ON THE CORRECTNESS OF BOUNDARY CONDITIONS FOR CERTAIN LINEAR DIFFERENTIAL OPERATORS

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For ordinary linear differential expressions $\tau$ of order $2n$ defined on a real interval $I$, the problem of determining which linear homogeneous boundary conditions give rise to well-posed differential operators in $L^2(I)$ is considered. For the case $I = [0, \infty)$, it is shown that all the operators obtained by imposing $n$ linearly independent (complex) boundary conditions at $0$ are well-posed, under appropriate conditions on the coefficients of $\tau$. The regular case, $I = [0,1]$, is also discussed.

The problem of correctly assigning boundary conditions to formal differential expressions arising from physical models, and elsewhere, is of central importance in applications of differential operator theory. For definiteness, consider the differential expression $\tau$ defined by

$$\tau y(x) = (-1)^n y^{(2n)} + \sum_{r=0}^{n-1} (p_n - r(x)) y^{(r)}(x), \quad x \in I, \quad (1)$$

where $I \subset \mathbb{R}$ and the coefficients $p_i(\cdot), 1 \leq i \leq n$, are complex-valued and locally Lebesgue integrable on $I$. We associate with $\tau$ the usual maximal and minimal operators, $T_1$ and $T_0$ respectively, in $L^2(I)$ as follows (see [12]): Let $f^{[i]}$ denote the $i$th quasi-derivative of a function $f$ (see [12, p. 49]). The operator $T_1$ is then given by

$$\mathcal{D}(T_1) = \{f \in L^2(I): f^{[i]}, 0 \leq i \leq 2n-1, \text{ are locally absolutely continuous, and } \tau f \in L^2(I)\}$$

$$T_1 f = \tau f, \quad f \in \mathcal{D}(T_1),$$

while $T_0$ is defined to be the closure of the operator $T_1$ given by
\[ D(T_0) = \{ f \in D(T_1) : f \text{ vanishes outside some compact interval } \ [a, b] \subset (I) \} \]

\[ T_0'f = \tau f, \quad f \in D(T_0). \]

The essence of the problem of assigning boundary conditions is roughly the following: one must choose the boundary conditions so that the associated restriction, \( T_1 \), of \( T_1 \) has domain optimally large in some suitable sense. In the best of cases this means that the spectrum of the operator \( T \) allows something like an eigenfunction expansion theory. If there are too few boundary conditions, one can expect the point spectrum of \( T \) to fill out the complex plane; and if there are too many boundary conditions the point spectrum of the adjoint operator may do likewise (in which case the residual spectrum of \( T \) may cover the complex plane). Clearly, a minimum requirement on the spectrum of \( T \) is that the resolvent set, \( \rho(T) \), be non-empty. The following necessary condition for this to occur forms a convenient starting point for our discussion:

**Lemma** [2, p. 1311]. Let \( T \) be an operator obtained from \( \tau \) by imposing a (possibly empty) set of boundary conditions on \( D(T_1) \), and let \( \lambda \in \rho(T) \). Then the number of linearly independent boundary conditions defining \( T \) is equal to the number of linearly independent solutions of the equation \( \tau y = \lambda y \) that belong to \( L^2(\mathbb{I}) \).

Our main concern here is to investigate the converse result; i.e., to determine conditions under which an operator \( T \), obtained from \( \tau \) by imposing the number of boundary conditions specified in the theorem, has non-empty resolvent. That the converse is not true in general may be seen from the following example:

Let \( T \) be defined by

\[ D(T) = \{ f \in L^2[0,1] : f' \text{ is absolutely continuous,} \]

\[ f'' \in L^2[0,1], \text{ and } f(0) + f(1) = f'(0) - f'(1) = 0 \}

\[ T f(x) = -f''(x), \quad 0 \leq x \leq 1, \quad f \in D(T). \]

Here, the associated characteristic polynomial for the eigenvalues of \( T \) is identically zero, giving \( \rho_0(T) = \mathbb{C} \), where \( \rho_0(T) \) denotes the point spectrum of \( T \). Clearly it is of interest to document precisely when such pathological cases occur, as one would expect, among other things, that any attempt at a numerical solution of a boundary value problem involving such an operator, would fail. On the other hand, it should be noted that even if one knows that \( \rho(T) \) is not empty, the associated spectral theory can still be extremely complicated ([11, §5.4; 10; 9]).

In the sequel, we denote the regularity field of an operator \( T \), \( T_0 \subset T \subset T_1 \), by \( \pi(T) \); the essential spectrum of \( T \) is denoted by \( E_0(T) \), and the residual spectrum by \( \rho_0(T) \) (c.f. [8, §2]). An extension \( T \) of \( T_0 \) is called well-posed if \( \pi(T) \) is not empty. It is known (see [8, §3]) that \( T \) is non-well-posed if and only if \( \pi(T_0) \subset \rho_0(T) \).

For general \( \tau \), one can observe asymptotic properties of \( \tau \) and \( \rho(T) \) and \( \rho_0(T) \).

In particular, if \( \tau \) is compact, \( \rho(T) \) and \( \rho_0(T) \) are well-defined.

Then we have the Theorem that the extension is well-posed if...

Proof. If \( \pi(T_0) \subset \rho_0(T) \), then...
We consider firstly the so-called regular case in which we take $I = [0,1]$ for simplicity, and assume $p_i(\cdot) \in L[0,1]$, $1 \leq i \leq n$. Notice that $\pi(T_0) = \mathbb{C}$, and thus an extension $T$ of $T_0$ is non-well-posed if and only if $P_\sigma(T) = \mathbb{C}$. Given matrices $A = (a_{rs})$ and $B = (b_{rs})$ of order $2n$ and with complex entries, define the operator $T_{AB}$ by

$$D(T_{AB}) = \{f \in L^2[0,1]: \frac{d^2}{ds^2} f(s), 0 \leq i \leq 2n-1, \text{locally absolutely continuous, } \tau f \in L^2[0,1], \text{ and}$$

$$\sum_{s=1}^{2n} a_{rs} f(s-1)(0) + b_{rs} f(s-1)(1) = 0 \text{ for } 1 \leq r \leq 2n\}$$

$$T_{AB} f = \tau f, \ f \in D(T_{AB}).$$

For general $n$, rather little is known about which extensions $T_{AB}$ are well-posed. One can reduce the problem to the case $p_i(\cdot) = 0$, $1 \leq i \leq n$, by means of known asymptotic formulae for the solutions $y(x,\lambda)$ of $(\tau - \lambda)y = 0$ valid for fixed $x$ and $|\lambda| \to \infty$ (c.f. [11,12]). For separated boundary conditions the extensions $T_{AB}$ are always well-posed ([11, Lemma 3, p. 94]). For $n = 1$ it is not difficult to show that $T_{AB}$ is non-well-posed if and only if

$$\begin{bmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{bmatrix} = \begin{bmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{bmatrix} = 0,$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = 0,$$

and

$$\begin{bmatrix} a_{12} & b_{11} \\ a_{22} & b_{21} \end{bmatrix} - \begin{bmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{bmatrix} = 0.$$
\( \lambda \not\in \mathcal{C} \cup [0, \omega) \) lies in \( \mathcal{P}_0(T_0) \). For such \( \lambda \) consider, then, the equation,

\[
(-1)^n f(2n) = \lambda f.
\]

(4)

Let \( \rho \) denote the \( 2n \)th root of \( (-1)^n \lambda \) satisfying \( \pi/2 < \arg \rho < \pi/2 + n/n \). Then the distinct \( 2n \)th roots of \( (-1)^n \lambda \) are given by \( \nu_i = \rho \epsilon^{-1} \), \( 1 \leq i \leq 2n \), where \( \epsilon = \exp(\pi/n) \). Any eigenfunction of \( T_0 \) must be of the form

\[
f(x) = c_1 \exp(\rho x) + c_2 \exp(\rho \epsilon x) + \cdots + c_n \exp(\rho \epsilon^{n-1} x)
\]

(5)

for appropriate constants \( c_1, \ldots, c_n \). Using (5), one can show that the characteristic equation for the eigenvalues of \( T_0 \) has the form \( \Delta(\lambda) = \det(M) = 0 \), where \( M = (m_{ij}) \) and

\[
m_{ij}(\rho) = \sum_{s=1}^{2n} a_{is} \epsilon^{(i-1)s} \epsilon^{-s}.
\]

We can write \( M = AG \), where \( G = (g_{ij}) \) the \( 2n \times n \) matrix with \( g_{ij} = (\rho \epsilon^{j-1})^{i-1} \) in the \( i \)th row and \( j \)th column. In this case the formula for the determinant of the product is

\[
\det(AG) = \sum_{1 \leq s_1 < s_2 < \cdots < s_{n-1} \leq 2n} \det A_{s_1 \ldots s_n} \cdot \det G_{s_1 \ldots s_n}
\]

(6)

where \( A_{s_1 \ldots s_n} \) denotes the \( n \times n \) matrix consisting of columns \( s_1, s_2, \ldots, s_n \) of \( A \), and \( G_{s_1 \ldots s_n} \) denotes the \( n \times n \) matrix consisting of rows \( s_1, s_2, \ldots, s_n \) of \( G \). The equation for the eigenvalues of \( T_0 \) thus becomes

\[
\sum_{1 \leq s_1 < s_2 < \cdots < s_{n-1} \leq 2n} \left( \prod_{i<j} (\epsilon^{s_i-1} - \epsilon^{s_j-1}) \right) \det A_{s_1 \ldots s_n} = 0
\]

(7)

Clearly, \( T_0 \) is well-posed if and only if equation (7) is not identically zero. As \( A \) is of rank \( n \), at least one of the determinants \( \det A_{s_1 \ldots s_n} \) is not zero.

Let \( s = \max(s_1 + s_2 + \cdots + s_n; \det A_{s_1 \ldots s_n} \neq 0) \). Then, by [11, Lemma 2, p. 91] the term \( \rho^{s-n} \) in (7) is non-trivial, and the result follows.

\# Remark. This result is of independent interest. In much of the qualitative spectral theory of non-selfadjoint differential operators, one is forced (at least implicitly) to make artificial assumptions in order to exclude the "bad" estimates (see e.g. [1, p. 11, 15; 4, 7, 19]). It is therefore very useful to know precisely when such extensions cannot occur.

Provided the coefficients \( p_i, 1 \leq i \leq n \), are not too large, Theorem 1 can be extended to cover more general operators \( T \). More precisely we have

\# Theorem 2. If \( p_i = q_i + r_i \), where \( q_i \in \mathcal{L}^m([0, \omega]) \), and \( r_i \in \mathcal{L}([0, \omega]) \), \( 1 \leq i \leq n \), then the extension \( T_0 \) of \( T_0 \) defined by (3) is well-posed for every choice of the boundary matrix \( A \).

\# Remark. This includes the case \( p_i \in \mathcal{L}^1([0, \omega]) \), \( 1 < m_i < \omega \), \( 1 \leq i \leq n \), as one can always write an \( r \)-integrable function, \( 1 < r < \omega \), as the sum of an integrable function and a bounded function.
Proof. The proof is divided into several stages. Following [12, 8.22.2] we write the equation \( ty = \lambda y \) in system form as

\[
\frac{dY}{dx} = A(x) Y
\]

where \( Y = Y(x, \rho) = (y, y[1], \ldots, y[2n-1])^T, \) and \( A(x) = A_0(x) + A_1(x) \) where \( A_0 \) and \( A_1 \) are defined in [12, p. 176], and we have set \( p_0 = 1 \). Let

\[
Y = BU
\]

where

\[
B = \begin{bmatrix}
1 & & & 1 \\
\mu_1 & & & \mu_{2n} \\
& \ddots & & \\
-\mu_1 & & & -\mu_{2n} \\
(-1)^{n-1} & & & (-1)^{n-1} \mu_{2n-1} \\
\end{bmatrix}
\]

and \( \mu_i, 1 \leq i \leq 2n, \) are the distinct \( 2n^{th} \) roots of \((-1)^n \lambda \) defined earlier. Set \( B^{-1} = (\beta_{ij}) \) and define \( A_2 \) and \( A_3 \) to be the matrices obtained from \( A_1 \) by replacing the elements \( p_i, 1 \leq i \leq n, \) by \( q_i \) and \( r_i, \) respectively. The system (8) then becomes

\[
\frac{dU}{dx} = (W + C + F)U
\]

where \( W = \text{diag}(\mu_1, \ldots, \mu_{2n}), \) \( C = B^{-1}A_2B = (c_{ij}), \) \( F = R^{-1}A_3B = (f_{ij}), \) and

\[
c_{ij} = c_{ij}(x, \rho) = \sum_{k=1}^{n} \beta_{i,n+k}(\rho)q_k(x)\mu_{ij}^{n-k}
\]

\[
f_{ij} = f_{ij}(x, \rho) = \sum_{k=1}^{n} \beta_{i,n+k}(\rho)r_k(x)\mu_{ij}^{n-k}
\]

It is not hard to see that the functions \( c_{ij}(x, \cdot) \) and \( f_{ij}(x, \cdot) \) are analytic for fixed \( x \geq 0. \) Also for \( |\rho| \geq 1 \) we have

\[
|c_{ij}(x, \rho)| \leq K_n \sum_{k=1}^{n} |q_k(x)|; \quad |f_{ij}(x, \rho)| \leq L_n \sum_{k=1}^{n} |r_k(x)|
\]

where \( K_n \) and \( L_n \) are independent of \( x \) and \( \rho. \)

Our initial goal is to determine the asymptotics of certain solutions of (11), from which we easily obtain the behaviour of solutions of \( ty = \lambda y \) via (8).

Before doing this we digress for a moment. Define \( H = (h_{ij}) \) by

\[
H = C - (\int_{0}^{x} F)W + W(\int_{0}^{x} F) - (\int_{0}^{x} F)W(\int_{0}^{x} F)
\]

and consider, for \( 1 \leq i \leq n, \) the solutions \( V_i = (v_{ij}) \) of the integral equations
\[ v_{ij}(x, \rho) = \int_0^x e^{j(x-\xi)} 2\pi \sum_{k=1}^{2n} h_{ik}(\xi, \rho)v_{ik}(\xi, \rho) d\xi \]
\[ v_{ij}(x, \rho) = \int_0^x e^{j(x-\xi)} 2\pi \sum_{k=1}^{2n} h_{jk}(\xi, \rho)v_{ik}(\xi, \rho) d\xi \text{ if } j \neq i, 1 \leq j \leq n \] (14)
\[ v_{ij}(x, \rho) = -\int_0^x e^{j(x-\xi)} 2\pi \sum_{k=1}^{2n} h_{jk}(\xi, \rho)v_{ik}(\xi, \rho) d\xi \text{ if } n + 1 \leq j \leq 2n. \]

Observe that the solutions \( V_i, 1 \leq i \leq n \), all satisfy the equation
\[ \frac{dV}{dx} = (W + H)V. \] (11)

We now adapt the techniques of Kaminura [7] to show that the solutions \( V_i = V_i(x, \rho), 1 \leq i \leq n \), of (14) exist and have components square integrable in \([0, \infty)\), for suitable values of \( \rho \). To do this we require analogues of certain inequalities used in [7]. Suppose that \( y \in L^2(0, \infty) \), and that \( \mu \) is a complex number whose real part, \( \gamma \), is positive. Then
\[ \int_0^\infty |e^{-\mu t} y(t)| dt \leq (2\gamma)^{-\frac{1}{2}} e^{-\gamma x} \int_0^\infty |y(t)|^2 dt \] (15)
\[ \int_0^\infty \left( \int_0^x e^{\mu(x-t)} y(t) dt \right)^2 dx \leq \gamma^{-2} \int_0^\infty |y(t)|^2 dt \] (16)
\[ \int_0^\infty \left( \int_0^x e^{-\mu(x-t)} y(t) dt \right)^2 dx \leq \gamma^{-2} \int_0^\infty |y(t)|^2 dt \] (17)

The first of these, (15), is a direct consequence of the Cauchy-Schwarz inequality.

The other inequalities may be obtained from Young's inequality (see, for example, [15, p. 32]); using the notation of [15], one obtains (16) by setting
\[ f(t) = e^{\mu t}, \quad t \leq 0 \]
\[ = 0, \quad t > 0 \]
and
\[ g(t) = y(t), \quad t \geq 0 \]
\[ = 0, \quad t < 0 \]
together with \( r = q = 2 \) and \( p = 1 \); (17) is obtained by replacing \( \mu \) above by the function
\[ h(t) = e^{-\mu t}, \quad t \geq 0 \]
\[ = 0, \quad t < 0. \]

Let \( L \) denote the Cartesian product of \( 2n \) copies of \( L^2(0, \infty) \); with the usual inner product topology, \( L \) is a Hilbert space. Define \( S: L \to L \) by
\[ (Sf)_i(x) = \int_0^x e^{j(x-\xi)} 2\pi \sum_{k=1}^{2n} h_{ik}(\xi)f_k(\xi) d\xi, \quad 1 \leq i \leq n \]
\[ -\int_0^x e^{j(x-\xi)} 2\pi \sum_{k=1}^{2n} h_{ik}(\xi)f_k(\xi) d\xi, \quad n + 1 \leq i \leq 2n \]

where \( f = (f_1, \ldots, f_n) \) and \( \alpha \) does not in \( \mathbb{R}^n \).

(we assume that as a bounded operator in the form \((I - S)V_1 \), \( V_1 \) is a vector in \( R^{2n} \)), solution \( V_1(\cdot, \rho) \) is independent, an \( (c.f. \ [7]) \)

Consider \( n \) independent solutions \( \gamma \) of (11).

One can show that (9), (11), and (15) are satisfied.

Finally, by modifying [5], (15), (16), (17), and (21) can be treated as a system of damped wave equations.

Also, by (9), (11), and (15), we have\( G(\lambda) \to 0 \), where
where \( f = (f_1, \ldots, f_{2n}) \in L \). Using (13), (16), and (17), one can show that \( S \) is a bounded linear operator in \( L \) with

\[
\|S\| \leq G(\lambda) = \gamma^{-1} \alpha_n
\]

where \( \alpha_n \) does not depend on \( \lambda \) and

\[
\gamma = \min\{|\Re \mu_i|: 1 \leq i \leq 2n\} = |\lambda|^{1/2n} \sin\left(\frac{\arg \lambda}{2n}\right)
\]

(we assume that \(-\pi < \arg \lambda \leq \pi\)). Thus for \( \lambda \) such that \( G(\lambda) < 1 \), \((I - S)^{-1}\) exists as a bounded operator on \( L \). If we now write the integral equations (14) in the form \((I - S)V_i = E_i\) where \( E_i = \exp(\mu_i x) e_i \) (\( e_i \) denoting the \( i \)-th standard basis vector in \( \mathbb{R}^{2n} \)), then it is clear that for each \( i, 1 \leq i \leq n \), (14) has a unique solution \( V_i(\cdot, \rho) \subseteq L \) for all \( \lambda \) such that \( G(\lambda) < 1 \). These solutions are linearly independent, and for each fixed \( i \) the components, \( v_{ij}, 1 \leq j \leq 2n \), of \( V_i \) satisfy (c.f. [7])

\[
\lim_{G(\lambda) \to 0} v_{ij}(0, \rho) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

Consider now equation (11). For \( 1 \leq i \leq n \), set

\[
U_i(x, \rho) = (I + \int_0^x F(t)dt)V_i(x, \rho)
\]

(21)

One can show directly that the vectors \( U_i(\cdot, \rho) \subseteq L \) form a set of \( n \) linearly independent solutions of (11). In addition, it is clear that the components, \( u_{ij}, 1 \leq j \leq 2n \), of \( U_i \) also satisfy (20).

Finally, consider the solutions of the equation \( \gamma y = \lambda y \) for \( \lambda \in \mathbb{C} - [0, \infty) \).

By modifying [5, Theorem 9, p. 138] along the lines of [14, Theorem 3.2] one can show that \( \pi(T_0) = \pi(T_A) = [0, \infty) \), and hence that \( \pi(T_0) = \mathbb{C} - [0, \infty) \). Furthermore, it is known ([6, Theorem C and equation (2.8)]) that for any \( \lambda \in \pi(T_0) \) there are precisely \( n \) square integrable solutions of the equation \( \gamma y = \lambda y \). Define \( Y_i = (y_{ij}) = BU_i, 1 \leq i \leq n \), where \( U_i \) are the solutions of (11) defined by (21). Set \( y_k = y_{k1} \) for \( k = 1, 2, \ldots, n \). Then the functions \( y_k, k = 1, 2, \ldots, n \), constitute \( n \) linearly independent square integrable solutions of the equation \( \gamma y = \lambda y \); the quasi-derivatives \( y_{kj}^{[j-1]}, 1 \leq j \leq 2n \), are given by \( y_k^{[j-1]} = y_{kj} \).

Also, by (9), (10), (20) we have for \( 1 \leq k \leq n \),

\[
\begin{align*}
y_k(0, \rho) &= 1 + o(1) \\
y_k^{[1]}(0, \rho) &= \mu_k[1 + o(1)] \\
&\vdots \\
y_k^{[n]}(0, \rho) &= \mu_k^{-1}[1 + o(1)] \\
y_k^{[n+1]}(0, \rho) &= -\mu_k[1 + o(1)] \\
&\vdots \\
y_k^{[2n-1]}(0, \rho) &= (-1)^{n-1} \mu_k^{2n-1}[1 + o(1)]
\end{align*}
\]

as \( G(\lambda) \to 0 \), where \( G(\lambda) \) is defined by (18).
We can now complete the proof of the theorem. Arguing as in the proof of Theorem 1, one can deduce that the eigenvalues $\lambda$ of $T_A$ are given by the roots of $D(\lambda) = 0$, where

$$D(\lambda) = \begin{vmatrix} a_{1s} y_1^{[s-1]}(0, \rho) & \cdots & a_{ns} y_1^{[s-1]}(0, \rho) \\ \vdots & \ddots & \vdots \\ a_{1s} y_n^{[s-1]}(0, \rho) & \cdots & a_{ns} y_n^{[s-1]}(0, \rho) \end{vmatrix}$$

Using (22) we then have that

$$D(\lambda) = \Delta(\lambda)[1 + o(1)]$$

where $\Delta(\lambda)$ is the corresponding determinant in the proof of Theorem 1. As $\Delta(\lambda)$ is never identically zero, it follows that $D(\lambda)$ has the same property, for any choice of the boundary matrix $A$.

Remarks. 1. One can deduce from the proof of Theorem 2 that the eigenvalues of $T_A$ are enclosed by a curve in the $\lambda$-plane of the form $G(\lambda) = \text{constant}$. For example, when $n = 1$ and $\lambda = r \exp(i\theta)$, the curve is of the form $r^{\frac{1}{2}}|\sin(\theta/2)| = k$.

This curve encloses the entire non-negative half-axis, and is in fact asymptotic to it for $\arg \lambda$ approaching 0 or $2\pi$. This behaviour is consistent with the fact that there are known examples (see e.g. [3]), in the second order case, of operators with $L^2[0,\infty)$-coefficients $(r > 1)$ having an unbounded set of positive eigenvalues. If $q_i = 0$, $1 \leq i \leq n$, the appropriate asymptotic formulae for the solution values $y_k^{[j-1]}(0, \rho)$ are valid for $|\lambda| \to \infty$, and the point spectrum is thereby confined to some disc.

2. One can infer rather more about the spectra of the operators $T_A$. Firstly, the point spectrum is always discrete in $\mathbb{C} - [0, \infty)$, and has all its limit-points on $[0, \infty)$. Furthermore, the eigenvalue equation for $T_A^*$ has the same general form as that for $T_A$. Consequently, similar remarks to the above apply to $R(T_A)$. Thus, in general $R_0(T_A) \cup P_0(T_A)$ is discrete and confined to a neighbourhood of the non-negative real axis in the $\lambda$-plane; i.e., $\mathbb{C} - [0, \infty) \subset \sigma(T_A)$, with the possible exception of a discrete set.

3. Finally, we observe that this theory may also be extended to cover formal differential expressions of the form

$$\tau_1 y(x) = (-1)^n y^{(2n)} + \sum_{r=0}^{2n-1} p_r(x)y^{(r)}, \quad 0 \leq x < \infty.$$ 

In this case one can obtain similar results by combining Theorem 1 with the asymptotic formulae for solutions of $\tau_1 y = \lambda y$ given in [1, 52].

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REFERENCES


