On the number of $L^2$-solutions of second order linear differential equations*

Ian Knowles

University of the Witwatersrand, Johannesburg and Department of Mathematics, University of Dundee

(Communicated by W. N. Everitt)

(MS received 13 October 1976. Revised MS received 15 July 1977. Read 14 November 1977)

Synopsis

Let $d$ denote the dimension of the vector space consisting of all solutions of the equation $-(p(t)y')' + q(t)y = 0$, $a < t < \infty$, that lie in the function space $L^2[a, \infty)$. By means of certain bounds on the solutions of this equation, sufficiency criteria are obtained for the cases $d = 0$ and $d = 2$.

1. Introduction

We will be concerned throughout with linear second order differential equations of the form

$$\tau y(t) = -(p(t)y'(t))' + q(t)y(t) = 0, \quad a \leq t < \infty \quad (1.1)$$

where the coefficient functions $p$ and $q$ are real valued with $p$ positive and locally absolutely continuous, and $q$ locally integrable on $[a, \infty)$.

It is well known from basic theory that the vector space consisting of all solutions of (1.1) that lie in the Hilbert function space $L^2[a, \infty)$ has dimension $d$, where $0 \leq d \leq 2$. Our main objective in this paper will be to obtain sufficiency criteria on $p$ and $q$ that determine the value of $d$ for the cases $d = 0$ and $d = 2$.

There is already a large and continually growing literature on the $L^2$-properties of solutions of (1.1). The interest, of course, is not surprising in view of the central rôle played by this equation in non-relativistic quantum mechanics, wherein $\tau$ represents the formal (one-dimensional) energy operator. According to the theory, one must find a suitable self-adjoint realization of $\tau$ acting in a suitable Hilbert space, usually taken to be $L^2[a, \infty)$. If we define $T$ in $L^2[a, \infty)$ by

$$\mathcal{D}(T) = \{u \in L^2[a, \infty]: \quad u(a) = 0, \quad u' \in AC_{loc}[a, \infty), \quad and \quad \tau u \in L^2[a, \infty]\}$$

$$Tu = \tau u \quad for \quad u \in \mathcal{D}(T)$$

then it is known [25, §18.3] that $T$ is self-adjoint if and only if $d \leq 1$. Here, and in the sequel, we use $AC_{loc}[a, \infty)$ to denote the set of all functions that are locally absolutely continuous on $[a, \infty)$. In terminology from [33], we say that $\tau$ is of

* This paper was assisted in publication by a grant from the Carnegie Trust for the Universities of Scotland.
limit-point type at $\infty$ (abbreviated, $\tau$ is $LP$) if $d \leq 1$, and of limit-circle type at $\infty$ (abbreviated, $\tau$ is $LC$) if $d = 2$.

Recently, very strong results for the $LP$ case were given by [9 and 27]. These extend most of the known results, including those given in [2, 3, 6, 10, 14, 15, 17, 22, 28 and 31].

Criteria for the $LC$ case have been given in [5, 7, 11, 13, 16, 18, 24, 26 and 30]; in addition, a rather earlier criterion due to [19] is now known to generalize all of these results. A simple proof of this important result appears as a by-product of the estimates in section 2.

Returning now to the $L'$-solution problem, observe that in the $LP$ case there is a further sub-division, depending on whether $d = 0$ or $d = 1$. The case $d = 0$ is of interest because this implies that $T$, whose spectrum must be real, does not have zero as an eigenvalue. This case has been investigated by various authors [1, 5, 12, 29, 36]. More generally, this is related to the problem of locating those eigenvalues of $T$ that are embedded in the continuous spectrum [see 8 for a survey of these results].

In view of the large number of $LP$ (i.e. $d \leq 1$) criteria available, one could ask: what additional condition(s) must be added to a $LP$ criterion in order to guarantee that $d = 0$? A partial answer to this question will be attempted in section 3.

The main tool for pursuing these investigations is a generalization of the following technique for obtaining bounds on the solutions of (1.1) which seems to be due to [23; see also 11, 21]. Let $y$ be any real-valued solution of (1.1), with $p = 1$, and define

$$E(t) = y^2(t) + (y'(t))^2.$$ 

If we differentiate $E$, estimate $E'/E$ and then integrate the result over $[a, t]$, there results the following inequality:

$$E(a) \exp \left\{ - \int_a^t |1 + q(r)| \, dr \right\} \leq E(t) \leq E(a) \exp \left\{ \int_a^t |1 + q(r)| \, dr \right\}.$$ 

One observes immediately that if the right hand bound is in $L^1[a, \infty)$ then $d = 2$, while if the left hand bound is outside $L^1[a, \infty)$ and $y'$ is not 'too large', then $d = 0$. The actual bounds that we will need are derived in section 2 (see inequality (2.8)) and then applied to the above problems in section 3.

Finally, note that the method is quite general and applies, with appropriate modifications, to finite as well as infinite singularities.

2. Bounds for the Solutions

Before proceeding to the main result of this section, a brief digression is necessary to recant the following facts. Given an arbitrary positive function $\psi$, with $\psi' \in AC_{loc}[a, \infty)$, it is known [5, p. 1500] that, under the (unitary) change of variables defined by $(t \to s, y \to Y)$

$$s(t) = \int_a^t \psi^2(r) \, dr,$$

$$Y(s) = y(t)(\psi(t))^{-1}$$

(2.1)
the equation (1.1) becomes

\[ \tau Y(s) = -(P(s)Y'(s))' + Q(s)Y(s) = 0, \quad 0 \leq s < s(\infty), \tag{2.2} \]

where

\[ P(s) = p(t)\psi^4(t) \]

\[ Q(s) = q(t) - (p(t)\psi'(t))(\psi(t))^{-1} \tag{2.3} \]

Throughout we use the convention that a dash denotes differentiation with respect to the variable following in brackets, when the meaning is not clear.

In order to ensure greater generality, and except where stated otherwise, we will now work exclusively with the transformed equation (2.2). Results applying to the original equation are then easily recovered via the transformation relations (2.1).

Let \( \psi \) be as above, and let \( \theta \) represent another arbitrary positive function in \( AC_{\text{loc}}(a, \infty) \). For any real-valued solution \( Y \) of (2.2) define,

\[ E(s) = \theta^2(t)[P(s)Y'(s)]^2 + Y^2(s) \tag{2.4} \]

After some computation, one can easily verify that

\[ E'(s)(E(s))^{-1} = A(s)\xi^2 + 2B(s)\xi \eta + C(s)\eta^2 \]

where \( \xi = \theta PY'E^{-1/2}, \eta = YE^{-1/2}, \xi^2 + \eta^2 = 1 \) and for all \( s \geq 0, \)

\[ A(s) = 2\theta'(t)[\theta(t)\psi^2(t)]^{-1} \]

\[ B(s) = \theta(t)P(s)^{-1} + \theta(t)Q(s) \]

\[ C(s) = 0. \]

Following [20], we now note that

\[ \lambda_0(s) \leq E'(s)(E(s))^{-1} \leq \lambda_1(s) \tag{2.5} \]

where for each \( s \geq 0, \)

\[ \lambda_1(s), \lambda_0(s) = [A(s) \pm (A^2(s) + 4B^2(s))^{1/2}] / 2 \]

are the eigenvalues of the matrix

\[
\begin{bmatrix}
  A(s) & B(s) \\
  B(s) & 0
\end{bmatrix}
\]

Integrating (2.5) with respect to \( s \) over \([0, s(t)]\) and using (2.1), (2.3) then gives

\[ K\theta(t)\exp \left\{ -\int_a^t F(r) \, dr \right\} \leq E(s) \leq K\theta(t)\exp \left\{ +\int_a^t F(r) \, dr \right\} \tag{2.6} \]

where

\[ K = E(0)(\theta(\infty))^{-1}, \]

and

\[ F = [(\theta'\theta^{-1})^2 + ((p\theta^2)^{-1} + \theta\psi^2(q - (p\psi')\psi^{-1}))^2]^{1/2} \tag{2.7} \]

In order to simplify the notation, we assume without loss of generality that \( K = 1 \) (i.e. renormalize \( Y \)), and we replace \( \theta \) by \( \omega(t) = \theta(t)\psi^2(t) \). With these
changes, and integrating (2.6) with respect to \( s \) over \([0, s(\infty))\) we finally obtain,

\[
\int_a^\infty \omega(t) \exp \left[ -\int_a^t F(r) \, dr \right] \, dt \leq \int_0^{s(\infty)} E(s) \, ds \leq \int_a^\infty \omega(t) \exp \left[ \int_a^t F(r) \, dr \right] \, dt
\]

where

\[
F = \left[ \frac{\omega'}{\omega} - \frac{2\psi'}{\psi} \right]^2 + \left| p^{-1} \omega^{-1} + \omega \{ q - \psi^{-1}(p\psi') \} \right|^2
\]

As an immediate application after noting that \( y \in L^2[a, \infty) \) if and only if \( Y \in L^2[0, s(\infty)) \), the right hand inequality in (2.8) gives:

\[\text{THEOREM 2.1. [19].} \quad \text{If there exist positive functions } \omega \text{ and } \psi \text{ with } \omega \text{ and } \psi' \text{ in } AC_{\text{loc}}[a, \infty) \text{ and}
\]

\[
\int_a^\infty \omega(t) \exp \left[ \int_a^t F(r) \, dr \right] < \infty
\]

then \( \tau \) is \( LC \).

\[\text{Remarks. The exact formulation of this Theorem given in [19] may be recovered by making the substitutions } s = \omega^{-2} \text{ and } \phi = -\psi'\psi^{-1}. \text{ The generality of this result is by no means obvious, due to the large choice available for } \omega \text{ and } \psi. \text{ That it contains the } LC \text{ results in [5, 7, 16 and 26] is noted in [18; see also 16]; some of the remaining results mentioned in the introduction are derived in [19], while the rest are more or less obvious special cases.}\]

3. CRITERIA FOR THE NON EXISTENCE OF \( L^2[a, \infty) \) SOLUTIONS

Turning our attention now to the left hand inequality in (2.8), let us define

\[
W(s) = P^{1/2}(s) \theta(t) = p^{1/2}(t)\omega(t)
\]

We observe immediately from (2.8) that if

(A) \[
\int_a^\infty \omega(t) \exp \left[ -\int_a^t F(r) \, dr \right] \, dt = \infty, \quad \text{and}
\]

(B) \[
P^{1/2}WY' \in L^2[0, s(\infty)) \text{ whenever } \tau_1 Y = 0 \quad \text{and} \quad Y \in L^2[0, s(\infty)),
\]

then (2.2) has no solution in \( L^2[0, s(\infty)) \). Since the mapping \( y \rightarrow Y \) given by (2.1) is known to be unitary, one can infer that equation (1.1) has no solutions in \( L^2[a, \infty) \). Thus, in order to isolate suitable criteria, we need conditions on \( P \) and \( Q \) in (2.2) which guarantee that (B) holds.

Fortunately, results such as these are readily available as many of the known \( LP \) results use the technique [due to 34] of first proving that (B) holds, and then proceeding via a reductio ad absurdum argument using the Wronskian condition [for examples see 4, 9, 17, 22, 27, 28, 12, 32, and 34].

To illustrate these remarks, we consider the following special case of the result in [27]:

\[
\int_a^\infty \omega(t) \exp \left[ -\int_a^t F(r) \, dr \right] \, dt \leq \int_0^{s(\infty)} E(s) \, ds \leq \int_a^\infty \omega(t) \exp \left[ \int_a^t F(r) \, dr \right] \, dt
\]
**Theorem 3.1.** [27]. Assume that \( s(\infty) = \infty \). If there exists a non-negative function \( W \) in \( AC_{loc}([0, \infty)) \) and a decomposition \( Q = Q_1 + Q_2 \) of \( Q \) such that for all \( s \geq 0 \)

\[
\begin{align*}
& (i) \quad P^{1/2}(s)|W'(s)| \leq K_1 \\
& (ii) \quad -Q_1(s)W^2(s) \leq K_2 \\
& (iii) \quad P^{-1/2}(s)W(s)\left|\int_0^s Q_2(r) \, dr\right| \leq K_3
\end{align*}
\]

then statement (B) above holds.

Returning now to the original equation (1.1), we combine (2.1), (2.3), (3.1), Theorem 3.1 and the above remarks to obtain

**Theorem 3.2.** If there exist positive functions \( \omega \) and \( \psi \) with \( \omega, \psi' \) in \( AC_{loc}(a, \infty) \), and a decomposition \( q = q_1 + q_2 \) of \( q \) such that

\[
\begin{align*}
& (i) \quad P^{1/2}[(p^{1/2}\omega)'] \leq K_1 \\
& (ii) \quad -\{q_1 - (p\psi')(\psi^{-1})(p^{1/2}\omega)^2 \leq K_2 \\
& (iii) \quad \omega\psi^{-2}\int_a^t q_2\psi^2 \leq K_3 \\
& (iv) \quad \int_a^\infty \psi^2 = \infty, \\
& (v) \quad \int_a^\infty \omega(t) \exp\left[-\int_a^t F(r) \, dr\right] dt = \infty
\end{align*}
\]

where \( F \) is given by (2.9), then (1.1) has no solution in \( L^2([a, \infty)). \)

A few remarks are in order. The case \( p = \psi = 1 \) provides a good comparison with the original LP criterion; one obtains the appropriate result in [27] by deleting the exponential factor in (v). As will be seen presently, it is mainly this factor that severely limits the behaviour of \( q \) if we wish \( d \) to be zero. Since several of the corollaries of Theorem 3.2 are 'best possible' in various senses, it would appear that the presence of this factor cannot be overlooked.

The case \( q_2 = 0 \) in Theorem 3.1 is the well known Levinson–Sears result as being rather weak compared, for example, to the conditions in [9 and 27]. It is thus a measure of the generality of this method that we can obtain many of the known results in the literature for \( d \) to be zero from Theorem 3.2 with \( q_2 = 0 \), without much recourse to the full result, or its equally strong analogues. It would appear however, that in order to locate eigenvalues of \( \tau \) other than zero, one does need to make use of these stronger results.

**Corollary 3.3.** Assume that, for some integer \( n \geq 0 \) and some constant \( k > 0 \) and all sufficiently large \( t \),

\[
|q(t) + (\frac{1}{4} + k^2)t^{-2}| \leq kt^{-2}\left[2 + \sum_{r=1}^n (\log t \log_2 t \ldots \log_m t)^{-1}\right] + \sigma(t) \tag{3.2}
\]

where \( \log_2 t = \log (\log t) \), \( \log_m t = \log (\log_{m-1} t) \) for \( m \geq 3 \), and \( \sigma \) is any function.
satisfying
\[ t \sigma(t) \in L^1[a, \infty) \quad (3.3) \]
for some suitable \( a > -\infty \). Then equation (1.1), with \( p = 1 \), has no solution in \( L^2[a, \infty) \).

Remarks 1. This is a generalization of the corresponding result in [29]. When \( n = 0, k = 1 \) and \( \sigma = 0 \), (3.2) reduces to
\[-(13/4)t^{-2} \leq q(t) \leq (3/4)t^{-2}\]
which shows that the lower bound of \((-3/4t^2)\) given in [29] can be decreased. The upper bound is best possible in the sense that if \( q(t) \geq Mt^{-2} \) for some \( M > 3/4 \), then \( d \geq 1 \) [see 29]. More recently, it was shown in [1] that for \( d \) to be zero, it is enough to assume that
\[-(4h_0 - \frac{1}{4})t^{-2} \leq q(t) \leq \frac{3}{4}t^{-2}\]
where \( h_0 \approx 3.046 \) is a root of the transcendental equation
\[(4h - 1)^{1/2} \log h - 2 \arcsin(1/2h^{1/2}) = \pi.\]
This result is best possible in the sense that, given \( m > (16h_0 - 1)/4 \), there exists a function \( q \) satisfying
\[-mt^{-2} \leq q(t) \leq (3/4)t^{-2}\]
such that (1.1), with \( p = 1 \), has a solution in \( L^2[a, \infty) \). Note however that Corollary 3.3 allows for conditions of the form
\[ \alpha t^{-2} \leq q(t) \leq \beta t^{-2}\]
where \( \alpha = -(k^2 + 2k + (1/4)) \) and \( \beta = -(k^2 - 2k + (1/4)) \), \( k > 0 \); in this case \( \alpha(k) \) and \( \beta(k) \to -\infty \) as \( k \to \infty \).

2. A condition very similar to (3.2), but valid as \( t \to 0^+ \), was shown in [11] to be sufficient for \( \tau \) to be \( LC \) at 0. It seems to be a consequence of this method that (roughly) conditions which make \( d = 2 \) near a zero singularity (i.e. \( LC \) at 0), make \( d = 0 \) at \( \infty \).

3. The condition
\[ \int_a^\infty t^3q^2(t) \, dt < \infty \]
given in [36] is also contained in Corollary 3.3 [see 29]; furthermore, this result is best possible in the sense that neither
\[ \int_a^\infty t^{3-\epsilon}q^2(t) \, dt < \infty \quad (\epsilon > 0) \]
nor
\[ \int_a^\infty t^3q^{2+\epsilon}(t) \, dt < \infty \quad (\epsilon > 0) \]
are sufficient alone or together for \( d \) to be zero.
Proof of Corollary 3.3. In theorem 3.2 set $p = 1$, $q = q_1 + q_2$ where $q_1 = q + \sigma$ and $q_2 = -\sigma$, $\psi(t) = (t/k)^{1/2}$ and $\omega(t) = t/k$. Conditions (i)–(iv) of the theorem are readily seen to be satisfied. To verify condition (v), note first that,

$$F(t) = |k^{-1}t(q(t) + (1/4)t^{-2}) + kt^{-1}| \leq t^{-1}
\left[2 + \sum_{r=1}^{n} (\log t \ldots \log r)^{-1}\right] + k^{-1}t\sigma(t).$$

Consequently

$$\int_{a}^{t} F(r) \, dr \leq 0(1) + 2 \log t + \sum_{r=2}^{n+1} \log r.$$

Thus, for some constant $C > 0$,

$$\int_{a}^{\infty} \omega(t) \exp\left[-\int_{a}^{t} F(r) \, dr\right] \, dt \geq C \int_{a}^{\infty} (t \log t \ldots \log_{n+1} t)^{-1} \, dt = \infty,$$

and the proof is completed.

To illustrate remark 2 above, we also have the following criteria which compare with similar conditions, given in [11], that are sufficient for $\tau$ to be $LC$ at $0$.

**Corollary 3.4.** The differential equation (1.1), with $p = 1$, can have no solutions in $L^2[a, \infty)$ if either of the following conditions on $q$ are satisfied for large $t$:

(i) $|q(t) + k^2t^{-n}| \leq (nk/2)t^{-n/2}$ for some $k > 0$ and $0 \leq n \leq 4$;
(ii) $|q(t) + k^2e^{nt/2}| \leq kt^{-1}e^{-n2t}$ for some $k > 0$, $n \geq 0$.

**Proof (i).** We use Theorem 3.2 with $p = 1$, $\omega(t) = \psi^2(t) = k^{-1}t^{(n-2)/2}$, $q = q_1 + q_2$ where $q_1 = q$ and $q_2 = 0$. Since $q(t) = 0(t^{-n})$, once again only condition (v) need be verified. We are given that

$$|k^{-1}t^{(n-2)/2}q(t) + kt^{(2-n)/2}| \leq 2^{-n}t^{-n},$$

Hence, for some $C > 0$ and $D > 0$,

$$\int_{a}^{\infty} \omega(t) \exp\left[-\int_{a}^{t} F(r) \, dr\right] \, dt \geq C \int_{a}^{\infty} t^{(n-2)/2}t^{-n/2} \exp (-Dt^{(n-4)/2}) = \infty.$$

(ii) Use the Theorem again with $p = \psi = 1$, $\omega(t) = k^{-1}e^{-n2t}$ and $q_1 = q$, $q_2 = 0$, proceeding as before.

**Corollary 3.5.** [35; 30, p. 125]. If $p = 1$, and $q' < 0$, $q'' < 0$ (or $q'' > 0$), $\lim_{t \to -\infty} q(t) = -\infty$, $q'(t) = 0(|q(t)|^\alpha)$ for some $\alpha$, $0 \leq \alpha < 3/2$ and

$$\int_{a}^{\infty} |q(t)|^{-1/2} = \infty,$$

then (1.1) has no solutions in $L^2[a, \infty)$.

**Proof.** We can assume that $q < 0$ on $[a, \infty)$. In Theorem 3.2 set $\omega = \psi^2 = (-q)^{-1/2}$, $q_1 = q + \psi^{-1}\psi''$ and $q_2 = -\psi^{-1}\psi''$. The result follows easily if we can
show that
\[ \int_a^\infty |q(t)|^{-\alpha} < \infty. \]
Let \( M = \sup_{a \leq t < a} |q(t)| \). Then, if \( q''(t) < 0 \) for \( t \geq a(q'' > 0 \) is similar)
\[
\int_a^t |(-q)^{-1/4}(-q)^{-1/4}|^2 \\
\leq (1/4) \int_a^t |q|^{-3/2}|q''| + (5/16) \int_a^t |q|^{-5/2}|q'|^2 \\
= (1/4) |q'(t)| q(t)^{-3/2} - |q'(a)| q(a)^{-3/2} \\
+ (11/16) \int_a^t |q|^{-5/2}|q'|^2 \\
\leq (M/4) |q(t)|^{a-3/2} + (11M/16) \int_a^t (-q)^{a-(5/2)}|q'|
\]
Since \( \alpha < 3/2 \) this last expression is easily seen to be bounded.

One can easily formulate analogues of Theorem 3.2 by simply replacing Theorem 3.1 with one of its alternatives. In this context the collection of results that require the function \( W \) in Theorem 3.1 to satisfy a monotonicity condition instead of condition (i) are worthy of mention. Typical of these results is that given in [28] for the case \( P = 1 \) (and \( Q_2 = 0 \)) in equation (2.2). This may be extended as follows:

**Theorem 3.6.** Let \( s(\infty) = \infty \) and suppose that there exists a positive continuous function \( W \) and a decomposition \( Q = Q_1 + Q_2 \) of \( Q \) such that on \([0, \infty)\),

(i) \( W \) is monotone non-increasing,
(ii) \( -Q_1 W^2 \leq K_1 \), and
(iii) \( P^{-1/2}(s) W(s) \left| \int_0^s Q_2 \right| \leq K_2 \).

Suppose further that

(iv) \( P \) is monotone non-increasing.

Then
\[
\int_0^\infty P(r)[Y'(r)]^2 W^2(2r) \, dr < \infty \tag{3.4}
\]
for any solution \( Y \) of (2.2) of class \( L^2[0, \infty) \).

**Proof.** Let \( Y \) represent any \( L^2[0, \infty) \)-solution of (2.2). Define
\[
\gamma(s) = \int_0^s P^{-1/2} \tag{3.5}
\]
and note that, from (iv), \( \gamma(s) \to \infty \) as \( s \to \infty \). We first integrate by parts and use
(2.2):\[
\int_0^s (\gamma(s) - \gamma(r))^2 P(r)[Y'(r)]^2 \, dr
\]
\[
= -\gamma^2(s)P(0)Y'(0)Y(0) - \int_0^s (\gamma(s) - \gamma(r))^2 Q(r)Y^2(r) \, dr
\]
\[
+ 2\int_0^s (\gamma(s) - \gamma(r))P^{1/2}(r)Y'(r)Y(r) \, dr.
\]
Next, move the second integral on the right hand side across and complete the square in the resultant integrand.
\[
\int_0^s [(\gamma(s) - \gamma(r))P^{1/2}(r)Y'(r) - Y(r)]^2 \, dr
\]
\[
= -\gamma^2(s)P(0)Y'(0)Y(0) - \int_0^s (\gamma(s) - \gamma(r))^2 Q(r)Y^2(r) \, dr
\]
\[
+ \int_0^s Y^2(r) \, dr.
\]
Then, add a further term \(\int_0^s Y^2(r) \, dr\) to both sides and use the inequality \((a + b)^2 \leq 2(a^2 + b^2)\). This gives, after some rearrangement,
\[
\int_0^s (\gamma(s) - \gamma(r))^2 P(r)[Y'(r)]^2 \, dr
\]
\[
\leq -2\gamma^2(s)P(0)Y'(0)Y(0) - 2\int_0^s (\gamma(s) - \gamma(r))^2 Q(r)Y^2(r) \, dr
\]
\[
+ 4\int_0^s Y^2(r) \, dr
\]
Noting that \(Q = Q_1 + Q_2\), we consider separately the integral involving \(Q_2\).
\[
-2\int_0^s (\gamma(s) - \gamma(r))^2 Q_2(r)Y^2(r) \, dr
\]
\[
= -2\left[ (\gamma(s) - \gamma(r))^2 Y^2(r) \int_0^s Q_2 \right]_0^s
\]
\[
+ 2\int_0^s \left\{ 2Y'(r)Y(r)(\gamma(s) - \gamma(r))^2 - 2(\gamma(s) - \gamma(r))P^{-1/2}(r)Y^2(r) \right\} \int_0^s Q_2 \, dr
\]
\[
\leq 4K_2\int_0^s \{(\gamma(s) - \gamma(r))^2 P^{1/2}(r)|Y'(r)Y(r)|W^{-1}(r) + (\gamma(s) - \gamma(r))Y^2(r)W^{-1}(r) \} \, dr
\]
using condition (iii),
\[
\leq 4K_2\int_0^s (\gamma(s) - \gamma(r))^2 P^{1/2}(r)|Y'(r)Y(r)|W^{-1}(r) \, dr
\]
\[
+ 2K_2\int_0^s (\gamma(s) - \gamma(r))^2 Y^2(r)W^{-2}(r) \, dr + 2K_2\int_0^s Y^2(r) \, dr
\]
using the inequality \(2ab \leq a^2 + b^2\). Noting that \(\gamma(s) \to \infty\) as \(s \to \infty\), we now
combine (3.6), condition (iii) and (3.7), and divide by $\gamma^2(s)$ throughout to obtain

$$
\int_0^s \left[1 - \gamma(r)(\gamma(s))^{-1}\right]^2 P(r)[Y'(r)]^2 \, dr \\
\leq K/2 + 2(K_1 + K_2) \int_0^s \left[1 - \gamma(r)(\gamma(s))^{-1}\right]^2 Y^2(r) W^{-2}(r) \, dr \\
+ 4K_2 \int_0^s \left[1 - \gamma(r)(\gamma(s))^{-1}\right]^2 P^{1/2}(r)[Y'(r)]|Y(r)|W^{-1}(r) \, dr \\
\leq K/2 + 2(K_1 + K_2) \int_0^s Y^2(r) W^{-2}(r) \, dr \\
+ 4K_2 \int_0^s \left[1 - \gamma(r)(\gamma(s))^{-1}\right] P^{1/2}(r)[Y'(r)]|Y(r)|W^{-1}(r) \, dr
$$

where $0 \leq 1 - \gamma(r)(\gamma(s))^{-1} \leq 1$, and $K$ is independent of $s$. Before continuing, we recall that the constant $K_2$ in condition (iii) can be chosen as small as we like, by rescaling $W$ if necessary. In particular we can choose $W$ so that $4K_2 = 1$. Then we again use the inequality $2ab \leq a^2 + b^2$, this time on the integrand of the second integral of the right hand side to obtain

$$
\int_0^s \left[1 - \gamma(r)(\gamma(s))^{-1}\right]^2 P(r)[Y'(r)]^2 \, dr
\leq K + (4K_1 + 2) \int_0^s Y^2(r) W^{-2}(r) \, dr
$$

valid for all $s$ large enough.

Now let $\gamma^{-1}$ denote the inverse function to $\gamma$, i.e. the unique non-decreasing function such that $\gamma^{-1}(\gamma(s)) = \gamma(\gamma^{-1}(s)) = s$ for all $s \geq 0$. Then $\gamma^{-1}(s) \to \infty$ as $s \to \infty$, and for all $0 \leq r \leq \gamma^{-1}(s/2)$,

$$
4[1 - (\gamma(r)/s)]^2 \geq 1
$$

Hence

$$
\int_0^{\gamma^{-1}(s/2)} P(r)[Y'(r)]^2 \, dr \\
\leq 4 \int_0^{\gamma^{-1}(s/2)} [1 - (\gamma(r)/s)]^2 P(r)[Y'(r)]^2 \, dr \\
\leq 4 \int_0^{\gamma^{-1}(s)} [1 - (\gamma(r)/s)]^2 P(r)[Y'(r)]^2 \, dr \\
\leq 4K + (16K_1 + 8) \int_0^{\gamma^{-1}(s)} Y^2(r) W^{-2}(r) \, dr
$$

by inequality (3.8) with $\gamma^{-1}(s)$ replacing $s$.

$$
= C_1 + C_2 \int_0^{\gamma^{-1}(s)} Y^2(r) W^{-2}(r) \, dr
$$

(3.10)

Define

$$
\alpha(u) = \int_0^u P(r)[Y'(r)]^2 \, dr
$$

$$
\beta(u) = C_1 + C_2 \int_0^u Y^2(r) W^{-2}(r) \, dr.
$$
Then (3.10) becomes $\alpha(u) \leq \beta(l(u))$ for all $0 \leq u < \infty$ where $l(u) = \gamma^{-1}(2\gamma(u))$. Note that, trivially, $l(u) \geq u$ for all $u$ and

$$l'(u) = 2\gamma'(u)[\gamma'(l(u))]^{-1} = 2P^{-1/2}(u)P^{1/2}(l(u))$$

Since $P$ is positive and non-increasing, it follows that $0 \leq l'(u) \leq 2$, and therefore (since $l(0) = 0$) that $u \leq l(u) \leq 2u$ for all $u \geq 0$. As $\beta$ is non-decreasing, we then have

$$\alpha(u) \leq \beta(2u) \quad (3.11)$$

for all $u \geq 0$. The standard argument given, for example, in [28] may now be used to complete the proof.

$$\int_0^u P(r)[Y'(r)]^2W^2(2r) \, dr$$

$$= \int_0^u W^2(2r) \, d\alpha(r)$$

$$= \alpha(u)W^2(2u) + \int_0^u \alpha(r) \, d\{-W^2(2r)\}$$

$$\leq \beta(2u)W^2(2u) + \int_0^u \beta(2r) \, d\{-W^2(2r)\}$$

by (3.11) since $-W^2(2r)$ is non-decreasing,

$$= \beta(0)W^2(0) + \int_0^u W^2(2r)W^{-2}(2r)Y^2(2r) \, dr$$

$$= \beta(0)W^2(0) + \int_0^u Y^2(2r) \, dr,$$

from which (3.4) follows immediately.

The following variation of Theorem 3.2 may now be deduced from the remarks made at the beginning of this section. It should be noted however that, instead of (3.1), we now define

$$W(2s) = P^{1/2}(s)\theta(t) = p^{1/2}(t)\omega(t)$$

Hence if $s = h(t)$, $W(s) = p^{1/2}(k(t))\omega(k(t))$ where

$$k(t) = h^{-1}(h(t)/2) \quad (3.12)$$

and $k'(t) = \psi^2(t)/2\psi^2(k(t))$.

**Theorem 3.7.** Suppose that there exist positive functions $\omega$ and $\psi$ with $\omega$ and $\psi'$ in $AC_{loc}[a, \infty)$ and a decomposition $q = q_1 + q_2$ of $q$ such that

(i) $p^{1/2}\omega$ is monotone non-increasing and continuous, and $p\psi^4$ is monotone non-increasing,

(ii) $-\{q_1(t) - (p(t)\psi'(t))(\psi(t))^{-1}\}p(k(t))\omega^2(k(t)) \leq K_1$, $t \geq a$,

(iii) $p^{-1/2}(t)\psi^2(t)p^{1/2}(k(t))\omega(k(t)) \left| \int_a^t q_2 \psi^4 \right| \leq K_2$, $t \geq a$,
(iv) \[ \int_{a}^{\infty} \psi^2 = \infty, \]

(v) \[ \int_{a}^{\infty} \omega(t) \exp \left[ -\int_{a}^{t} F(r) \, dr \right] \, dt = \infty, \]

where \( F \) is given by (2.9), and \( k \) is given by (3.12). Then (1.1) has no solutions of class \( L^2[\alpha, \infty) \).

**Acknowledgments**

The author acknowledges, with gratitude, financial support for the year 1976 from the University of the Witwatersrand in the form of a Council Fellowship. He would also like to thank the referees for their helpful comments, and Professor W. N. Everitt for the opportunity to visit Dundee in the period January-August 1976, during which time this work was completed.

**References**

24 V. B. Lidskii. On the number of square integrable solutions of the system of differential equations

*(Issued 15 September 1978)*