mathcal{S}^-_n \tag{3.7}$$

is a symplectomorphism, i.e. T extends diffeomorphically to $\mathcal{R}^+$.

4. Quadratic forms and invariant cone fields

The study of Lyapunov exponents was carried out using a method developed by Wojtkowski in the string of papers [W85, W88, W91, LW92, W00]. This method has been successfully implemented to derive that an arbitrary number of falling balls has non-zero Lyapunov exponents almost everywhere [S96]. The basic tools of the Lyapunov exponent machinery were further advanced and are inevitable in the study of ergodicity of Hamiltonian systems [LW92]. We are therefore going to formulate the fundamentals of this method and how it applies to the system of falling balls.
The standard symplectic form $\omega = \sum_{i=1}^N dq_i \wedge dp_i$ is given by

$$\omega(v_1, v_2) = \langle v_{1,1}, v_{2,2} \rangle - \langle v_{2,1}, v_{1,2} \rangle ,$$

where $v_i = (v_{i,1}, v_{i,2}) \in \mathbb{R}^N \times \mathbb{R}^N$, $i = 1, 2$. A Lagrangian subspace V is a subspace of dimension N which is the ω-orthogonal complement to itself, i.e. the symplectic form is zero for every input from V [LM87, Definition 6.4]. It is equivalently the subspace of maximal dimension on which ω vanishes. Note, that for two transversal Lagrangian subspaces V_1, V_2, every vector $v \in \mathbb{R}^N \times \mathbb{R}^N$ has a unique decomposition $v = v_1 + v_2$, $v_1 \in V_1$, $v_2 \in V_2$.

Definition 4.1. For two transversal Lagrangian subspaces V_1, V_2 we define the cone between V_1 and V_2 by

$$\mathcal{C}_{V_1,V_2} = \{v \in \mathbb{R}^N \times \mathbb{R}^N : \ \omega(v_1, v_2) > 0, \ v = v_1 + v_2, \ v_i \in V_i, \ i = 1, 2\} \cup \{\vec{0}\}.$$

Definition 4.2. The quadratic form Q_{V_1,V_2}, or Q_{V_1,V_2}-form, associated to a pair of transversal Lagrangian subspaces V_1, V_2 is given by

$$\begin{aligned} Q_{V_1,V_2} : \mathbb{R}^N \times \mathbb{R}^N &\to \mathbb{R}, \\ v &\mapsto \omega(v_1, v_2), \end{aligned}$$

where $v = v_1 + v_2$, $v_i \in V_i$, $i = 1, 2$.

Observe, that the quadratic Q_{V_1,V_2}-form is indefinite with signature (N, N) on $\mathbb{R}^N \times \mathbb{R}^N$. With the definitions above, the quadratic form can be used to define the cone

$$\mathcal{C}_{V_1,V_2} = \{v \in \mathbb{R}^N \times \mathbb{R}^N : \ Q_{V_1,V_2}(v) > 0\} \cup \{\vec{0}\}.$$

The complementary cone of $\mathcal{C}_{V_1,V_2}$ is given by

$$\mathcal{C}'_{V_1,V_2} = \{v \in \mathbb{R}^N \times \mathbb{R}^N : \ Q_{V_1,V_2}(v) < 0\} \cup \{\vec{0}\}.$$

The arguably simplest expression of Q_{V_1,V_2} can be obtained by associating it to the standard Lagrangian subspaces given by

$$L_1 = \mathbb{R}^N \times \{\vec{0}\}, \quad L_2 = \{\vec{0}\} \times \mathbb{R}^N. \tag{4.1}$$

For this choice of transversal Lagrangian subspaces we will abbreviate $Q = Q_{L_1,L_2}$ and $\mathcal{C} = \mathcal{C}_{L_1,L_2}$. Further, for $v = v_1 + v_2$, the Q-form reads

$$Q(v) = \langle v_1, v_2 \rangle .$$

In [W90a], Wojtkowski introduced two coordinate transformations, $i = 1, \ldots, N$,

$$h_i = \frac{p_i^2}{2m_i} + m_i q_i, \quad v_i \ = \frac{p_i}{m_i}, \tag{4.2}$$

and

$$\begin{aligned} (\xi_0, \xi_1, \ldots, \xi_{N-1})^T &= A^{-1}(h_1, h_2, \ldots, h_N)^T \\ (\eta_0, \eta_1, \ldots, \eta_{N-1})^T &= A^T(v_1, v_2, \ldots, v_N)^T, \end{aligned} \tag{4.3}$$

where A is an invertible matrix depending only on the masses [W90a, p. 520]. In order to keep calculations concise and lucid, we will work exclusively in (ξ, η)-coordinates.

The energy manifold, its tangent space and the Hamiltonian vector field take the form

$$\begin{aligned} E_c &= \{(\xi, \eta) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} : \ H(\xi, \eta) = \xi_0 = c\}, \\ \mathcal{T}E_c &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} : \ \nabla_{(\xi,\eta)} H(\delta\xi, \delta\eta) = \delta\xi_0 = 0\}, \\ X_H(\xi, \eta) &= (0, \ldots, 0, -1, 0, \ldots, 0). \end{aligned}$$

Intersecting the standard Lagrangian subspaces (4.1) in $(\delta\xi, \delta\eta)$-coordinates with the tangent space of the energy manifold and quotienting out the flow direction gives

$$\begin{aligned} \mathbf{L}_1 &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : \ \delta\xi_0 = 0, \ \delta\eta_i = 0, \ i = 0, \ldots, N-1\} \simeq \mathbb{R}^{N-1}, \\ \mathbf{L}_2 &= \{(\delta\xi, \delta\eta) \in \mathbb{R}^N \times \mathbb{R}^N : \ \delta\eta_0 = 0, \ \delta\xi_i = 0, \ i = 0, \ldots, N-1\} \simeq \mathbb{R}^{N-1}. \end{aligned} \tag{4.4}$$

Thus, the Q-form given by $\mathbf{L}_1$, $\mathbf{L}_2$ reduces to $\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$ and now amounts to

$$Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle = \sum_{i=1}^{N-1} \delta\xi_i \delta\eta_i, \tag{4.5}$$

with no further restrictions, when inserting a vector from $\mathbf{L}_1 \oplus \mathbf{L}_2$.
In these coordinates, the derivative of the flow $d\phi^t$ equals the identity map. Thus, only the derivatives of the collision maps $d\Phi_{i,i+1}$ are relevant to the dynamics in tangent space. Since $\delta\xi_0 = 0$, $\delta\eta_0 = 0$ we can reduce the derivatives of the collision maps to $(2N-2 \times 2N-2)$-matrices. In these coordinates they are given by

$$d\Phi_{0,1} = \begin{pmatrix} \mathrm{id}_{N-1} & 0 \\ B & \mathrm{id}_{N-1} \end{pmatrix}, \ d\Phi_{i,i+1} = \begin{pmatrix} D_i & F_i \\ 0 & D_i^T \end{pmatrix}, \ i = 1, \ldots, N-1, \tag{4.6}$$

where $B = (b_{m,n})_{m,n=1}^{N-1}$, $F_i = (f_{m,n})_{m,n=1}^{N-1}$ have the structure of the zero matrix, except for the entries $b_{1,1} = \beta$, $f_{i,i} = -\alpha_i$ and $D_i = (d_{m,n})_{m,n=1}^{N}$ has the structure of the identity matrix, except for the following entries in the i-th row

$$d_{i,i-1} = 1 - \gamma_i, \quad d_{i,i} = -1, \quad d_{i,i+1} = 1 + \gamma_i.$$

The terms $\alpha_1, \ldots, \alpha_N$ and β in the matrices are non-negative and given by

$$\beta = -\frac{2}{m_1 v_1^-}, \quad \alpha_i = \frac{2m_i m_{i+1}(m_i - m_{i+1})(v_i^- - v_{i+1}^-)}{(m_i + m_{i+1})^2}. \tag{4.7}$$

Observe, that the strict inequality $m_1 > \ldots > m_N$ of the mass configurations implies, that $\alpha_i > 0$, since $v_i^- - v_{i+1}^- > 0$.
Using the quadratic form Q, we define the open cone $\mathcal{C}$ and the complementary cone $\mathcal{C}'$ associated to the Lagrangian subspaces $\mathbf{L}_1$, $\mathbf{L}_2$ by

$$\begin{aligned} \mathcal{C} &= \{(\delta\xi, \delta\eta) \in \mathbf{L}_1 \oplus \mathbf{L}_2 : \ Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle > 0\} \cup \{\vec{0}\}, \\ \mathcal{C}' &= \{(\delta\xi, \delta\eta) \in \mathbf{L}_1 \oplus \mathbf{L}_2 : \ Q(\delta\xi, \delta\eta) = \langle \delta\xi, \delta\eta \rangle < 0\} \cup \{\vec{0}\}. \end{aligned}$$

The cone field $\{\mathcal{C}(x) : \ x \in \mathcal{M}\}$ is constant and therefore continuous in $\mathcal{M}$. Denote by $\overline{\mathcal{C}}$ the closure of the cone $\mathcal{C}$.

Definition 4.3. 1. The cone $\mathcal{C}$ is called *invariant* at $x \in \mathcal{M}$, if

$$d_x T \overline{\mathcal{C}} \subseteq \overline{\mathcal{C}},$$

2. The cone $\mathcal{C}$ is called *strictly invariant* at $x \in \mathcal{M}$, if

$$d_x T \overline{\mathcal{C}} \subseteq \mathcal{C},$$

3. The cone $\mathcal{C}$ is called *eventually strictly invariant* at $x \in \mathcal{M}$, if there exists a positive integer $k = k(x) \geq 1$, such that

$$d_x T^k \overline{\mathcal{C}} \subseteq \mathcal{C},$$

4. The map d_xT is called *Q-monotone*, if

$$Q(d_xTv) \geq Q(v),$$

for all $v \in \mathbf{L}_1 \oplus \mathbf{L}_2$.
5. The map d_xT is called *strictly Q-monotone*, if

$$Q(d_xTv) > Q(v),$$

for all $v \in \mathbf{L}_1 \oplus \mathbf{L}_2 \setminus \{\vec{0}\}$.
6. The map d_xT is called *eventually strictly Q-monotone*, if there exists a positive integer $k = k(x) \geq 1$, such that

$$Q(d_xT^kv) > Q(v),$$

for all $v \in \mathbf{L}_1 \oplus \mathbf{L}_2 \setminus \{\vec{0}\}$.

In the definition above, statements 1, 2, 3 are equivalent to statements 4, 5, 6 [LW92, Section 4]. In order to obtain non-zero Lyapunov exponents we repeat Wojtkowski's criterion [W85], which links eventual strict Q-monotonicity to non-zero Lyapunov exponents

Q-Criterion (Theorem 5.1, [W85])**.** *If d_xT is eventually strictly Q-monotone for μ-a.e. x, then all Lyapunov exponents are non-zero almost everywhere.*

The derivative d_xT is Q-monotone for every $x \in \mathcal{M}$ and any number of falling balls [W90a]. Simányi established that N, $N \geq 2$, falling balls have non-zero Lyapunov exponents for μ-a.e. $x \in \mathcal{M}$, by verifying the Q-criterion [S96].
Observe, that the coordinate transformation (4.3) is Q-isometric, i.e.

$$Q(\delta\xi, \delta\eta) = Q(A^{-1}\delta h, A^T\delta v) = Q(\delta h, \delta v),$$

which represents a change of basis inside of both Lagrangian subspaces. Therefore, it does not make a difference in terms of the Q-form's value whether we operate in $(\delta h, \delta v)$ or $(\delta\xi, \delta\eta)$-coordinates.
We close this subsection by formulating the (strict) unboundedness property and the least expansion coefficients, which will be used to establish criteria for ergodicity. The *least expansion coefficients* σ, $\sigma_{\mathcal{C}'}$, for $n \geq 1$, are defined as

$$\sigma(d_xT^n) = \inf_{0\neq v\in\mathcal{C}(x)} \sqrt{\frac{Q(d_xT^nv)}{Q(v)}}, \quad \sigma_{\mathcal{C}'}(d_xT^{-n}) = \inf_{0\neq v\in\mathcal{C}'(x)} \sqrt{\frac{Q(d_xT^{-n}v)}{Q(v)}}.$$

Definition 4.4. 1. The sequence $(d_{T^nx}T)_{n\in\mathbb{N}}$ is called *unbounded*, if

$$\lim_{n\to+\infty} Q(d_xT^nv) = +\infty, \ \forall \ v \in \mathcal{C}(x) \setminus \{\vec{0}\}.$$

2. The sequence $(d_{T^nx}T)_{n\in\mathbb{N}}$ is called *strictly unbounded*, if

$$\lim_{n\to+\infty} Q(d_xT^nv) = +\infty, \ \forall \ v \in \overline{\mathcal{C}(x)} \setminus \{\vec{0}\}.$$

The least expansion coefficient and the property of strict unboundedness relate to each other in the following way

Theorem 4.1 (Theorem 6.8, [LW92])**.** *The following assertions are equivalent:*

(1) *The sequence $(d_{T^nx}T)_{n\in\mathbb{N}}$ is strictly unbounded.*

(2) $\lim_{n\to\infty}\sigma(d_xT^n) = \infty$.

Remark 4.1. The strict unboundedness property can also be stated in negative time, i.e.

$$\lim_{n\to-\infty} Q(d_xT^n v) = -\infty,\ \forall\ v \in \overline{\mathcal{C}'(x)} \setminus \{\vec{0}\}.$$

Following the proof of [LW92, Theorem 6.8], Theorem 4.1 also extends to this case, i.e.

(1) The sequence $(d_{T^n x}T)_{n\in\mathbb{Z}_-}$ is strictly unbounded.
(2) $\lim_{n\to\infty} \sigma_{\mathcal{C}'}(d_xT^{-n}) = \infty$.

5. Ergodicity

Due to the theory of Katok-Strelcyn [KS86] we know that our phase space decomposes into at most countably many components on which the conditional smooth measure is ergodic. The strategy to prove ergodicity involves two steps:

(1) Proving local ergodicity (or the Local Ergodic Theorem), which implies that every ergodic component is a (mod 0) open set.
(2) Proving that the set of sufficiently expanding points (Definition 5.1) is arcwise connected and of full measure, which implies that any two (mod 0) open ergodic components can be connected with each other, such that their intersection is of positive μ-measure.

The validity of both points above proves the existence of only one ergodic component of full measure.

5.1. **Local Ergodicity.**

We use the Local Ergodic Theorem (LET) of [LW92] and begin with the definition of a sufficiently expanding point.

Definition 5.1. A point $p \in \mathcal{M}$ is called *sufficient* (or *sufficiently expanding*) if there exists a neighbourhood $\mathcal{U} = \mathcal{U}(p)$ and an integer $\mathtt{N} = \mathtt{N}(p) > 0$ such that either

(3) $\mathcal{U} \cap \mathcal{S}^-_{\mathtt{N}} = \varnothing$ and $\sigma(d_yT^{\mathtt{N}}) > 3$, for all $y \in T^{-\mathtt{N}}\mathcal{U}$, or
(4) $\mathcal{U} \cap \mathcal{S}^+_{\mathtt{N}} = \varnothing$ and $\sigma_{\mathcal{C}'}(d_yT^{-\mathtt{N}}) > 3$, for all $y \in T^{\mathtt{N}}\mathcal{U}$.

Note, that in the sufficiency definition the requirements $\mathcal{U} \cap \mathcal{S}^-_{\mathtt{N}} = \varnothing$ in (3) and $\mathcal{U} \cap \mathcal{S}^+_{\mathtt{N}} = \varnothing$ in (4) additionally demand, that the orbit meets no singular manifold in the first $\mathtt{N}(p) - 1$ iterates.
The LET amounts to showing that around a sufficient point, it is possible to find an open neighbourhood, which lies (mod 0) in one ergodic component.

Local Ergodic Theorem. *Let $p \in \mathcal{M}$ be a sufficient point and let $\mathcal{U} = \mathcal{U}(p)$ be the neighbourhood from Definition 5.1. Suppose conditions (C1) - (C5) below are satisfied.*

(C1) $(Regularity\ of\ singularity\ manifolds)$: *The sets $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$, $n \geq 1$, are regular subsets.*[4]
(C2) $(Non-contraction\ property)$: *There exists $\zeta > 0$, such that*
  (a) for every $n \geq 1$, $x \in \mathcal{M} \setminus \mathcal{S}^+_n$, and $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}(x)}$, we have

$$\|d_xT^n(\delta\xi, \delta\eta)\| \geq \zeta\|(\delta\xi, \delta\eta)\|,$$

[4] The definition of a regular subset can be found in Definition 6.1.

(b) for every $n \geq 1$, $x \in \mathcal{M} \setminus \mathcal{S}_n^-$, *and* $(\delta\xi, \delta\eta) \in \overline{\mathcal{C}'(x)}$, *we have*

$$\|d_x T^{-n}(\delta\xi, \delta\eta)\| \geq \zeta \|(\delta\xi, \delta\eta)\|.$$

(C3) (*Continuity of Lagrangian subspaces*) : *The ordered pair of transversal Lagrangian subspaces* $(L_1(x),\, L_2(x))$ *varies continuously in* $\operatorname{int}\mathcal{M}$.

(C4) (*Chernov – Sinai ansatz*) : *For* $\mu_{\mathcal{S}^\mp}$*-a.e.* $x \in \mathcal{S}^\mp$,

$$\lim_{n\to\pm\infty} Q(d_x T^n v) = \pm\infty,$$

for every $v \in \overline{\mathcal{C}(x)} \setminus \{0\}$, *if* $x \in \mathcal{S}^-$ *and for every* $v \in \overline{\mathcal{C}'(x)} \setminus \{0\}$, *if* $x \in \mathcal{S}^+$.

(C5) (*Proper alignment*) : *There exists* $\mathtt{M} \geq 0$, *such that for every* $x \in \mathcal{S}^+$ *resp.* $\mathcal{S}^-$, *we have* $d_x T^{-\mathtt{M}} v_x^+$ *resp.* $d_x T^{\mathtt{M}} v_x^-$ *belong to* $\overline{\mathcal{C}'(T^{-\mathtt{M}}x)}$ *resp.* $\overline{\mathcal{C}(T^{\mathtt{M}}x)}$, *where* v_x^+ *resp.* v_x^- *are the characteristic lines*[5] *of* $\mathcal{T}_x\mathcal{S}^+$ *resp.* $\mathcal{T}_x\mathcal{S}^-$.

Then, the open neighbourhood $\mathcal{U}(p)$ *is contained (mod 0) in one ergodic component.*

Chernov postulated in [Ch93] a weaker condition to verify the proper alignment condition (C5). Indeed, the proper alignment condition implies the transversality condition pointwise [HT20, Lemma 5.2].
Denote by $W^u(x)$ resp. $W^s(x)$ the unstable resp. stable manifold at point x.

(C5') (Transversality): For $\mu_{\mathcal{S}^\pm}$-a.e. x, the stable manifold $W^s(x)$ resp. unstable manifold $W^u(x)$ is transversal to $\mathcal{S}^-$ resp. $\mathcal{S}^+$.

Note that the transversality condition only has to hold on a set of full measure with respect to the measure $\mu_{\mathcal{S}^\pm}$.

5.2. **Abundance of sufficiently expanding points.** The notion of a sufficiently expanding point was given in Definition 5.1. Once local ergodicity is established we deduce that every ergodic component is (mod 0) open. One possibility to obtain a single ergodic component is

Theorem 5.1 (Abundance of sufficiently expanding points)**.** *The set of sufficiently expanding points has full measure and is arcwise connected.*

The abundance of sufficiently expanding points can be proven at once by requiring the strict unboundedness property (cf. Theorem 2.1), the transversality of $\mathcal{S}_m^-$, $\mathcal{S}^+$, for every $m \in \mathbb{N}$ and the explicit construction of the neighbourhood lying in one ergodic component from the LET in the beginning of Section 8 in [LW92].

Proof of Theorem 5.1. Recall that a point $x \in \mathcal{M}$ is sufficient if there exists a positive integer $\mathtt{N} = \mathtt{N}(x) > 0$, such that either (3) or (4) from Definition 5.1 are satisfied. Due to Theorem 2.1, Theorem 4.1 and Remark 4.1, $\sigma(d_xT^n)$ and $\sigma_{\mathcal{C}'}(d_xT^{-n})$ diverge to infinity for every $x \in \mathcal{M} \setminus \mathcal{D}$. Therefore, every orbit which experiences at most one singular collision satisfies either

(5) $\sigma(d_xT^{\mathtt{N}(x)}) > 3$, $T^k\mathcal{U} \cap \mathcal{S}^+ = \varnothing$, $0 \leq k \leq \mathtt{N}(x)$, or

(6) $\sigma_{\mathcal{C}'}(d_xT^{-\mathtt{N}(x)}) > 3$, $T^{-k}\mathcal{U} \cap \mathcal{S}^- = \varnothing$, $0 \leq k \leq \mathtt{N}(x)$.

[5] The characteristic line $v_x^\pm$ is a vector of $\mathcal{T}_x\mathcal{S}^\pm$ that has the property of annihilating every other vector $w \in \mathcal{T}_x\mathcal{S}^\pm$ with respect to the symplectic form ω, i.e. $\omega(v_x^\pm, w) = 0$, $\forall\, w \in \mathcal{T}_x\mathcal{S}^\pm$. Alternatively stated, it is the ω-orthogonal complement of $\mathcal{T}_x\mathcal{S}^\pm$. Note, that in symplectic geometry the ω-orthogonal complement of a codimension one subspace is one dimensional.

It follows that the only non-sufficient orbits lie in a subset of double singular collisions. The singularities $\mathcal{S}_m^-$ and $\mathcal{S}^+$ are transversal for every $m \in \mathbb{N}$, thus, the points of double singular collisions form a set of (at least) codimension two. Hence, there is an arcwise connected set of measure one such that the least expansion coefficient is larger than three. For the last part, the proof follows the beginning of Section 8 in [LW92]:
Without loss of generality assume that $\sigma_{\mathcal{C}'}(d_x T^{\mathtt{N}}) > 3$. We can choose a small enough neighbourhood $\mathcal{U}$ around the point x such that $T^{\mathtt{N}} : T^{-\mathtt{N}}\mathcal{U} \to \mathcal{U}$ is a diffeomorphism. This implies that $\mathcal{U} \cap \mathcal{S}_{\mathtt{N}}^- = \varnothing$ and $T^{-\mathtt{N}}\mathcal{U} \cap \mathcal{S}_{\mathtt{N}}^+ = \varnothing$. Further, the functional $y \mapsto \sigma(d_y T^{\mathtt{N}})$ is continuous on $\mathcal{U}$ and by making $\mathcal{U}$ smaller, if necessary, we obtain $\sigma(d_y T^{\mathtt{N}}) > 3$, for every $y \in T^{-\mathtt{N}}\mathcal{U}$. $\square$

6. Regularity of singularity sets

We give the definition of a regular set with the dimensionality adjusted to the system of falling balls. The LET requires that $\mathcal{S}_n^\pm$ are regular sets for every $n \in \mathbb{N}$.

Definition 6.1. A compact subset $X \subset \mathcal{M}$ is called *regular*, if

(1) $X = \bigcup_{i=1}^n I_i$, where I_i are compact subsets, with $I_i = \overline{\mathrm{int}(I_i)}$,
(2) $\dim I_i = 2N - 3$,
(3) $I_i \cap I_j \subset \partial I_i \cup \partial I_j$, $i, j = 1, \ldots, n$,
(4) $\partial I_i = \bigcup_{j=1}^m H_{i,j}$, where $\dim H_{i,j} = 2N - 4$ and $H_{i,j}$ is compact.

For the proof of the regularity of singular manifolds we will use the following slightly weaker version of [LW92, Lemma 7.7].

Lemma 6.1. *Assume the validity of*

(R1) $T : \ \mathcal{M} \setminus \mathcal{S}^+ \to \mathcal{M} \setminus \mathcal{S}^-$ *is a diffeomorphism,*
(R2) the transversality of singularity manifolds $\mathcal{S}_m^-$, $\mathcal{S}^+$ *for every* $m \in \mathbb{N}$.
(R3) the invariance of the cone field $\{\mathcal{C}(x) : \ x \in \mathcal{M}\}$,

then $\mathcal{S}_n^+$, $\mathcal{S}_n^-$, $n \geq 1$, *are regular subsets.*

Conditions (R1), (R3) have already been verified in Sections 3, 4. In the original version of [LW92, Lemma 7.7], the transversality condition (R2) is replaced by the proper alignment condition (C5). In the proof of [LW92, Lemma 7.7], condition (C5) is used to establish the (R2). Therefore, the proof of Lemma 6.1 can be carried out in exactly the same way as the proof of [LW92, Lemma 7.7].

7. Uniform lower bound of velocity differences

The investigation regarding a uniform lower bound of velocity differences $v_i^- - v_{i+1}^-$, for any $i \in \{1, \ldots, N-1\}$, is of main interest for the strict unboundedness property. Denote by $(i, i+1)$ the collision between ball i and ball $i+1$, i.e. when $q_i = q_{i+1}$. Let $x = x(t) \in \mathcal{M}_{i+1}$, $i \in \{1, \ldots N-1\}$. The velocity difference $v_i^-(t) - v_{i+1}^-(t)$ is non-negative and due to the collision laws (3.4), changes sign after the collision, i.e.

$$0 < v_i^-(t) - v_{i+1}^-(t) = -(v_i^+(t) - v_{i+1}^+(t)). \tag{7.1}$$

The Hamiltonian equations imply, that during free flight this quantity remains preserved (3.2). Using (3.2), (3.4), we see that the term $v_i^-(t) - v_{i+1}^-(t)$ is only affected by a $(i-1,i)$ resp. $(i+1,i+2)$ collision when being expanded backwards, i.e.

$$
\begin{aligned}
0 < v_i^-(t) - v_{i+1}^-(t) &= (1+\gamma_{i-1})(v_{i-1}^-(t_c) - v_i^-(t_c)) \\
&+ (v_i^-(t_c) - v_{i+1}^-(t_c)), \text{ resp.} \\
0 < v_i^-(t) - v_{i+1}^-(t) &= (1-\gamma_{i+1})(v_{i+1}^-(t_c) - v_{i+2}^-(t_c)) \\
&+ (v_i^-(t_c) - v_{i+1}^-(t_c)),
\end{aligned} \tag{7.2}
$$

where $t_c < t$ is the collision time of the $(i-1,i)$ resp. $(i+1,i+2)$ collision. Since we stopped our expansion right before a $(i-1,i)$ resp. $(i+1,i+2)$ collision, we have

$$
v_{i-1}^-(t_c) - v_i^-(t_c) > 0, \quad v_{i+1}^-(t_c) - v_{i+2}^-(t_c) > 0. \tag{7.3}
$$

Formula (7.2) can be generalized in the following way. Let $t_1 < t_2$ be collision times of two successful $(i,i+1)$ collisions and $m,n \in \mathbb{N}$. Assume that in between those two $(i,i+1)$ collisions we have m $(i-1,i)$ collisions and n $(i+1,i+2)$ collisions, with collision times $r_1,\ldots,r_m$ and $u_1,\ldots u_n$. Expanding only the $(i,i+1)$ velocity difference backwards without changing the appearing $(i-1,i)$ and $(i+1,i+2)$ velocity differences, we obtain for $i \geq 2$

$$
\begin{aligned}
0 &< v_i^-(t_2) - v_{i+1}^-(t_2) \\
&= (1+\gamma_{i-1}) \textstyle\sum_{j=1}^m \big(v_{i-1}^-(r_j) - v_i^-(r_j)\big) \\
&+ (1-\gamma_{i+1}) \textstyle\sum_{l=1}^n \big(v_{i+1}^-(u_l) - v_{i+2}^-(u_l)\big) \\
&+ v_i^+(t_1) - v_{i+1}^+(t_1).
\end{aligned} \tag{7.4}
$$

In between two $(1,2)$ collisions, we assume to have one floor collision, m full returns to the floor of the lowest ball and n $(2,3)$ collisions, again with collision times $r_1,\ldots,r_m$ and $u_1,\ldots u_n$. Expanding $(1,2)$ at t_2 backwards yields

$$
\begin{aligned}
0 &< v_1^-(t_2) - v_2^-(t_2) \\
&= 2 \textstyle\sum_{j=1}^m 2j v_1^+(r_j) \\
&+ (1-\gamma_2) \textstyle\sum_{l=1}^n \big(v_2^-(u_l) - v_3^-(u_l)\big) \\
&+ 2\sqrt{(v_1^+(t_1))^2 + 2q_1(t_1)} + v_1^+(t_1) - v_2^+(t_1).
\end{aligned} \tag{7.5}
$$

If there is at least one floor collision between two $(1,2)$ collisions, then the square root term $2\sqrt{(v_1^+(t_1))^2 + 2q_1(t_1)}$ appears in (7.5). The latter is part of the time the lowest ball needs to fall to the floor after a $(1,2)$ collision.[6] [7]
Remember, that

$$
v_1^+(r_j) > 0, \ \forall\, j \in \{1,\ldots,m\}, \tag{7.6}
$$

since this is the velocity of the first ball right after taking off from the floor.
At the heart of this work lies the following

---

[6]The exact time the lowest ball needs to fall to the floor is $v_1^+(t_1) + \sqrt{(v_1^+(t_1))^2 + 2q_1(t_1)}$.

[7]Note, that we can have a floor collision between two $(1,2)$ collision without a full return of the lowest ball to the floor, i.e. the square root term is present in (7.5) but $m = 0$ in the first sum.

Theorem 7.1. *For every energy value $c > 0$, there exists a constant $C = C(c) > 0$, such that for every $x \in \mathcal{M} \setminus \mathcal{D}$, and every $i \in \{1, \ldots, N-1\}$, there exists a divergent sequence of collision times $(\mathtt{t}_n)_{n\in\mathbb{N}} = (\mathtt{t}_n(x,i))_{n\in\mathbb{N}} : v_i^-(\mathtt{t}_n) - v_{i+1}^-(\mathtt{t}_n) \geq C$.*

Outline of the proof: We start describing a certain collision pattern. Since every collision happens infinitely often, this collision pattern can be found (non-uniquely) infinitely often in every orbit. The time interval of this pattern in the proof below is given by $[t_{-(1,2)}, t_{(1,2)}]$. At $t_{-(1,2)}$, $t_{(1,2)}$ we have a $(1,2)$ collision and somewhere in between is at least one $(0,1)$ collision.
We start investigating the implications of letting every pre-collisional velocity difference of $(1,2)$ collisions in $[t_{-(1,2)}, t_{(1,2)}]$ decrease. Using the above formulas (7.2) - (7.5) this will amount to having every ball position and every absolute value of each velocity decreasing at time $t_{-(1,2)}$.
Therefore, by choosing every $(1,2)$ velocity difference in $[t_{-(1,2)}, t_{(1,2)}]$ sufficiently small, this results in a contradiction since the energy of this orbit would be smaller than the energy of the system $c = H(q,p)$. Hence, the velocity difference $v_1^- - v_2^-$ of at least one $(1,2)$ collision in $[t_{-(1,2)}, t_{(1,2)}]$ is bounded from below.
Repeatedly using the above formulas, we obtain lower bounds for at least one velocity difference $v_i^- - v_{i+1}^-$, for every $i \in \{1, \ldots, N-1\}$. Since this collision pattern appears infinitely often along every orbit we can extend these considerations obtaining the result from Theorem 7.1.

Proof. Pick an arbitrary $(1,2)$ collision and mark the time as $t_{-(1,2)}$. Then, pick the next $(2,3)$ collision in the future and mark the time as $t_{-(2,3)}$. Continuing this procedure for the next $(3,4), \ldots (N-1,N)$ collisions, gives us collision times $t_{-(3,4)}, \ldots, t_{-(N-1,N)}$. After that we pick the first $(0,1)$ collision and mark its collision time with t_0. We now reverse the order of collisions after t_0 and mark the future collision times of the first consecutively appearing $(N-1,N), \ldots, (1,2)$ collisions as $t_{(N-1,N)}, \ldots, t_{(1,2)}$. Note, that in the intervals $[t_{-(i,i+1)}, t_{-(i+1,i+2)}]$, $i \in \{1, \ldots, N-2\}$, exactly one $(i+1, i+2)$ collision occurs, while in the interval $[t_0, t_{(N-1,N)}]$ resp. $[t_{(i,i+1)}, t_{(i-1,i)}]$, $i \in \{2, \ldots, N-1\}$, exactly one $(N-1,N)$ resp. $(i-1,i)$ collision occurs, but there is no restriction on other collisions happening.
The collision times of each $(i, i+1)$ collision, including floor collisions, induce a partition $\mathcal{P}_i$ of the time interval $[t_{-(1,2)}, t_{(1,2)}]$: For every $i \in \{2, \ldots N-1\}$, there exists a positive integer $n = n(i) \geq 2$, such that the collision times of all the $(i, i+1)$ collisions in the interval $[t_{-(i,i+1)}, t_{(i,i+1)}]$ are given by $s_{i,1}, \ldots, s_{i,n}$, with $s_{i,1} := t_{-(i,i+1)}$ and $s_{i,n} := t_{(i,i+1)}$. For $i = 1$, $n = n(1) \geq 0$, and by default $s_{1,0} = t_{-(1,2)}$, $s_{1,n+1} = t_{(1,2)}$. For $i = 0$, $s_{0,1}, \ldots, s_{0,n}$, $n = n(0) \geq 1$, are simply the collision times of the lowest ball with the floor in the open interval $(t_{-(1,2)}, t_{(1,2)})$.
We augment the collision time sequences by a first element $s_{i,0} := t_{-(1,2)}$ and a last element $s_{i,n+1} := t_{(1,2)}$, which yields the partitions $\mathcal{P}_i = \bigcup_{k=0}^{n} [s_{i,k}, s_{i,k+1}]$, for every $i \in \{0, \ldots, N-1\}$.
We will first observe how decreasing the $(1,2)$ velocity differences

$$0 < v_1^-(s_{1,k}) - v_2^-(s_{1,k}), \ \forall\, k \in \{0, \ldots, n(1)+1\}, \tag{7.7}$$

decreases the positions and absolute values of velocities in the interval $[t_{-(1,2)}, t_{(1,2)}]$. For the latter to take effect, we will first apply (7.5) to quantify how many $(2,3)$ collisions and floor returns of the lowest ball are in between two successful $(1,2)$ collisions. We introduce, for $i \in \{0, \ldots, N-1\}$, $j \in \{1, \ldots, N-1\}$, $k \in \{0, \ldots, n\}$, where $n = n(j)$, the functional

$$\begin{aligned} \mathsf{c}_i : \mathcal{P}_j &\to \mathbb{N} \\ [s_{j,k}, s_{j,k+1}] &\mapsto \mathsf{c}_i([s_{j,k}, s_{j,k+1}]) =: \mathsf{c}_{i,j,k}. \end{aligned}$$

The term $\mathsf{c}_{i,j,k}$ counts how many $(i, i{+}1)$ collisions appear in the interval $[s_{j,k}, s_{j,k+1}]$ of the partition $\mathcal{P}_j$, i.e. in between two successful $(j, j+1)$ collisions happening at time $s_{j,k}$ and $s_{j,k+1}$. Applying this notation, we expand the velocity differences in (7.7) backwards and according to (7.5) obtain for every $k \in \{1, \dots, n(1)+1\}$

$$\begin{aligned} 0 &\leq v_1^-(s_{1,k}) - v_2^-(s_{1,k}) \\ &= 2\textstyle\sum_{j=1}^{\mathsf{c}_{0,1,k}} 2jv_1^+(s_{0,g_0(j)}) \\ &+ (1-\gamma_2)\textstyle\sum_{l=1}^{\mathsf{c}_{2,1,k}} \big(v_2^-(s_{2,g_2(l)}) - v_3^-(s_{2,g_2(l)})\big) \\ &+ 2\sqrt{(v_1^+(s_{1,k-1}))^2 + 2q_1(s_{1,k-1})} + v_1^+(s_{1,k-1}) - v_2^+(s_{1,k-1}), \end{aligned} \tag{7.8}$$

where the functions $g_0(j) \in \{1, \dots, n(0)\}$ and $g_2(l) \in \{1, \dots, n(2)\}$ enumerate the collision times subindices. Using (7.8) together with (7.3), (7.6) and our assumption, that each $(1,2)$ velocity difference in (7.7) decreases, implies that the quantities

$$v_2^-(s_{2,k}) - v_3^-(s_{2,k}),\ \forall\ k \in \{1, \dots, n(2)\}, \tag{7.9a}$$

$$v_1^+(s_{0,k}),\ \forall\ k \in \{1, \dots, n(0)\}, \tag{7.9b}$$

$$v_1^+(s_{1,k-1}),\ q_1(s_{1,k-1}),\ \forall\ k \in \{1, \dots, n(1)+1\}, \tag{7.9c}$$

decrease as well. We repeat step (7.8), by expanding the remaining velocity differences $v_i^-(s_{i,k}) - v_{i+1}^-(s_{i,k})$, for all $i \in \{2, \dots, N-1\}$, $k \in \{2, \dots, n(i)\}$ backwards. Using again the assumption that every $(1,2)$ velocity difference in (7.7) decreases, its implication on (7.9a) and (7.3), (7.4), leads to the decrease of

$$v_i^-(s_{i,k}) - v_{i+1}^-(s_{i,k}),\ \forall\ i \in \{2, \dots, N-1\},\ k \in \{2, \dots, n(i)\}. \tag{7.10}$$

Every pre-collisional velocity difference $v_i^- - v_{i+1}^-$ occurring in $[t_{-(1,2)}, s_{i,1})$ resp. $(s_{i,n}, t_{(1,2)}]$ can be expanded forward resp. backward and by using (7.10) will decrease in the same way as the velocity differences before.
If the next ball to ball collision is $(i, i+1)$, $i \in \{1, \dots, N-1\}$, the collision time is given by

$$\frac{q_{i+1} - q_i}{v_i - v_{i+1}}.$$

If the denominator $v_i - v_{i+1}$ decreases, q_{i+1}, q_i, has to decrease as well, otherwise the collision time would be unbounded, which would result in arbitrarily large velocities and contradict the finite energy assumption. We conclude, that if every $(1,2)$ velocity difference in (7.7) decreases, then every ball to ball pre-collisional velocity difference, floor collision velocity and position, at time $t_{-(1,2)}$, decrease to a positive value, which can be made as small as needed. Due to the continuity of $H(q,p)$, we deduce that if every $(1,2)$ velocity difference in (7.7) becomes sufficiently small, our orbit would break through the constant energy surface. Hence, there exists a constant $C_1 > 0$ and at least one $k \in \{0, \dots, n(1)+1\}$, such that

$$v_1^-(s_{1,k}) - v_2^-(s_{1,k}) \geq C_1. \tag{7.11}$$

In order to obtain the existence of a constant $C > 0$ and at least one $(i, i+1)$ collision, such that $v_i^- - v_{i+1}^- \geq C$, for all $i \in \{1, \dots N-1\}$, we first pick the previous resp. next $(2,3)$ collision before resp. after the $(1,2)$ collision in (7.11). Let the past resp. future $(2,3)$ collision happen at t_p resp. t_f. Using (7.4) we expand $v_2^-(t_f) - v_3^-(t_f)$ backwards and obtain

$$\begin{aligned} 0 &< v_2^-(t_f) - v_3^-(t_f) \\ &= (1+\gamma_1) \textstyle\sum_{j=1}^m \big(v_1^-(r_j) - v_2^-(r_j)\big) \\ &+ (1-\gamma_3) \textstyle\sum_{l=1}^n \big(v_3^-(u_l) - v_4^-(u_l)\big) \\ &+ v_2^+(t_p) - v_3^+(t_p), \end{aligned}$$

where $r_1, \dots, r_m$ resp. $u_1, \dots, u_n$ are the collision times of the $(1,2)$ resp. $(3,4)$ collisions in between the two $(2,3)$ collisions occurring at times t_p, t_f. Note, that the reason we denoted these collision times as r_j resp. u_l (and not $s_{1,j}$ resp. $s_{3,l}$) is because one of the $(2,3)$ collisions may lie outside of $[t_{-(1,2)}, t_{(1,2)}]$. This depends on the position of the $(1,2)$ collision at time $s_{1,k}$ from (7.11).
Assuming that both $(2,3)$ velocity differences in the past and future are arbitrarily small yields a contradiction since $v_1^-(s_{1,k}) - v_2^-(s_{1,k}) \geq C_1$. Hence, there exists a constant $C_2 > 0$, such that either $v_2^-(t_f) - v_3^-(t_f) \geq C_2$ or $v_2^-(t_p) - v_3^-(t_p) \geq C_2$. Successfully continuing this procedure we find positive constants $C_1, \dots, C_{N-1} > 0$ and at least one $(i, i+1)$ collision, for all $i \in \{1, \dots, N-1\}$, such that

$$v_i^- - v_{i+1}^- \geq \min\{C_1, \dots, C_{N-1}\}. \tag{7.12}$$

It follows from the way we obtained (7.12), that the collision times of all $(i, i+1)$ collisions satisfying (7.12) do not necessarily belong to $[t_{-(1,2)}, t_{(1,2)}]$.
The above steps can be repeated, thus, creating infinitely many compact intervals with a sequence of constant positive lower bounds for at least one $v_1^- - v_2^-$ per compact interval. This holds along every orbit. Those lower bounds have a global minimum, i.e.

$$\min_{x \in \mathcal{M}} \min_{n \in \mathbb{N}} \; v_1^-(\mathtt{t}_n(x,1)) - v_2^-(\mathtt{t}_n(x,1))$$

exists. Otherwise the velocities in (7.10) would decrease to a value small enough causing energy loss, hence, a contradiction. For this global lower bound we can repeat the steps from the last paragraph to obtain a global lower bound, say $C > 0$, for every pre-collisional velocity difference. □

8. Strict unboundedness

Liverani and Wojtkowski formulated in [LW92, Theorem 6.8] equivalent conditions to the strict unboundedness property, that we state again here.

Theorem 8.1. *For every $x \in \mathcal{M}$, the sequence $(d_{T^n x}T)_{n\in\mathbb{N}}$ is strictly unbounded if and only if*

$$\text{(8.1a)} \qquad \textit{The sequence } (d_{T^n x}T)_{n\in\mathbb{N}} \textit{ is unbounded.}$$

$$\text{(8.1b)} \qquad \begin{aligned} &\textit{There exist } k_1, k_2 \in \mathbb{N}, \textit{ such that } Q(d_xT^{k_1}(\delta\xi, 0)) > 0 \\ &\textit{and } Q(d_xT^{k_2}(0, \delta\eta)) > 0, \textit{ for all } (\delta\xi, 0) \in L_1, (0, \delta\eta) \in L_2. \end{aligned}$$

Point (8.1b) is equivalent to d_xT being eventually strictly Q-monotone, for every $x \in \mathcal{M}$ [LW92]. We will prove strict unboundedness for every $x \in \mathcal{M} \setminus \mathcal{D}$ and split its proof into two parts: First, we prove that (8.1a) is equivalent to point (8.1b) and, second, we establish the validity of point (8.1a).

8.1. **Prerequisites.** It is immanent for our goal to investigate how the Q-form behaves when undergoing a floor or ball to ball collision. Using the collision maps (4.6) and the definition of the Q-form (4.5), we obtain for any $i \in \{1, \dots, N-1\}$

$$Q(d_x\Phi_{0,1}(\delta\xi, \delta\eta)) = Q(\delta\xi, \delta\eta) + \beta\delta\xi_1^2, \tag{8.2a}$$

$$Q(d_x\Phi_{i,i+1}(\delta\xi, \delta\eta)) = Q(\delta\xi, \delta\eta) + \alpha_i\delta\eta_i^2, \tag{8.2b}$$

where the coefficients α_i, β were defined in (4.7).
Let $(t_n)_{n\in\mathbb{N}} = (t_n(x))_{n\in\mathbb{N}}$ be the sequence of future collision times starting at x. For $n \in \mathbb{N}$, let $\mathtt{b}_{x,n}$ count the ball to ball collisions, starting from x up to the n-th iterate $T^n x$ and let, similarly, $\mathtt{f}_{x,n}$ count the $(0,1)$ collisions starting from x up to the n-th iterate $T^n x$.
Using estimates (8.2a), (8.2b), we can expand the Q-form of the n-th iterate backwards, i.e.

$$\begin{aligned} Q(d_xT^n(\delta\xi, \delta\eta)) = Q(\delta\xi, \delta\eta) &+ \textstyle\sum_{k=1}^{\mathtt{b}_{x,n}} \alpha_{p_1(k)}(t_{p_2(k)})\delta\eta_{p_1(k)}^2(t_{p_2(k)}) \\ &+ \textstyle\sum_{m=1}^{\mathtt{f}_{x,n}} \beta(t_{p_2(m)})\delta\xi_1^2(t_{p_2(m)}), \end{aligned} \tag{8.3}$$

where $p_1 : \mathbb{N} \to \{1, \dots, N\}$ gives the subindex component of the vector element and $p_2 : \mathbb{N} \to \mathbb{N}$ gives the collision time index.
On $\mathbb{R}^{N-1}$, we define the norm

$$\|\delta\eta\|_{CW}^2 = \sum_{i=1}^{N-2} \frac{(\delta\eta_{i+1} - \delta\eta_i)^2}{m_i}.$$

It was introduced by Cheng and Wojtkowski in [ChW91, (11)]. It is invariant with respect to the submatrices D_i, D_i^T of the ball to ball collision map derivatives given in (4.6), which implies that in a ball to ball collision $\|\cdot\|_{CW}$ is preserved.

8.2. **Equivalence.** This subsection contains the proof of the equivalence of points (8.1a), (8.1b). Recursively define $(\delta\xi_n, \delta\eta_n) = dT(\delta\xi_{n-1}, \delta\eta_{n-1})$, with $(\delta\xi_0, \delta\eta_0) = (\delta\xi, \delta\eta)$ and $x_n = T^n x$. We recall some results from [S96], which we translate from $(\delta h, \delta v)$ to $(\delta\xi, \delta\eta)$-coordinates. Consider the neutral space

$$\mathcal{N}_x = \{(\delta\xi, 0) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1} : \ Q(d_xT^n(\delta\xi, 0)) = 0, \ \forall\, n \geq 0\}.$$

This space contains all the vectors of $\mathbb{R}^{N-1} \times \{\vec{0}\}$, for which Q remains zero along the whole orbit starting at x. Let $t_{(0,1)+k} = t_{(0,1)+k}(x)$, $k \in \mathbb{N}$, denote the collision time of the $(k+1)$-st floor collision time in the future. It was explained in [W90a, S96], that $(\delta\xi, \delta\eta) \in \mathcal{N}_x$ if and only if $\delta\xi_1(t_{(0,1)+k}) = 0$, for every $k \in \mathbb{N}$. In [S96, Sublemma 3.8], Simányi introduced the norm

$$\|\delta\xi\|_S^2 = \langle A^T DA\delta\xi, \delta\xi\rangle,$$

where $D = \mathrm{diag}(m_1, \ldots, m_N)$ is the diagonal matrix with masses as diagonal elements and A is the matrix introduced in (4.3). This norm is point independent and time invariant for every vector of the neutral space. Thus, for every $(\delta\xi_n, 0) \in \mathcal{N}_{x_n}$, there exists a constant positive value $d = d\big((\delta\xi_n, 0)\big) \in \mathbb{R}_+$, such that

$$\|\delta\xi_n\|_S^2 = d, \ \forall\ n \geq 0. \tag{8.4}$$

With this information we can prove the equivalence of the unboundedness condition (8.1a) and the eventual strict Q-monotonicity (8.1b).

Theorem 8.2. *The sequence $(d_{T^n x}T)_{n\in\mathbb{N}}$ is unbounded for $x \in \mathcal{M}$ if and only if the map d_xT is eventually strictly Q-monotone for $x \in \mathcal{M}$.*

Proof. If d_xT is eventually strictly Q-monotone then $\mathcal{N}_x = \{\emptyset\}$ and therefore there exists a subsequence $(k_n)_{n\in\mathbb{N}}$, such that

$$\delta\xi_1(t_{(0,1)+k_n}) \neq 0, \ \forall\ n \in \mathbb{N}. \tag{8.5}$$

Assume that $(d_{T^n x}T)_{n\in\mathbb{N}}$ is not unbounded, i.e. there exists $x \in \mathcal{M}$, $(\delta\xi, \delta\eta) \in \mathcal{C}(x)$ and $E > 0$, such that

$$\lim_{n\to\infty} Q(d_xT^n(\delta\xi, \delta\eta)) = E. \tag{8.6}$$

Note, that the limit E must be positive, since $(\delta\xi, \delta\eta)$ lies in the interior of the cone and the Q-form is increasing on every orbit.
When the orbit experiences a $(0,1)$ collision the value of the Q-form changes according to (8.2a). If $v_{\max} > 0$ is the maximal possible velocity of a ball, then β has a uniform lower bound, i.e.,

$$\beta = -2/m_1 v_1^- > 2/m_1 v_{\max} =: \beta_{min}. \tag{8.7}$$

Using Q-monotonicity, in combination with estimate (8.2a) and (8.7), we can modify formula (8.3) in order to obtain the estimate

$$Q(d_xT^n(\delta\xi, \delta\eta)) > Q(\delta\xi, \delta\eta) + \beta_{\min} \sum_{k=1}^{\mathtt{f}_{x,n}} \delta\xi_1^2(t_{(0,1)+k}). \tag{8.8}$$

We follow from (8.6) and (8.8), that the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\mathtt{f}_{x,n}} \delta\xi_1^2(t_{(0,1)+k})$$

exists, which necessarily implies that

$$\lim_{k\to\infty} \delta\xi_1(t_{(0,1)+k}) = 0.$$

But this contradicts (8.5) and therefore $(d_{T^n x}T)_{n\in\mathbb{N}}$ is unbounded.
Assume now that $(d_{T^n x}T)_{n\in\mathbb{N}}$ is unbounded along an orbit where d_xT is not eventually strictly Q-monotone, i.e. $\mathcal{N}_x \neq \{\emptyset\}$. Every vector in $v \in \mathcal{C}(x)$ can be uniquely decomposed into $v = v_1 + v_2$, where $v_i \in \mathbf{L}_i$, $i = 1, 2$. Consider the projection

$$\begin{aligned} \pi_1 : \mathcal{C}(x) &\to \mathbf{L}_1, \\ v &\mapsto \pi_1(v) = v_1. \end{aligned}$$

Due to our assumption and since $\mathcal{N}_x \subseteq \mathbf{L}_1$, we have $\pi_1(v) \in \mathcal{N}_x$. Using the Cauchy-Schwarz inequality and (8.4), we estimate

$$\begin{aligned} Q(d_xT^n(\delta\xi,\delta\eta)) = \langle \delta\xi_n, \delta\eta_n\rangle &\leq \|\delta\xi_n\|_2\|\delta\eta_n\|_2 \\ &\leq \mathtt{C}_1\mathtt{C}_2\|\delta\xi_n\|_S\|\delta\eta_n\|_{CW} = \mathtt{C}_1\mathtt{C}_2 d\|\delta\eta_n\|_{CW}, \end{aligned}$$

where $\mathtt{C}_1$, $\mathtt{C}_2$ are constants arising from the equivalence of norms. Letting $n \to \infty$ in the previous inequality and using the validity of unboundedness yields

$$\lim_{n\to\infty} \|\delta\eta_n\|_{CW} = \infty.$$

In a ball to ball collision the value of the norm $\|\cdot\|_{CW}$ does not change. Thus, $\|\delta\eta_n\|_{CW}$ can only grow in a floor collision. From (4.6), we obtain

$$\delta\xi_i^+ = \delta\xi_i^-, \quad \delta\eta_j^+ = \delta\eta_j^-, \quad \delta\eta_1^+ = \beta\delta\xi_1^- + \delta\eta_1^-, \tag{8.9}$$

where $i \in \{1,\ldots,N\}$, $j \in \{2,\ldots,N\}$. Therefore, $\|\delta\eta_n\|_{CW}$ increases if and only if $\delta\xi_1^- \neq 0$. This contradicts our assumption $\mathcal{N}_x \neq \{\emptyset\}$, which implies that d_xT is eventually strictly Q-monotone. ☐

8.3. **Unboundedness.** For later estimates we want to combine the subsequences obtained from Theorem 7.1 into one.

Definition 8.1. For every $x \in \mathcal{M} \setminus \mathcal{D}$, we define the subsequence $(\mathtt{s}_n)_{n\in\mathbb{N}} = (\mathtt{s}_n(x))_{n\in\mathbb{N}}$ as the union of subsequences $(\mathtt{t}_n(x,i))_{n\in\mathbb{N}}$ over $i \in \{1,\ldots,N\}$, where the elements are numbered consecutively with respect to the order of occurrence in time.

To establish unboundedness, we need to prove property (8.1a) of Theorem 8.1, namely,

$$\lim_{n\to\infty} Q(d_xT^n(\delta\xi,\delta\eta)) = \infty, \ \forall\, x \in \mathcal{M}, \ \forall\, x \in \mathcal{C}(x). \tag{8.10}$$

We remind the reader, that the subtle difference to strict unboundedness is that the divergence in (8.10) needs to hold only for every vector from the interior of the cone.
At collision times $(\mathtt{t}_n(x,i))_{n\in\mathbb{N}}$, obtained from Theorem 7.1, the coefficient α_i in (8.2b), has the uniform lower bound

$$\Lambda := \min_{\{1,\ldots,N-1\}} \frac{2m_i m_{i+1}(m_i - m_{i+1})}{(m_i + m_{i+1})^2} C. \tag{8.11}$$

We use the lower bound (8.11) of α_i along the subsequence $(\mathtt{t}_n(x,i))_{n\in\mathbb{N}}$ together with Q-monotonicity and subsequence $(\mathtt{s}_n(x))_{n\in\mathbb{N}}$ defined in Definition 8.1, to modify (8.8) into the estimate

$$Q(d_xT^n(\delta\xi,\delta\eta)) > Q(\delta\xi,\delta\eta) + \Lambda \sum_{k=1}^{\mathtt{b}_{x,n}} \delta\eta_{p_1(k)}^2(\mathtt{s}_k) \tag{8.12}$$

where $p_1 : \mathbb{N} \to \{1,\ldots,N\}$ gives the subindex component of the vector element.

Theorem 8.3. *The sequence $(d_{T^n x}T)_{n\in\mathbb{N}}$ is unbounded for every $x \in \mathcal{M} \setminus \mathcal{D}$.*

Proof. Assume on the contrary that (8.10) is violated, i.e. there exists $E > 0$, $x \in \mathcal{M}$, $(\delta\xi,\delta\eta) \in \mathcal{C}(x)$, such that

$$\lim_{n\to\infty} Q(d_xT^n(\delta\xi,\delta\eta)) = E. \tag{8.13}$$

We deduce from (8.12) and (8.13), that the limit

$$\lim_{n\to\infty}\sum_{k=1}^{\mathtt{b}_{x,n}}\delta\eta^2_{p_1(k)}(\mathtt{s}_k)$$

exists, which necessarily implies that

$$\lim_{n\to\infty}\delta\eta_{p_1(n)}(\mathtt{s}_n)=0. \tag{8.14}$$

Since every collision happens infinitely often, every component of the $\delta\eta$-part of the vector $(\delta\xi,\delta\eta)$ converges to zero along the subsequence $(\mathtt{t}_n(x,i))_{n\in\mathbb{N}}$, i.e.

$$\lim_{n\to\infty}\delta\eta_i(\mathtt{t}_n(x,i))=0,\ \forall\ i\in\{1,\dots,N\}. \tag{8.15}$$

It follows from our assumption and Theorem 8.2 that along the orbit used in (8.13), d_xT is not eventually strictly Q-monotone, i.e. $\delta\xi_1(t_{(0,1)+k})=0$, for every $k\in\mathbb{N}$. Point (8.9) and the invariance of $\|\cdot\|_{CW}$ in ball to ball collisions imply that the sequence $\|\delta\eta_n\|_{CW}$ remains constant, i.e. $\|\delta\eta_n\|_{CW}=\|\delta\eta\|_{CW}$, for every $n\in\mathbb{N}$. Since $(\delta\xi,\delta\eta)$ lies in the interior of the cone, it follows that $\|\delta\eta\|_{CW}>0$. But this contradicts (8.15), thus, $(d_{T^nx}T)_{n\in\mathbb{N}}$ is unbounded for every $x\in\mathcal{M}\setminus\mathcal{D}$. □

8.4. **Chernov-Sinai ansatz.** For the Chernov-Sinai ansatz to be valid, we need that the orbit for $\mu_{\mathcal{S}^\pm}$-a.e. $x\in\mathcal{S}^\pm$ emerging from the sinuglarity manifold is strictly unbounded. Due to the strict unboundedness property (cf. Theorem 2.1) being valid for every $x\in\mathcal{M}\setminus\mathcal{D}$ this certainly holds.

9. Transversality

We know [BBNV12, HT19], that arbitrarily many $(0,1)$ collisions can occur in finite time. This is why we prefer to formulate the subsequent results regarding the transversality condition (C5') in terms of the flow, because we rather deal with finite times than arbitrarily many derivative map compositions.

In [LW92], it was verified that every point of the floor collision singularity manifolds is properly aligned and, therefore, the transversality condition holds for these singularity manifolds. It remains to prove the latter for the triple collision singularity manifolds only.

We will restrict the discussion and proof of the transversality condition to triple singularity manifolds $\mathcal{S}^-$ only since the former is completely equivalent for $\mathcal{S}^+$.

The proof of the transversality condition relies on a compactness argument which utilizes

- (K1) the strict unboundedness property (cf. Theorem 2.1), for every $m\in\mathbb{N}$.
- (K2) the transversality of singularity manifolds $\mathcal{S}^-$, $\mathcal{S}^+_m$
- (K3) the continuity of $x\mapsto Q(d_x\phi^t v_x^-)$ for every $t\geq 0$, $x\in\mathcal{S}^-\setminus\mathcal{D}$.

The characteristic line v_x^- in (q,p)-coordinates for the triple collision singularity is given by

$$\begin{aligned} v_x^- = \Big\{(\delta q,\delta p)\in\mathcal{T}_x\mathcal{S}^- : \ & \delta q_1=\ldots=\delta q_N=0,\\ & \delta p_i=\delta p_{i+2}\big((\frac{p_{i+2}}{m_{i+2}}-\frac{p_i}{m_i})(\frac{p_{i+1}}{m_{i+1}}-\frac{p_i}{m_i})^{-1}-1\big),\\ & \delta p_{i+1}=\delta p_{i+2}(\frac{p_{i+2}}{m_{i+2}}-\frac{p_i}{m_i})(\frac{p_{i+1}}{m_{i+1}}-\frac{p_i}{m_i})^{-1},\\ & \delta p_j=0, j\notin\{i,i+1,i+2\}\Big\}.\end{aligned}$$

Due to the velocity restriction in $\mathcal{M}$, $(q,p) \mapsto v^-_{(q,p)}$ is a continuous function and this remains true when switching to (h,v) or (ξ,η)-coordinates via the symplectomorphisms given in (4.2), (4.3).
Our strategy for the proof of (C5'), is to prove the proper alignment condition on a compact set K of full $\mu_{\mathcal{S}^-}$-measure, which then implies the transversality condition.

Lemma 9.1. *The transversality condition (C5') holds.*

Proof. As a result of (K2), the regularity of singularity manifolds (C1) holds (cf. Lemma 6.1), and therefore, non-double singular points $\mathcal{S}^- \setminus \mathcal{D}$ form an open set of full $\mu_{\mathcal{S}^-}$-measure. On this set, the derivative $d\phi^t$ is defined for every $t \geq 0$. Since $\mu_{\mathcal{S}^-}$ is a regular measure, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathcal{S}^- \setminus \mathcal{D}$, such that

$$\mu_{\mathcal{S}^-}(K) \geq 1 - \varepsilon. \tag{9.1}$$

Consider the set

$$A = \{x \in K : \; Q(v^-_x) \geq 0\}$$

of all properly aligned points in K. This set is closed and due to $A \subset K$ compact. Let $\partial B_{\|\cdot\|}(0,1)$ be the compact ball of unit radius, with respect to the norm $\|\cdot\|$, in tangent space. Since Q is homogeneous (of degree two), we can restrict, without loss of generality, the characteristic line v^-_x to the compact unit circle, i.e.

$$v^-_x \in \partial B_{\|\cdot\|}(\vec{0},1), \; \forall \; x \in K.$$

Due to the strict unboundedness property (K1) and the compactness of A, $\overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(\vec{0},1)$, we can find for every constant $E_0 > 0$, a time $\mathtt{T}_0 = \mathtt{T}_0(E_0) > 0$, such that for all $x \in A$, $v^-_x \in \overline{\mathcal{C}(x)} \cap \partial B_{\|\cdot\|}(\vec{0},1)$, we have $Q(d_x\phi^{\mathtt{T}_0} v^-_x) \geq E_0$. Otherwise this would contradict the strict unboundedness property (K1). From the continuity of $x \mapsto Q(d_x\phi^{\mathtt{T}_0} v^-_x)$ (K3), we obtain the existence of a constant $E_1 \in (0, E_0]$ and a set $N_1 = \{x \in K : \; Q(d_x\phi^{\mathtt{T}_0} v^-_x) \geq E_1\}$, such that $\mu_{\mathcal{S}^-}(N_1 \setminus A) > 0$. Due to strict unboundedness (K1), we can find a uniform time $\mathtt{T}_1 \geq \mathtt{T}_0$ on the compact set N_1, such that $N_1 = \{x \in K : \; Q(d_x\phi^{\mathtt{T}_1} v^-_x) \geq E_0\}$. Now we repeat the above argument beginning with N_1 in place of A. Since K is compact, we obtain finitely many constants $\mathtt{T}_0 \leq \ldots \leq \mathtt{T}_L$ and sets $N_1 \subset \ldots \subset N_L$, such that $N_L = K$, where $N_L = \{x \in K : \; Q(d_x\phi^{\mathtt{T}_L} v^-_x) \geq E_0\}$, i.e. every point in K becomes properly aligned after $\mathtt{T}_L$ time.
For the sake of convenience of the reader we formulate the proof details of the last statement explicitly: Assume on the contrary, that there exists a point $y \in K$, such that $Q(d_y\phi^t v^-_y) < 0$, for every $t \geq 0$. Due to the continuity of $y \mapsto Q(d_y\phi^t v^-_y)$ (K3), there exists an open neighbourhood V around y satisfying $V = K \setminus N_L$ and $Q(d_z\phi^t v^-_z) < 0$, for every $t \geq 0$, $z \in V$. Using the definition of N_L and continuity (K3) again, we arrive at the contradiction $N_L \cap V \neq \emptyset$.
Since the proper alignment condition implies Chernov's transversality condition pointwise and due to (9.1), we derive Chernov's transversality condition for a set K of full $\mu_{\mathcal{S}^-}$-measure. □

References

[BBNV12] Bálint, P., Borbély, G., Némedy Varga, A.; *Statistical properties of the system of two falling balls*, Chaos 22, 026104 (2012)

[BChSzT02] Bálint, P., Chernov, N., Szász, D., Tóth, I.P.; *Multi-dimensional semi-dispersing billiards: singularities and the fundamental theorem*, Annales Henri Poincaré 3, 451-482 (2002)

[ChW91] Cheng, J., Wojtkowski, M.; *Linear stability of a periodic orbit in the system of falling balls*, appeared in The Geometry of Hamiltonian Systems (ed. Tudor Ratiu), MSRI Publications, Springer Verlag, 53-71 (1991)

[Ch91] Chernov, N.; *The ergodicity of a Hamiltonian system of two particles in an external field*, Physica D, Volume 53, Issues 2–4, 233-239 (1991)

[Ch93] Chernov, N.; *Local ergodicity in hyperbolic systems with singularities*, Functional Analysis & Applications, 27:1 (1993)

[ChS87] Chernov, N., Sinai, Y.; *Ergodic properties of some systems of two-dimensional discs and three-dimensional spheres*, Russ. Math. Surv. 42:3, 181-207 (1987)

[HT19] Hofbauer-Tsiflakos, M.; *Towards the ergodicity of three falling balls*, PhD thesis (2019)

[HT20] Hofbauer-Tsiflakos, M.; *Certain systems of three falling balls satisfy the Chernov-Sinai ansatz*, Studia Mathematica 250 163-192 (2020)

[HT21] Hofbauer-Tsiflakos, M.; *A conditional proof of the non-contraction properties for N falling balls*, *In preparation* (2021)

[KS86] Katok, A., Strelcyn, J.-M.; *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lecture Notes in Mathematics, Volume 1222, Springer-Verlag (1986)

[KSSz90] Krámli, A., Simányi, N., Szász, D.; *A transversal fundamental theorem for semi-dispersing billiards*, Communications in Mathematical Physics, Volume 129, Issue 3, 535–560 (1990)

[LM87] Libermann, P., Marle C-M.; *Symplectic geometry and analytical mechanics*, D. Reidel Publishing Company (1987)

[LW92] Liverani, C., Wojtkowski, M.; *Ergodicity of Hamiltonian systems*, Dynamics Reported 27:1, 130-202 (1995)

[S70] Sinai, Y. *Dynamical systems with elastic reflections*, Russian Math. Surveys 25, 137-189 (1970)

[S96] Simányi, N.; *The characteristic exponents of the system of falling balls*, Communications in Mathematical Physics, Volume 182, Issue 2, 457-468 (1996)

[S21] Simányi, N.; *Toward the Proof of Wojtkowski's Ergodicity Conjecture*, *In preparation* (2021)

[W85] Wojtkowski, M.; *Invariant families of cones and Lyapunov exponents*, Ergodic Theory and Dynamical Systems, 5, 145-161 (1988)

[W88] Wojtkowski, M.; *Measure theoretic entropy of the system of hard spheres*, Ergodic Theory and Dynamical Systems, 8, 133-153 (1988)

[W90a] Wojtkowski, M.; *A system of one dimensional balls with gravity*, Communications in Mathematical Physics, Volume 126, Issue 3, 507-533 (1990)

[W90b] Wojtkowski, M.; *The system of one-dimensional balls in an external field. II*, Communications in Mathematical Physics, Volume 127, Issue 2, 425-432 (1990)

[W91] Wojtkowski, M.; *Systems of classical interacting particles with nonvanishing Lyapunov exponents*, in: Lyapunov Exponents (Oberwolfach 1990), L. Arnold, H. Crauel and J.-P. Eckmann (eds.), Lecture Notes in Math. 1486, Springer, 243-262 (1991)

[W00] Wojtkowski, M.; *Monotonicity, J-Algebra of Potapov and Lyapunov Exponents*, Proceedings of Symposia in Pure Mathematics, Volume 69 (2000)

Michael Hofbauer-Tsiflakos

Current address: UAB, University Hall 4037, 1402 10th Ave S, 35294 Birmingham, AL

Email address: `hofbauer@uab.edu`

URL: `https://people.cas.uab.edu/~hofbauer/`