ON A LIMIT-CIRCLE CRITERION FOR SECOND-ORDER DIFFERENTIAL OPERATORS

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1. Introduction

We consider formally self-adjoint formal differential operators of the form

\[ \tau y(t) = (p(t)y'(t))' + q(t)y(t), \]

(1)

where \( p^{-1}(t) \) and \( q(t) \) are real-valued functions defined and locally Lebesgue integrable on the half-open interval \([a, b), b < \infty\) (i.e. integrable on compact subsets of \([a, b))\). Such an operator is said to be of limit-circle type at \( b \) if every solution of the differential equation \( \tau y(t) = 0 \) is in \( L^2[a, b) \), the space of functions square integrable on the interval \([a, b)\); otherwise \( \tau \) is said to be of limit-point type at \( b \). For convenience, we shall in the remaining text use the symbols LC and LP to denote the phrases 'of limit-circle type' and 'of limit-point type' respectively.

In their book (2), Dunford and Schwartz prove (p. 1409 Theorem 20) that the condition

\[ \int_a^b p^{-\frac{1}{4}}(t)q^{-\frac{1}{4}}(t) \, dt < \infty \]

(2)

is both necessary and sufficient for \( \tau \) to be LC at \( b \), provided that \( p(t) \) and \( q(t) \) are positive in \([a, b)\) and that

\[ \int_a^b |D(p, q)(t)| \, dt < \infty, \]

where

\[ D(p, q) = \frac{1}{4}(pq')^2p^{-\frac{3}{4}}q^{-\frac{3}{4}} + \{(pq)'p^{-\frac{3}{4}}q^{-\frac{1}{4}}\}'. \]

(3)

We shall prove that (2) is just one member of a family of conditions which are necessary and sufficient for limit-circle type, provided always that the functions \( p(t) \) and \( q(t) \) satisfy certain additional requirements in each case (see for example conditions (7) and (8)). The result of Pavlyuk in (5) is also generalized.

2. Preliminaries

We begin by stating, without proof, the comparison criteria of (4) and (1).
Consider the two formal operators \( \tau_1 \) and \( \tau_2 \) defined by
\[
\tau_1 x(s) = (P(s)x'(s))' + Q_1(s)x(s), \\
\tau_2 x(s) = (P(s)x'(s))' + Q_2(s)x(s),
\]
where \( P^{-1}(s), Q_1(s), \) and \( Q_2(s) \) are real-valued functions defined and locally Lebesgue integrable on \( I \).

**Lemma 1** (Halvorsen). If \( |Q_1 - Q_2| \leq x \in L^2(I) \) for every solution \( x(s) \) of the differential equation \( \tau_1 x(s) = 0 \), then \( \tau_1 \) is LC at an end-point of \( I \) if and only if \( \tau_2 \) is also LC at that end-point.

**Lemma 2** (Bellman). If \( |Q_1 - Q_2| \) is bounded almost everywhere on \( I \) then, as before, \( \tau_1 \) is LC at an end-point of \( I \) if and only if \( \tau_2 \) is LC at that end-point.

**Remark.** These results were originally given for the case \( P(s) = 1 \); the extensions to the general case are straightforward and will be omitted. They are also valid when \( P(s), Q_1(s), \) and \( Q_2(s) \) are complex-valued functions [cf. (4)].

As a further preliminary we must consider some work on isometric transformations of these differential operators. The following results are taken from (2) 1498. Assume, for example, that the formal operator \( \tau \) acts on a space of functions \( \mathcal{D} \) and that we transform \( \tau \) to the operator \( \hat{\tau} = U\tau U^{-1} \) (which acts on the space \( U(\mathcal{D}) \)), by means of the variable change \( t = h(s), h'(s) > 0 \), and the corresponding isometric transformation of functions,
\[
(Uf)(s) = f(h(s))(h'(s))^{\frac{1}{2}}.
\]
It can be shown that if \( h(s) \) is the inverse of the function
\[
s(t) = \int_a^t \psi^2(r) \, dr,
\]
then, for \( x(s) \in U(\mathcal{D}) \) and \( s \in [h^{-1}(a), h^{-1}(b)] \),
\[
\hat{\tau} x(s) = (P(s)x'(s))' + Q(s)x(s),
\]
where
\[
P(s) = p(h(s))\psi^4(h(s)), \\
Q(s) = q(h(s)) + \phi(h(s)), \\
\phi(t) = \psi^{-1}(t)(p(t)\psi'(t))'.
\]
Furthermore, \( \tau \) is LC at \( b \) if and only if \( \hat{\tau} \) is LC at \( h^{-1}(b) \).

### 3. The main result

**Theorem.** If \( p'(t) \) and \( q'(t) \) are locally absolutely continuous on \([a, b]\), and \( m \) and \( n \) are real numbers such that \( p^{-m}(t)q^{-n}(t) \) and \( p^{m-1}(t)q^n(t) > 0 \)
are locally Lebesgue integrable on \([a, b]\), and the function \(\alpha_{m,n}(t)\) defined by

\[
\alpha_{m,n} = \left[\frac{(m-1)p'q+npq}{4q}\right]' + p^{-1}\left[\frac{(m-1)p'q+npq}{4q}\right]^2
\]  

(6)

satisfies either

\[
|q(t) - p^{-m}(t)q^{-n}(t) + \alpha_{m,n}(t)| \leq K < \infty \quad \text{a.e. on } [a, b)
\]  

(7)

or

\[
\int_a^b p^{2(m-1)}(t)q^{2n}(t)|q(t) - p^{-m}(t)q^{-n}(t) + \alpha_{m,n}(t)| \, dt < \infty,
\]  

(8)

then the operator \(\tau\) is LC at \(b\) if and only if

\[
\int_a^b p^{2(m-1)}(t)q^{2n}(t) \, dt < \infty.
\]  

(9)

**Proof.** We use the transformation given (via equation (3)) by

\[
\phi(t) = \psi^{-1}(t)(p(t)\psi'(t))'
\]  

where \(\psi(t)\) is defined to be \(\{\psi^2(t)\}^\frac{1}{2}\). Since we have not assumed that \(p(t)\) and \(q(t)\) are positive, we proceed as follows:

\[
\phi(t) = (\psi^{-2})^{-1}\left\{p((\psi^2)')'\right\}
\]  

(10)

since

\[
2p.(\psi^2)^{-1} . (\psi^2)' = q^{-1}\{(m-1)p'q+npq\}'.
\]

The variable change \(t \rightarrow s\), where \(t = h(s)\), is given by the inverse of the function

\[
s(t) = \int_a^t p^{2(m-1)}(r)q^{2n}(r) \, dr,
\]  

(11)

It follows from (5) and (10) that the operator \(\tau\) is transformed to the operator \(\hat{\tau}\) defined by

\[
\hat{\tau}x(s) = \{p^{m}(h(s))q^{n}(h(s))x'(s)\}' + \{q(h(s)) + \alpha_{m,n}(h(s))\}x(s)
\]

on the interval \(J = [h^{-1}(a), h^{-1}(b)]\). On the same interval we define

\[
\hat{\eta}x(s) = \{p^{m}(h(s))q^{n}(h(s))x'(s)\}' + p^{-m}(h(s))q^{-n}(h(s))x(s)
\]

and note that the functions

\[
u(s) = \sin \int_c^s p^{-m}(h(r))q^{-n}(h(r)) \, dr,
\]

\[
u(s) = \cos \int_c^s p^{-m}(h(r))q^{-n}(h(r)) \, dr
\]
form a fundamental set of solutions for $\hat{\eta}x(s) = 0$. Thus, $\hat{\eta}$ is LC at $h^{-1}(b)$ if and only if

$$\int_a^b (u'(s)^2 + v'(s)^2) \, ds = \int_a^b ds = \int_a^b p^{(m-1)}(t)q^{kn}(t) \, dt < \infty.$$  

Hence, by (9), the proof will be complete when we show that the operators $\hat{\tau}$ and $\hat{\eta}$ are of the same type at $h^{-1}(b)$. We use Lemmas 1 and 2 with $\tau_1 = \hat{\eta}$ and $\tau_2 = \hat{\tau}$. Since

$$|Q_1(s) - Q_2(s)| = |q(h(s)) - p^{-m}(h(s))q^{-n}(h(s)) + \alpha_{m,n}(h(s))|,$$

the result is an immediate consequence of Lemma 2 if condition (7) applies. Otherwise, as every solution $z(s)$ of $\hat{\eta}x(s) = 0$ is bounded,

$$\int_a^b |Q_1(s) - Q_2(s)| |z^2(s)| \, ds = O(1) \int_a^b |Q_1(s) - Q_2(s)| \, ds$$

$$= O(1) \int_a^b p^{(m-1)}(t)q^{kn}(t) \times$$

$$\times |q(t) - p^{-m}(t)q^{-n}(t) + \alpha_{m,n}(t)| \, dt \quad \text{using (11)}$$

$$< \infty$$

and the theorem follows by Lemma 1.

4. The criterion of Dunford and Schwartz mentioned earlier can now be obtained by substituting $m = 0$ and $n = -1$ in (8) and observing that (cf. (3))

$$D(p, q) = \frac{1}{4} p^{-\frac{3}{2}} q^{-\frac{3}{2}} (pq)^{2} + 4((pq)^{-1} \cdot (\frac{1}{4} q^{-1}(pq)^{-1} ))'$$

$$= 4(pq)^{-\frac{1}{2}} p^{-\frac{1}{2}} q^{-\frac{1}{2}} (pq)^{'2} + 4(pq)^{-\frac{1}{2}} q^{-\frac{1}{2}} (pq)^{'} - 8(pq)^{-\frac{1}{2}} p^{-1} q^{-1}(pq)^{-1}$$

$$= -4(pq)^{-\frac{1}{2}} \alpha_{0, -1}.$$  

The recent result of W. N. Everitt in this direction [see (3)] is also a special case; in particular, Everitt proved that $\tau$ is LC at $b$ if $p(t)$ and $q(t)$ are positive and both $(pq)^{-\frac{1}{2}}$ and $(p(pq)^{-\frac{1}{2}})$ are in $L^2[a, b]$.

Now

$$(pq)^{-\frac{1}{2}} \cdot (pq)^{-\frac{1}{2}} (pq)^{'} = (pq)^{-\frac{1}{2}} q^{-1}(qp)^{'}$$

$$= 4(pq)^{-\frac{1}{2}} q^{-\frac{1}{2}} (pq)^{'} - 4(pq)^{-\frac{1}{2}} p^{-1} q^{-1}(pq)^{-1}$$

$$= -4(pq)^{-\frac{1}{2}} \alpha_{0, -1}$$

$$= D(p, q) \quad \text{from (12)}.$$

Consequently $D(p, q)$ must be in $L^1[a, b]$ and we in fact have a special case of the Dunford and Schwartz result.
The situation in which \( m = 1 \) and \( n = 0 \) is also of interest. In this case \( \alpha_{1,0}(t) \equiv 0 \) and, provided that
\[
|q(t) - p^{-1}(t)| \leq K < \infty
\]
almost everywhere on \([a, b)\), \( \tau \) is LC at \( b \) if and only if \( b < \infty \).

Finally, if \( p(t) \equiv 1 \) then
\[
-\alpha_{0,-1}(t) = \eta(q)(t) = \frac{1}{4}q''(t)q^{-1}(t) - \frac{5}{16}(q'(t)q^{-1}(t))^2,
\]
and we have, using (7):

**Corollary (cf. (5)).** If \( q(t) > 0 \) and \( |\eta(q)| \leq K < \infty \), then the formal operator \( \xi \) defined by \( \xi y(t) = y''(t) + q(t)y(t) \) on the interval \([a, b)\) is LC at \( b \) if and only if
\[
\int_a^b q^{-1}(t) \, dt < \infty.
\]

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**REFERENCES**

5. I. A. Pavlyuk, ‘Necessary and sufficient conditions for boundedness in the space \( L^2(0, \infty) \) for solutions of a class of linear differential equations’ (Ukrainian), *Dopovidi akad. nauk Ukrain. RSR* (1960) 156–8.