On the Extension Problem for Singular Accretive Differential Operators

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1. INTRODUCTION

A linear operator $T$ with domain $\mathcal{D}(T)$ in a Hilbert space $\mathcal{H}$ is said to be accretive if

$$(Ty, y) + (y, Ty) \geq 0$$ (1.1)

for all $y$ in $\mathcal{D}(T)$, and maximal accretive if it is accretive and has no proper accretive extension. Operators of this general type, in which the numerical range

$$\Theta(T) = \{(Tu, u) : u \in \mathcal{D}(T), \|u\| = 1\}$$ (1.2)

is restricted to lie in a half-plane, have proved very useful in applications (see [8, Sect. 7] for a partial survey, noting that $T$ is called dissipative if $-T$ is accretive in the above sense; see also [7, 14, 18, 23]).

In applications involving differential equations a major problem is that one is usually only given a formal differential expression and a function domain on which it gives rise to an accretive operator, and one has to prescribe a (possibly) larger domain on which the differential expression is maximal accretive. More generally, one would like to describe, explicitly, boundary conditions for all possible maximal accretive differential operators derivable from a given formal differential expression.
We consider here differential expressions \( \tau \) of the general form

\[
\tau y(x) = \frac{1}{\omega(x)} \sum_{j=0}^{n} (-1)^j (p_{n-j}(x) y^{(j)}(x))^{(j)}, \quad x \in I,
\]

where \( I \) is an interval of the real axis, and the coefficients \( \omega > 0, p_0, p_1, \ldots, p_n \) are real-valued and such that the minimal operator, \( T_0 \), associated with \( \tau \) (see Sect. 2) is accretive in the weighted Hilbert space, \( L^2_{\omega}(I) \), of functions \( f \) satisfying

\[
\int_I \omega(x)|f(x)|^2 \, dx < \infty.
\]

A solution to the above extension problem is given in [11] for the case that \( \tau \) in (1.3) is a regular differential operator in the sense of [20, Sect. 15], i.e., that the interval \( I \) is closed, and the coefficient functions \( \omega, p_0^{-1}, p_1, \ldots, p_n \) are Lebesgue integrable on \( I \). Our main aim here is to consider the case when the given operator \( \tau \) is singular, i.e., when either \( I \) is not closed or at least one of the functions mentioned above fails to be in \( L^1(I) \). The methods used here are similar to those of [11] in that we ultimately appeal to the abstract extension theory of Phillips ([25]; see [11, Sect. 2] for an outline). However, the technical difficulties are somewhat more severe in the singular case; in particular the proof of the crucial embedding result (Lemma 3.6) is more involved. In Section 2 we present some preliminary differential operator theory. The main results appear in Section 3, notably Theorem 3.11 and Corollary 3.12. The Appendix contains a detailed outline of the adjoint construction technique of Brown and Krall [4], including proofs for the singular case, which is not covered in [4].

It should be noted that we solve the extension problem for a specific class of singular differential expressions, namely those satisfying conditions (A2) and (A3) of Sections 2 and 3, (i.e., roughly, we assume that functions in the appropriate maximal domains all have finite energy integrals). In particular, all of the formal operators \( \tau \) above are of limit-point type at the singular end-point(s) of \( I \). It is not known what happens if these assumptions are dropped. Thus, the extension problem is open even for the simplest operators, \( \tau y = -y'' + q(x) y \), of limit-circle type.

Once again we acknowledge with gratitude the assistance given by Professor R.C. Brown in adapting the techniques in [4] to the present situation.
2. Preliminary Theory

Throughout the paper we consider formal differential expressions of the form (1.3) with coefficient functions \( \omega > 0, p_0 \neq 0 \) a.e., and

\[
\frac{1}{\omega}, \frac{1}{p_0}, p_1, \ldots, p_n \quad (2.1)
\]

assumed to be real and Lebesgue integrable in any closed subinterval of \( I \). The expression \( \tau \) is said to be regular on \( I \) if \( I \) is finite and closed, and the functions (2.1) are integrable on \( I \); otherwise \( \tau \) is said to be singular. In addition, the left-hand end-point, \( a \), of \( I \) is regular if \( a > -\infty \), \( I \) is closed at \( a \), and the functions (2.1) are integrable on every interval \([a, \beta] \subset I\), and singular otherwise, with a similar definition for the right-hand endpoint, \( b \), of \( I \). In lieu of further smoothness requirements on the coefficients, and following [20, Sect. 15.2] we define the formal quasi-derivatives (up to order \( 2n \)) of a function \( \gamma \) to be the functions \( \gamma^{(0)} = \gamma, \gamma^{[1]}, \ldots, \gamma^{[2n]} \) given by

\[
\gamma^{(i)}, \quad 0 \leq i \leq n - 1, \\
\gamma^{[i]} = p_0 \gamma^{(n)}, \quad i = n, \\
p_{i-n} \gamma^{(2n-i)} - \{ \gamma^{[i-1]} \} r, \quad n + 1 \leq i \leq 2n,
\]

where \( \gamma^{(i)} \) is the usual \( i \)th derivative. The expression \( \tau \) is then given by

\[
\tau(\gamma) = \omega^{-1} \gamma^{[2n]}. \quad (2.3)
\]

If \( p_{n-i} \) has \( i \) continuous derivatives, then \( \tau(\gamma) \) is also given by (1.3).

Let \( L^2_{\omega}(I) \) denote the Hilbert space of all complex-valued functions \( f \) on \( I \) satisfying (1.4). For \( \omega = 1 \), we denote \( L^2(I) \) by \( L^2(I) \). Set

\[
\mathcal{D} = \{ \gamma \in L^2_{\omega}(I): \gamma^{[k]} \text{ exists and is absolutely continuous on compact subintervals of } I, \ 0 \leq k \leq 2n - 1, \\
\text{and } \tau \gamma \in L^2_{\omega}(I) \}. \quad (2.4)
\]

We define the maximal operator for \( \tau \) in \( L^2_{\omega}(I) \), \( T_1 \), to be

\[
T_1 \gamma = \tau \gamma, \quad \gamma \in \mathcal{D}. \quad (2.5)
\]

The minimal operator for \( \tau \) in \( L^2_{\omega}(I) \), \( T_0 \), is defined to be the closure of the restriction of \( T_1 \) to the functions in \( \mathcal{D} \) that vanish outside a finite interval \([x, \beta] \) interior to \( I \). It is known [20, Sect. 17] that

\[
T_0^* = T_1. \quad (2.6)
\]
Observe also that the Lagrange formula is [20, p. 50],

\[ \omega \{ \bar{z} \tau (y) - y \bar{\tau} (z) \} = \frac{d}{dx} [y, z](x) \]

with the corresponding Green's formula

\[ \int_C \omega \{ \bar{z} \tau (y) - y \bar{\tau} (z) \} = [y, z](d) - [y, z](c), \quad (2.7) \]

where

\[ D(y, z)(x) = \sum_{i=1}^{n} y^{[2n-i]}(x) z^{[i-1]}(x), \quad (2.8) \]

\[ [y, z](x) = D(\bar{z}, y)(x) - D(y, \bar{z})(x). \]

For \( y, z \in \mathcal{D} \), we have the differentiated Dirichlet formula

\[ -\omega \tau (y) z + \sum_{i=0}^{n} p_{n-i} y^{(i)} z^{(i)} = \{ D(y, z)(x) \}^\prime. \quad (2.9) \]

For a later use we also need a version of the well-known Sobolev inequality:

**Lemma 2.1.** (c.f. [1, Theorem 3.9]). Let \( u \) and \( u^{(n)} \) be in \( L^2(J) \) for some real interval \( J \). Then for any \( x \) in \( J \) and any \( \varepsilon > 0 \) there is a constant \( K = K(\varepsilon) \) such that for any \( k \) with \( 0 \leq k \leq n - 1 \),

\[ |u^{(k)}(x)|^2 \leq \varepsilon \|u^{(n)}\|^2_J + K(\varepsilon) \|u\|^2_J, \quad (2.10) \]

where \( \| \cdot \|_J \) denotes the usual norm on \( L^2(J) \).

We assume throughout that

(A1) \( T_0 \) is an accretive operator on \( L^2(I) \), and

(A2) the Dirichlet integral for \( \tau \),

\[ \sum_{j=0}^{n} \int_I p_{n-j} y^{(j)} \bar{z}^{(j)}, \]

exists (possibly as an improper integral) for \( y, z \) in \( \mathcal{D} \), and forms a positive definite inner product on the maximal domain \( \mathcal{D} \); this inner product is
denoted by $(\cdot, \cdot)_D$ and the corresponding norm by $\|\cdot\|_D$. Assume further that for some $c > 0$,

$$\|u\|_D \geq c\|u\|, \quad u \in \mathcal{D}.$$ 

Observe that (A1) is a consequence of (A2). For if $u \in \mathcal{D}$ vanishes outside an interval $[a, b]$ interior to $I$, $(T_0u, u) = \|u\|_D^2 > 0$. This continues to hold for $u \in \mathcal{D}(T_0)$ and hence $T_0 \geq 0$. In fact, $T_0$ is actually symmetric here. Observe that (A2) is clearly satisfied if, for example, $p_i \geq 0$, $0 \leq i \leq n-1$, and $p_n \geq \varepsilon \omega$ on $I$, for some positive constant $\varepsilon$.

3. SINGULAR DIFFERENTIAL OPERATORS

In this case, in addition to (A1) and (A2), we also assume

(A3) At each singular end-point $c$ of $I$, $\lim_{\varepsilon \to c} D(y, z)(x) = 0$, for all $y, z$ in $\mathcal{D} = \mathcal{D}(T_1)$.

As an immediate consequence, we have from (2.9) that the integral in (A2) exists when $I = [a, b)$, with $a$ a regular endpoint, and also

$$(\tau y, z) = D(y, \tilde{z})(a) + (y, z)_D$$

for all $y, z$ in $\mathcal{D}$. (A3) is satisfied if, for example, $p_n \geq \varepsilon > 0$ and $p_r \geq 0$, $1 \leq r \leq n-1$. For an account of the relationships between conditions of type (A2) and (A3) we refer to [13, 16]. As the coefficient functions (2.1) are real, it follows from (A3) and [12] that the deficiency indices of $T_0$ are $(n, n)$ if $\tau$ is singular at only one endpoint of $I$, and $T_0$ has deficiency indices $(0, 0)$ if $\tau$ is singular at both endpoints of $I$. In the latter case, as $T_0$ is a nonnegative self-adjoint operator, and hence maximal accretive by (A1), the extension problem becomes trivial. Thus we may assume without loss that $I = [a, b)$, and $\tau$ is regular at $a$ and singular at $b$. In this case we have from [20, pp. 71, 78] that

$$\mathcal{D}(T_0) = \{ y \in \mathcal{D}: y^{(k)}(a) = 0, \ 0 \leq k \leq 2n-1 \}.$$  

Set $H = L^2_{\omega}[a, b) \times L^2_{\omega}[a, b)$ and $P = G(T_0)$. Under the indefinite inner product

$$Q(\tilde{u}, \tilde{v}) = (u_2, v_1) + (u_1, v_2),$$

where $\tilde{u} = \{u_1, u_2\}$ and $\tilde{v} = \{v_1, v_2\}$ are in $H$, and $(\cdot, \cdot)$ is the usual inner product on $L^2_{\omega}[a, b)$, $H$ is a Krein space. As in [11], for $\tilde{u} = \{u, T_0u\} \in P$ we have

$$Q(\tilde{u}, \tilde{u}) = 2\|u\|_D^2.$$
and thus $P$ is a closed positive subspace of $H$ with a trivial null space.

From (2.6) and [11, (2.16)] we have that $P' = G(-T_1)$, and from [11, (2.13)] $P' = M_+ \oplus M_-$ where $M_+ = P' \cap H_+$. Thus $\tilde{u} = \{u, -T_1u\} \in M_+$ if and only if $-T_1u = u$. As the deficiency indices of $T_0$ are $(n, n)$ it follows that

$$\text{nul}(-T_1 - 1) = \text{def}(-T_0 - 1) = n. \quad (3.3)$$

Consequently there exist $n$ real functions $w_1, w_2, \ldots, w_n$ in $\mathcal{D}$ orthogonal in $L^2_\omega[a, b]$, and such that

$$-T_1 w_i = w_i, \quad \|w_i\| = 1, \quad 1 \leq i \leq n.$$

Thus $M_+$, the linear span of $\{w_1, \ldots, w_n\}$, is a positive definite subspace of $H$.

The negative subspace, $M_-$, of $P'$ may be characterized as follows. Let $\tilde{u} = \{u, -T_1u\} \in M_-$. Then, if $\tilde{w}_i = \{w_i, -T_1w_i\} = \{w_i, w_i\}$ we have for $1 \leq i \leq n$,

$$0 = Q(\tilde{u}, \tilde{w}_i) = (-T_1u, w_i) + (u, -T_1w_i)$$
$$= [u, w_i](a) + 2(u, w_i)$$
$$= D(w_i, u)(a) - D(u, w_i)(a) + 2(u, w_i). \quad (3.4)$$

Set

$$\mathbf{u}_1(x) = [u(x), u^{[1]}(x), \ldots, u^{[n-1]}(x)]^T,$$
$$\mathbf{u}_2(x) = [u^{[2n-1]}(x), u^{[2n-2]}(x), \ldots, u^{[n]}(x)]^T,$$
$$\mathbf{w}(x) = [w_1(x), w_2(x), \ldots, w_n(x)]^T,$$
$$C(u, w) = [(u, w_1), (u, w_2), \ldots, (u, w_n)]^T,$$

$$W_1(x) = \begin{bmatrix} w_1(x) & \cdots & w_1^{[n-1]}(x) \\ \vdots & \ddots & \vdots \\ w_n(x) & \cdots & w_n^{[n-1]}(x) \end{bmatrix}, \quad W_2(x) = \begin{bmatrix} w_1^{[2n-1]}(x) & \cdots & w_1^{[n]}(x) \\ \vdots & \ddots & \vdots \\ w_n^{[2n-1]}(x) & \cdots & w_n^{[n]}(x) \end{bmatrix}$$

$$W_4(x) = \begin{bmatrix} w_1^{[2n-1]}(x) & \cdots & w_1^{[n]}(x) \\ \vdots & \ddots & \vdots \\ w_{2n-1}^{[2n-1]} & \cdots & w_{2n}^{[n]}(x) \end{bmatrix}. \quad (3.5)$$

Then $\tilde{u} = \{u, -T_1u\} \in M_-$ if and only if

$$W_1(a) \mathbf{u}_2(a) = W_2(a) \mathbf{u}_1(a) + 2C(u, w).$$
Note that \( \det W_1(a) \neq 0 \), since \( \{ w_i : i = 1, ..., n \} \) is linearly independent, and hence \( W^{-1}(a) \) exists. Consequently, \( \bar{u} = \{ u, -T_1 u \} \in M_\perp \) if and only if

\[
\mathbf{u}_2(a) = W_1^{-1}(a) \{ W_2(a) \mathbf{u}_1(a) + 2C(u, w) \}.
\] (3.6)

Define the space \( X \) by

\[
X = H_n \times \mathbb{C}^n \times \mathbb{C}^n,
\]

where \( H_n \) is defined to be the completion of \( \mathcal{D} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_D \). We identify \( P' \) with a subspace of \( X \) via the association

\[
\tilde{u} \leftrightarrow \hat{u},
\] (3.7)

where \( \tilde{u} = \{ u, -T_1 u \} \in P' \) and \( \hat{u} = \{ u, \mathbf{u}_1(a), \mathbf{u}_2(a) \} \in X \), the vectors \( \mathbf{u}_i(a) \) being defined by (3.5). For \( \hat{u} = \{ u, \mathbf{a}_1, \mathbf{a}_2 \} \) and \( \hat{v} = \{ v, \mathbf{b}_1, \mathbf{b}_2 \} \) in \( X \) (with \( u, v \in H_n \) and \( \mathbf{a}_i, \mathbf{b}_i \in \mathbb{C}^n \)) define an inner product \( Q_1(\cdot, \cdot) \) by

\[
Q_1(\hat{u}, \hat{v}) = - \langle \mathbf{a}_1, \mathbf{b}_2 \rangle - \langle \mathbf{a}_2, \mathbf{b}_1 \rangle - 2(u, v)_D.
\] (3.8)

where \( \langle \cdot, \cdot \rangle \) is the usual Euclidean inner product. Then, if \( \tilde{u}, \tilde{v} \in P' \) and \( \hat{u} \leftrightarrow \tilde{u} \in X, \hat{v} \leftrightarrow \tilde{v} \in X, \)

\[
Q(\tilde{u}, \tilde{v}) = ( -T_1 u, v ) + ( u, -T_1 v )
\]

\[
= -2(u, v)_D - \langle \mathbf{u}_2(a), \mathbf{v}_1(a) \rangle - \langle \mathbf{u}_1(a), \mathbf{v}_2(a) \rangle
\]

\[
= Q_1(\hat{u}, \hat{v}).
\]

Thus the correspondence (3.7) is a one-to-one inner product preserving map of \( P' \) into \( X \). We also have

**Lemma 3.1.** \( X \) is a Pontrjagin space of index \( n \) (i.e., a \( \Pi_n \) space).

**Proof.** This is a consequence of the decomposition

\[
X = X_+ \oplus X_-
\] (3.9)

where

\[
X_+ = \{ \{ 0, \mathbf{b}, -\mathbf{b} \} : \mathbf{b} \in \mathbb{C}^n \},
\]

\[
X_- = \{ \{ u, \mathbf{a}, \mathbf{a} \} : u \in H_n, \mathbf{a} \in \mathbb{C}^n \},
\]

and any \( \hat{u} = \{ u, \mathbf{a}, \mathbf{b} \}, u \in H_n, \mathbf{a}, \mathbf{b} \in \mathbb{C}^n \) can be written as \( \hat{u} = \hat{u}_+ + \hat{u}_- \), where

\[
\hat{u}_- = \{ u, \frac{1}{2}(\mathbf{a} + \mathbf{b}), \frac{1}{2}(\mathbf{a} + \mathbf{b}) \},
\]

\[
\hat{u}_+ = \{ 0, \frac{1}{2}(\mathbf{a} - \mathbf{b}), \frac{1}{2}(\mathbf{b} - \mathbf{a}) \}.
\] (3.10)
The strong norm on $X$ (with respect to (3.9)) is given for $\hat{u} = \{u, \alpha, \beta\}$ by
\[
\|u\|_X^2 = Q_1(\hat{u}_+, \hat{u}_+) - Q_1(\hat{u}_-, \hat{u}_-)
= \langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle + 2(u, u)_D.
\] (3.11)

We show eventually that $\hat{M}_-$, the intrinsic completion of $M_-$, may be identified with the strong closure of $M_-$ in $X$. We need several lemmas; the proofs are very close to those given for Lemmas 3.2, 3.3, 3.4, and 3.5, respectively, in [11].

**Lemma 3.2.** Denote by $L$ the $Q_1$-orthogonal complement of $M_+$ in $X$. Then $L$ is the set of all $\{u, \alpha_1, \alpha_2\}$ in $X$ such that
\[
\alpha_2 = W_1^{-1}(a)\{ W_2(a) \alpha_1 + 2C(u, w) \}.
\] (3.12)

**Lemma 3.3.** Under the identification (3.7), $M_+$ is a positive definite subspace of $X$ of dimension $n$.

**Lemma 3.4.** $L$ is a negative definite subspace of $X$.

**Lemma 3.5.** The strong topology (generated by (3.11)) and the intrinsic topology (generated by (3.8)) are equivalent on $L$, and $L$ is intrinsically closed.

Corresponding to [11, Lemma 3.6], we have

**Lemma 3.6.** Assume that
\[
p_i \geq 0, \quad 1 \leq i \leq n \quad \text{and} \quad \omega(x) \geq \mu, \quad p_0(x) \geq \mu \text{ for } x \in [a, a + \eta]
\] (3.13)
for some constants $\mu > 0$ and $\eta > 0$. Then the set
\[
B = \{u \in \mathcal{D}: \{u, -T_1 u\} \in M_\}
\]
is dense in $H_n$ in the norm $\| \cdot \|_D$.

**Proof.** As in the Proof of [11, Lemma 3.6], any $u \in H_n$ can be identified with a function having $(n-1)$ absolutely continuous derivatives on $[a, a + \eta]$. Also for any $\varepsilon > 0$ there exists a constant $K(\varepsilon) > 0$ such that
\[
|u^{[k]}(a)|^2 \leq \varepsilon \|u\|_D^2 + K(\varepsilon)\|u\|^2, \quad u \in H_n, \quad 0 \leq k \leq n - 1.
\]
Define
\[
h[u, v] = (u, v)_D + \langle \alpha_2, v_t(a) \rangle, \quad u, v \in H_n.
\]
where
\[ \alpha_2 = W^{-1}_2(a) \{ W_2(a) u_1(a) + 2C(u, w) \}. \]

By the method of [11, Lemma 3.6] it follows that \( h[\cdot, \cdot] \) is a closed sectorial form in \( L^2_\omega[a, b) \). Also the restriction \( S \) of \( T_1 \) to \( B \) satisfies
\[ (Su, v) = h[u, v], \quad u \in B, v \in H_n \]
on account of (3.6). \( S \) is therefore sectorial and since \( M_- \) is negative, \( S \) is also accretive. To complete the proof we prove that \( -1 \) lies in the resolvent set of \( S \).

Let \( f \in L^2_\omega[a, b) \) have compact support, but otherwise be arbitrary. We try to solve \( (\tau + 1) y = f \) so that \( y \in \mathcal{D}(S) \) and \( y = y(f) \) is bounded on \( L^2_\omega[a, b) \). Choose \( w_i, n + 1 \leq i \leq 2n \), to be linearly independent solutions of \( (\tau + 1) y = 0 \) satisfying
\[ w_i^{(k)}(a) = 0, \quad 0 \leq k \leq n - 1, n + 1 \leq i \leq 2n. \]

Each of the functions \( w_i, n + 1 \leq i \leq 2n \), as well as all linear combinations of them are not in \( \mathcal{D} \). To see this, let \( v \) be a nontrivial linear combination of these functions that lies in \( \mathcal{D} \). Then \( v^{(k)}(a) = 0, 0 \leq k \leq n - 1 \), and by (3.1)
\[ -\|v\|^2 = (T_1 v, v) = (v, v)_D; \]
thus, \( v = 0 \), a contradiction.

The general solution of \( (\tau + 1) y = f \) is given by (c.f. [20, Eq. (24), p. 59])
\[ y(x) = \sum_{k=1}^{2n} a_k w_k(x) + \sum_{k=1}^{2n} w_k(x) \int_a^x v_k \omega f, \]
where \( v_k \) is given by the Wronskian quotient,
\[ v_k(x) = (-1)^{2n+k} \frac{W(w_1, \ldots, w_{k-1}, w_{k+1} \ldots, w_{2n})(x)}{W(w_1, \ldots, w_{2n})(x)}. \]

As \( f \) has compact support, and no linear combination of the functions \( w_i, \quad n + 1 \leq k \leq 2n \), can lie in \( \mathcal{D} \), it follows that for these values of \( k \), \( a_k = -\int_a^x v_k \omega f \). Thus (3.16) becomes
\[ y(x) = \sum_{k=1}^n a_k w_k(x) + \sum_{k=1}^n w_k(x) \int_a^x \omega v_k f - \sum_{k=n+1}^{2n} w_k(x) \int_a^x \omega v_k f. \]
First, consider the operator $M$ defined on functions $f$ with compact support in $\mathcal{D}$ by

$$(Mf)(x) = \sum_{k=1}^{n} w_k(x) \int_{a}^{x} \omega v_k f - \sum_{k=n+1}^{2n} w_k(x) \int_{a}^{b} v_k f \omega.$$  \hspace{1cm} (3.19)

Observe that $(\tau + 1)(Mf) = f$, and that

$$(Mf)^{[k]}(a) = - \sum_{j=n+1}^{2n} w_j^{[k]}(a) \int_{a}^{b} \omega v_j f,$$  \hspace{1cm} (3.20)

for $0 \leq k \leq 2n - 1$. Thus for $n + 1 \leq r \leq 2n$, and setting $b_j = \int_{a}^{b} \omega v_j f$,

$$[Mf, w_r](a) = D(w_r, Mf)(a) - D(Mf, w_r)(a)$$

$$= - \sum_{k=1}^{n} w_r^{[2n-k]}(a) \sum_{j=n+1}^{2n} w_j^{[k-1]}(a) b_j$$

$$+ \sum_{k=1}^{n} w_r^{[k-1]}(a) \sum_{j=n+1}^{2n} w_j^{[2n-k]}(a) b_j$$

$$= 0, \quad \text{from (3.15).}$$ \hspace{1cm} (3.21)

Let $Q$ be the restriction of $\tau$ to the domain

$$\mathcal{D}(Q) = \{ y \in \mathcal{D} : [y, w_i](a) = 0, n + 1 \leq i \leq 2n \}. \hspace{1cm} (3.22)$$

It follows from standard theory [20, Theorem 4', p. 79] that $Q$ is a self-adjoint extension of $T_0$. As the latter is positive, the essential spectrum of $Q$ lies in $[0, \infty)$. Furthermore, if $y$ is a solution of $(\tau + 1) y = 0$ satisfying $[y, w_i](a) = 0, n + 1 \leq i \leq 2n$, then $y$ is a linear combination of $w_i, n + 1 \leq i \leq 2n$. To see this, let $y = \sum_{k=1}^{2n} \alpha_k w_k$. Then

$$\sum_{k=1}^{n} \alpha_k [w_k, w_i](a) = 0, \quad n + 1 \leq j \leq 2n,$$

i.e., $Aa = 0$, where $a = (\alpha_1, ..., \alpha_n)^T$ and $A$ is the $n \times n$ matrix with the entry $[w_k, w_j](a)$ in row $j$ and column $k$. From (3.15) one can show that $A = W_4(a)(W_1(a))^T$ where $W_4(a)$ is defined in (3.5). We know already that $\det W_1(a) \neq 0$. As $\{w_1, ..., w_{2n}\}$ are linearly independent solutions of $(\tau + 1) y = 0$, their Wronskian, $W$, is nonsingular. As the columns of $W$ involving $w_j, n + 1 \leq j \leq 2n$, are linearly independent, it follows from (3.15) again that $\det W_4(a) \neq 0$. Consequently $A$ is nonsingular, and $a = 0$, as required. Now, as $w_k, n + 1 \leq k \leq 2n$, and all their linear combinations, lie
outside $\mathcal{D}$, it follows that $-1$ cannot be an eigenvalue for $Q$. Thus $-1$ must lie in the resolvent set for $Q$, and hence $(Q + 1)^{-1}$ exists as a bounded linear operator defined on all of $L^2_\omega[a, b)$.

Returning to (3.19), it is clear from (3.21) that for $f$ with compact support, $Mf$ and $(Q + 1)^{-1}f$ are solutions of $(\tau + 1)y = f$ in $\mathcal{D}$ satisfying the same boundary conditions, and thus $Mf = (Q + 1)^{-1}f$.

We now choose $a_1, \ldots, a_n$ so that the function $y$ defined in (3.18) lies in $\mathcal{D}(S)$. From (3.4), $y \in \mathcal{D}(S)$ if and only if for $1 \leq i \leq n$, \[2(y, w_i) + (y, w_i)(a) = 0.\]

On substituting (3.18) into this identity we obtain
\[
\sum_{k=1}^{n} a_k(w_k, w_i) + 2(Mf, w_i) + \sum_{k=1}^{n} a_k[w_k, w_i](a) + [Mf, w_i](a) = 0.
\]

Now from (3.1) for $1 \leq k, i \leq n$, \[w_k, w_i](a) = (\tau w_i, w_k) - (w_i, \tau w_k) = 0.
\]

Similarly, as $\tau(Mf) = f - Mf$, \[[Mf, w_i](a) = (\tau w_i, Mf) - (w_i, \tau(Mf)) = -(w_i, f).
\]

Consequently, setting $a = (a_1, \ldots, a_n)^T$, \[(Ga)_i = -2(Mf, w_i) + (w_i, f), \quad i = 1, \ldots, n, \tag{3.23}
\]

where $G$ is the Gram matrix for $\{w_1, \ldots, w_n\}$. As $G$ is nonsingular, (3.23) determines $a$ uniquely. Also, when $f \in L^2_\omega[a, b)$ does not have compact support, one can replace $M$ by $(Q + 1)^{-1}$ in (3.23) and (3.18) so that \[y(x) = \sum_{k=1}^{n} a_k w_k(x) + (Q + 1)^{-1}f(x).
\]

Thus, from (3.23), $y = y(f)$ is a bounded function of $f$ on $L^2_\omega[a, b)$. This completes the proof.

**Lemma 3.7.** Assuming (3.13), the strong completion of $M_-$ in $X$ is given by

\[M = \{ [u, u_1(a), a_2] : u \in H_n, a_2 \in \mathbb{C}^n, \text{ and} \quad a_2 = W^{-1}_1(a)[W_2(a)u_1(a) + 2C(u, w)] \}.
\]

**Proof:** Using Lemma 3.6, the proof is very similar to the proof of [11, Lemma 3.7].
Lemma 3.8. Assuming (3.13), the boundary space \( \hat{H} \) as a subspace of \( X \) is given by

\[
\hat{H} = M + \oplus M = \{ \{ u, u_1(a), a \}; u \in H_n, a \in \mathbb{C}^n \}.
\]

Proof. It is clear from Lemmas 3.5 and 3.7 that \( \overline{M} = M \). Thus we only need show that each \( \hat{v} = \{ v, v_1(a), a \} \), where \( v \in H_n \) and \( a \in \mathbb{C}^n \) are arbitrary, may be written in the form

\[
\hat{v} = \sum_{i=1}^{n} a_i \hat{w}_i + \hat{u}
\]

for some constants \( a_i \), \( 1 \leq i \leq n \), and some \( \hat{u} = \{ u, u_1(a), a_2 \} \in M \). We choose \( u \in H_n \) so that

\[
u = v - \sum_{i=1}^{n} a_i w_i,
\]

where the numbers \( a_i \) are to be determined so that \( \hat{u} \in M \), i.e., so that for \( 1 \leq i \leq n \),

\[
2(u, w_i) = -D(w_i, u)(a) + \langle a_2, w_{1,i}(a) \rangle
\]

by (3.4). Note that from (3.24)

\[
\alpha = \sum_{i=1}^{n} a_i w_{2,i}(a) + a_2,
\]

where

\[
\alpha_2 = W_1^{-1}(a) \{ W_2(a) u_1(a) + 2C(u, w) \}.
\]

On solving (3.27) for \( \alpha_2 \) and substituting this and (3.25) into (3.26) we obtain, for \( 1 \leq i \leq n \),

\[
2(v, w_i) - 2 \sum_{j=1}^{n} a_j(w_j, w_i)
\]

\[
= -D(w_i, v)(a) + \sum_{j=1}^{n} a_j D(w_i, w_j)(a)
\]

\[
+ \langle a, w_{1,i}(a) \rangle - \sum_{j=1}^{n} a_j D(w_j, w_i)(a)
\]

\[
= -D(w_i, v)(a) + \langle a, w_{1,i}(a) \rangle
\]
as
\[
D(w_i, w_j)(a) - D(w_j, w_i)(a) = [w_j, w_i](a) \\
= (\tau w_i, w_j) - (w_i, \tau w_j) \quad \text{by (3.1)} \\
= 0.
\]

Thus, if \( a = (a_1, \ldots, a_n)^T \), and \( G \) is the Gram matrix for \( \{w_1, \ldots, w_n\} \), we have
\[
(Ga)_i = 2(v, w_i) + D(w_i, v)(a) - \langle a, w_1, (a) \rangle. \tag{3.29}
\]

As \( G \) is invertible, (3.29) defines the vector \( a \) uniquely. With this value for \( a \), and \( u, \alpha_2 \) given by (3.25) and (3.28), \( \hat{u} \in M \), as required.

We now prove

**Theorem 3.9.** Under the assumption (3.13), an operator \( T \) is a maximal accretive extension of \( T_0 \) if and only if its adjoint, \( T^* \), is a restriction of \( T \) to a domain of the form

\[
D(T^*) = \{ u \in \mathcal{D} : \langle u_1(a), \mu_1^{(i)} \rangle + \langle u_2(a), \phi_1^{(i)}(a) \rangle + 2(u, \phi_1^{(i)})_D = 0, \ 1 \leq i \leq n \}
\]

for some functions \( \phi_1^{(i)} \in H_n \), and vectors \( \mu_1^{(i)} \in \mathbb{C}^n \), \( 1 \leq i \leq n \), satisfying

\[
\langle \mu_1^{(i)}, \phi_1^{(i)}(a) \rangle + \langle \phi_1^{(i)}(a), \mu_1^{(i)} \rangle + 2(\phi_1^{(i)}, \phi_1^{(i)})_D \begin{cases} 
0 & \text{if } i \neq j, \\
\leq 0 & \text{if } i = j.
\end{cases} \tag{3.31}
\]

**Proof.** As \( \hat{H} \) is a \( \Pi_n \) space, all maximal positive subspaces of \( \hat{H} \) have dimension \( n \), and are generated by a set \( \{ \tilde{\phi}_i^{(i)} = \{ \phi_1^{(i)}, \phi_1^{(i)}(a), \mu_1^{(i)} \} : 1 \leq i \leq n \} \), where the \( \phi_1^{(i)} \) and \( \mu_1^{(i)} \) satisfy (3.31). Each maximal negative subspace, \( \hat{N} \), in \( \hat{H} \) is the \( Q_1 \)-orthogonal complement in \( \hat{H} \) of such a subspace. By [11, Theorem 2.2],

\[
N = \{ \{ u, -T_1 u \} : \hat{u} = \{ u, u_1(a), u_2(a) \} \in \hat{H} \} \\
\text{satisfies } Q_1(\hat{u}, \phi_1^{(i)}) = 0, \ 1 \leq i \leq n \}
\]

is the graph of the adjoint of a maximal accretive extension of \( T_0 \), and all such extensions are found in this way.

**Remark.** Examples of these extensions may be constructed by a similar method to that used in the remark following [11, Theorem 3.9].
Once again (cf. [11, Theorem 3.10]), the realization of $\hat{H}$ given in Lemma 3.8 is not unique. The next theorem shows that any maximal negative subspace of $X$ gives rise to a maximal accretive extension of $T_0$.

**Theorem 3.10.** Under the assumption (3.13), an operator $T$ is a maximal accretive extension of $T_0$ if and only if its adjoint, $T^*$, is a restriction of $T_1$ to a domain of the form

$$\mathcal{D}(T^*) = \{ u \in \mathcal{D}: \langle u_2(a), \eta^{(i)} \rangle + \langle u_1(a), \mu^{(i)} \rangle + 2(u, \phi^{(i)})_D = 0, 1 \leq i \leq n \}$$

(3.32)

for some functions $\phi^{(i)} \in H_n$ and vectors $\mu^{(i)}, \eta^{(i)} \in \mathbb{C}^n$, $1 \leq i \leq n$, satisfying the conditions

$$\begin{cases} \langle \mu^{(i)}, \eta^{(j)} \rangle + \langle \eta^{(i)}, \mu^{(j)} \rangle + 2(\phi^{(i)}, \phi^{(j)})_D = 0, & \text{if } i \neq j, \\ \leq 0, & \text{if } i = j. \end{cases} \quad (3.33)$$

**Proof.** From Theorem 3.9, it is only necessary to show that each choice of $\{\phi^{(i)} = \{\phi^{(i)}, \mu^{(i)}, \eta^{(i)}\}: 1 \leq i \leq n\}$ satisfying (3.33) defines a maximal accretive restriction of $T_1$.

For $\hat{\phi} = \{\phi, \mu, \eta\}$ in $X$ we define a mapping $h$ from $X$ to $\hat{H}$ by

$$h(\hat{\phi}) = \hat{\psi} = \{\phi + \theta, \mu, \eta + 2\theta_2(a)\}, \quad (3.34)$$

where $\theta$ satisfies

$$T_1 \theta = 0, \quad \theta_1(a) = \mu - \phi_1(a). \quad (3.35)$$

As before, in the proof of [11, Theorem 3.10], the conditions (3.35) define $\theta$ uniquely, as there are precisely $n$ linearly independent solutions of $\tau(\theta) = 0$ in $\mathcal{D}(T_1)$ in this case. Observe that, by construction, for $\hat{u} = \{u, u_1(a), u_2(a)\}$ in $P'$,

$$Q_1(\hat{u}, \hat{\phi}) - Q_1(\hat{u}, \hat{\psi}) = 0. \quad (3.36)$$

Also, the mapping $h$ is linear (from the uniqueness of $\theta$), and

$$Q_1(\hat{\psi}, \hat{\psi}) = Q_1(\hat{\phi}, \hat{\phi}) + 2(\theta, \theta)_D. \quad (3.37)$$

for any $\hat{\phi}$ in $X$. The latter identity follows from the identities

$$(\theta, \phi)_D = -\langle \theta_2(a), \phi_1(a) \rangle,$$

$$(\theta, \theta)_D = -\langle \theta_2(a), \theta_1(a) \rangle = -\langle \theta_1(a), \theta_2(a) \rangle,$$
valid for any \( \phi \) in \( H_n \) and \( \theta \) in \( \mathcal{D}(T_1) \) with \( T_1 \theta = 0 \); these may be derived from (3.1).

Consider now the maximal positive subspace \( B \) in \( X \) generated by \( \{ \phi^{(i)}: 1 \leq i \leq n \} \). It follows from (3.37) and the linearity of \( h \) that \( h(B) \subset \hat{H} \) is a positive subspace. Furthermore, by (3.37) again, \( h \) is one-to-one when restricted to a positive subspace of \( X \). Consequently \( h(B) \) has dimension \( n \) and is therefore a maximal positive subspace of \( \hat{H} \). The remainder of the proof now follows that of [11, Theorem 3.10].

We are now in a position to describe the maximal accretive extensions of \( T_0 \) explicitly, via Theorem A.8 of the Appendix. For a given \( \gamma \in \mathbb{C}^n \) and \( \phi = (\phi^{(1)}, \ldots, \phi^{(n)})^T, \phi^{(i)} \in H_n, \) the partial adjoints for \( \tau \) in this case are defined by

\[
\tau_k^+ z = z^{[k]}, \quad 0 \leq k \leq n - 1
\]

\[
= (z - \gamma^T \phi)^{[k]}, \quad n \leq k \leq 2n.
\] (3.38)

**Theorem 3.11.** An operator \( T \) is a maximal accretive extension of \( T_0 \) if and only if it has the form

\[
\mathcal{D}(T) = \{ z \in L_0^2[a, b]: \tau_k^+ z \text{ is absolutely continuous,} \quad 0 \leq k \leq 2n - 1, \frac{1}{\omega} \tau_{2n}^+ z \in L_0^2[a, b], \}
\]

\[
\tau_k^+ z(a) = \sum_{i=1}^n \xi_i \mu_{k+1}^{(i)}, 0 \leq k \leq n - 1,
\]

\[
\tau_k^+ z(a) = -\sum_{i=1}^n \xi_i \eta_{2n-k}^{(i)}, n \leq k \leq 2n - 1, \}
\]

\[
Tz = \frac{1}{\omega} \tau_{2n}^+ z, \quad z \in \mathcal{D}(T)
\]

for some \( \gamma \in \mathbb{C}^n, \mu^{(i)}, \eta^{(i)} \in \mathbb{C}^n, \phi^{(i)} \in H_n, \) \( 1 \leq i \leq n \), satisfying \( \gamma^T \phi = 2\xi^T \phi \), and (3.33).

The case \( n = 1 \) here is particularly simple. We consider the operator

\[
\tau y = (-p_0(x) y')' + p_1(x) y, \quad a \leq x < b,
\]

assuming that \( 1/p_0, p_1 \) are locally integrable on \( [a, b] \), \( \int_a^b p_0^{-1/2} = \infty \), \( p_0 \geq \mu_0 > 0 \) on \( [a, a + \eta] \), where \( \eta > 0 \), and \( p_1 \geq \mu > 0 \) on \( [a, b] \). These assumptions ensure that conditions (3.13), (A1), (A2), and (A3) are satisfied (see [10, Corollary 1, p. 88]). Then we have

**Corollary 3.12.** Under the above assumptions on \( \tau \), \( T \) is a maximal
accretive extension of $T_0$ if and only if $T$ is the Dirichlet extension of $T_0$, or it has the form

$$\mathcal{D}(T) = \left\{ z \in L^2[a, b]: z \text{ and } p_0 \left( z - \frac{2z(a)}{\alpha} \phi \right) \right\}$$

are locally absolutely continuous on $[a, b)$

$$\tau_+^ a z \in L^2[a, b], \text{ and}$$

$$\alpha p_0(a) \left( z - \frac{2z(a)}{\alpha} \phi \right)'(a) + \beta z(a) = 0 \right\},$$

$$Tz = \left( p_0(x) \left( z - \frac{2z(a)}{\alpha} \phi(x) \right)' + p_1(x) \left( z - \frac{2z(a)}{\alpha} \phi(x) \right) \right), \quad z \in \mathcal{D}(T)$$

for some $\alpha, \beta \in \mathbb{C}, \, \alpha \neq 0$, and $\phi$ in $H_1$ satisfying

$$\text{Re}(\beta \bar{\alpha}) + \int_a^b \left( p_0 |\phi'|^2 + p_1 |\phi|^2 \right) \leq 0. \quad (3.39)$$

In this case $T^*$ is the restriction of $T_1$ with domain

$$\mathcal{D}(T^*) = \{ y \in \mathcal{D}: \bar{\alpha} y(a) + \beta p_0(a) y'(a) + 2 \int_a^b \left( p_0 y' \bar{\phi}' + p_1 y \bar{\phi} \right) = 0 \}.$$  

\textbf{Proof.} This follows directly from Theorem 3.11. As $\hat{\phi} = \{ \phi, \alpha, \beta \}$ must be a one-dimensional subspace of $H_1 \times \mathbb{C} \times \mathbb{C}$, it follows from (3.39) that $\alpha$ or $\beta$ must be nonzero. Consequently one can eliminate $\xi$ from the parametric boundary conditions $\tau_+^ a z(a) = \xi \alpha, \tau_+^ b z(a) = - \xi \beta$ to obtain the (Dirichlet) boundary condition $z(a) = 0$ when $\alpha$ (and hence $\phi$) is zero, or, if $\alpha \neq 0$, the boundary condition $\alpha \tau_+^ a z(a) + \beta \tau_+^ b z(a) = 0$. \hspace{1em} \blacksquare

\textbf{Remarks.} (1) Observe that in Theorem 3.11 we must have the $\hat{\phi}^{(i)} \neq \hat{\phi}$ in $X$ in order to maintain the correct dimensions.

(2) In Corollary 3.12, on setting $\phi = 0, \alpha = 0, \beta \neq 0$ in (3.39) we get the Dirichlet extension of $T_0$, a positive (hence maximal accretive) self-adjoint operator. Similarly, the Neumann extension is given by $\phi = 0, \beta = 0, \alpha \neq 0$.

\textbf{APPENDIX}

We consider here in detail the method, due to Brown and Krall [4], by which one obtains the maximal accretive extensions of $T_0$, given the
specific form of their adjoints. The treatment follows that of [4,5], except for the modifications needed to handle quasi-differential expressions.

First, we consider the theory for the regular case in which the expression $\tau$ is defined on a finite closed interval $[a, b]$ of the real axis. Let $S$ be the restriction of $T_1$ defined by the conditions

$$U_i y = -\langle y_2(b), \rho^{(i)} \rangle - \langle y_1(b), v^{(i)} \rangle + \langle y_2(a), \eta^{(i)} \rangle + \langle y_1(a), \mu^{(i)} \rangle + 2(y, \phi^{(i)})_D = 0, \quad 1 \leq i \leq 2n \quad (A.1)$$

where $\rho^{(i)}, v^{(i)}, \eta^{(i)}, \mu^{(i)}$ denote fixed vectors in $\mathbb{C}^n$, $1 \leq i \leq 2n$, and the $\phi^{(i)}, 1 \leq i \leq 2n$, denote given functions in $H_n$. Assume also that $\mathcal{D}(S)$ is dense in $L^2_0[a, b]$. The latter assumption simplifies the presentation somewhat, and is sufficient for our present needs, as the operators $S$ will be closed and maximal accretive, and therefore densely defined by [25, Theorem 2.3]. Our objective here is to determine the specific form of the operator $S^*$.

Let $\mathcal{D}_0 = \mathcal{D}(T_0) \cap \mathcal{D}(S)$, where $\mathcal{D}(T_0)$ is given by [20, p. 62],

$$\mathcal{D}(T_0) = \{ y \in \mathcal{D}: y^{[k]}(a) = y^{[k]}(b) = 0, 0 \leq k \leq 2n - 1 \}.$$ 

Using (2.9), for $y$ in $\mathcal{D}_0$, the conditions (A.1) become

$$U_i r = 2(r, \phi^{(i)}) = 0 \quad (A.2)$$

for $1 \leq i \leq 2n$ and $r = (1/\omega) y^{[2n]}$. Let $\{u_k: 1 \leq k \leq 2n\}$ be a fixed fundamental set of solutions for the equation $\tau u = 0$. Then we have

**Lemma A.1.** Assume that the $u_k$ are real, and let $r \in L^2_0[a, b]$. Then

$$\hat{U}_i r = 0, \quad (u_i, r) = 0, \quad 1 \leq i \leq 2n, \quad (A.3)$$

if and only if $r = (1/\omega) y^{[2n]}$ for some $y$ in $\mathcal{D}_0$.

**Proof.** If $r = (1/\omega) y^{[2n]}$ for some $y$ in $\mathcal{D}_0$, then it is clear from (A.2) that $\hat{U}_i r = 0, 1 \leq i \leq 2n$. Also, by (2.7) and since $\tau u_i = 0, 1 \leq i \leq 2n$,

$$(r, u_i) = \int_a^b u_i(x) y^{[2n]}(x) d\omega = \int_a^b (\tau u_i)(x) y(x) d\omega = 0.$$ 

Conversely, if $r$ satisfies (A.3), then

$$y(x) = \sum_{i=1}^{2n} u_i(x) \int_a^x v_i(x) \omega r.$$ 

By standard theory (see, e.g., [20, p.59]), $(1/\omega) y^{[2n]} = r$. Moreover, as $(u_i, r) = 0, 1 \leq i \leq 2n, r \in \mathcal{N}(T_1)^\perp = \mathcal{R}(T_0)$; consequently $y \in \mathcal{D}(T_0)$. Finally,
as $\mathcal{D}_0 r = 0$, $1 \leq i \leq 2n$, and $y \in \mathcal{D}(T_0)$, it follows that $y \in \mathcal{D}(S)$ and the proof is complete.

**Remark.** Lemma A.1 may also be stated as

$$L^2_0[a, b] = \mathcal{A}(S_0) \oplus \mathcal{G},$$

where $\mathcal{G}$ denotes the subspace spanned by $\{u_i, \phi^{(i)}: 1 \leq i \leq 2n\}$ and $S_0$ denotes the restriction of $S$ to $\mathcal{D}_0$.

**Lemma A.2.** [18, p. 7]. Suppose $\lambda, \psi_1, ..., \psi_n$ are linear functionals defined on a linear space. If $\ker \lambda \supset \bigcap_{i=1}^n \ker \psi_i$, then $\lambda$ is a linear combination of the functionals $\psi_i$.

Let $\gamma \in \mathbb{C}^{2n}$ and $\phi^{(i)} \in H_n$, $1 \leq i \leq 2n$, be given. The formal partial adjoint expressions for $\tau$ are defined by

$$\tau^+_k z = z^{(k)}_k, \quad 0 \leq k \leq n - 1,$$

$$\tau^+_k z = (z - \gamma^T \phi)^{(k)}, \quad n \leq k \leq 2n.$$  \hspace{1cm} (A.4)

**Lemma A.3.** Let $z \in \mathcal{D}(S^*)$. Then there exists a vector $\gamma \in \mathbb{C}^{2n}$ such that $\tau_k^+ z$, $0 \leq k \leq 2n - 1$, are absolutely continuous on $[a, b]$ and $S^* z = (1/\omega) \tau_{2n}^+ z$.

**Proof.** Observe first that by (2.9) and assumption (A.2), the positive self-adjoint restriction $T_D$ of $T_1$ with Dirichlet boundary conditions is invertible. Thus $\mathcal{A}(T_D) = \mathcal{A}(T_1) = L^2_0[a, b]$. Let $y \in \mathcal{D}_0$ and choose $p$ in $\mathcal{D}$ so that $\tau p = S^* z$. Then for $z$ in $\mathcal{D}(S^*)$

$$(z, \tau y) = (z, Sy)$$

$$= (S^* z, y)$$

$$= (\tau p, y)$$

$$= (p, \tau y) \quad \text{by (2.7)} \hspace{1cm} (A.5)$$

i.e., $(z - p, \tau y) = 0$.

Set $\lambda(f) = (f, z - p)$. Then by (A.5), the kernel of $\lambda$ contains the set $\{\tau y: y \in \mathcal{D}_0\}$, which by Lemma A.1 is exactly the intersection of the kernels of the functionals

$$\psi_i(r) = (r, \phi^{(i)}), \quad i = 1, ..., 2n,$$

$$\psi_{2n+i}(r) = (r, u_i), \quad i = 1, ..., 2n.$$
Consequently, by Lemma A.2, there exist constants $c_1, \ldots, c_{2n}$ and a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{2n})^T \in \mathbb{C}^{2n}$ such that

$$\lambda = \sum_{i=1}^{2n} c_i \psi_i + \sum_{i=1}^{2n} \gamma_i \phi^{(i)},$$

i.e.,

$$z - p = \sum_{i=1}^{2n} c_i u_i + \sum_{i=1}^{2n} \gamma_i \phi^{(i)},$$

i.e.,

$$z - \gamma^T \Phi = p + \sum_{i=1}^{2n} c_i u_i.$$

Consequently, $z - \gamma^T \Phi \in \mathcal{D}$, and

$$\frac{1}{\omega} \tau_{2n} z = \tau \left( p + \sum_{i=1}^{2n} c_i u_i \right) = \tau(p) \quad \text{(as } \tau u_i = 0\text{)}$$

$$= S^* z. \quad (A.6)$$

**Lemma A.4.** (Green’s formula). Let $y \in \mathcal{D}$, let $\tau_k^+ z$, $0 \leq k \leq 2n - 1$, be absolutely continuous on $[a, b]$ and $(1/\omega) \tau_{2n}^+ z$ be in $L^2_\omega[a, b]$. Then

$$\int_a^b (\tau y - y \tau_{2n}^+ z) \omega = \left[ D((z - \gamma^T \Phi), y) - D(y, z) \right]_a^b + (y, \gamma^T \Phi)_D. \quad (A.7)$$

**Proof.**

$$\int_a^b \omega(\tau y - y \tau_{2n}^+ z) = \int_a^b (\tau y \gamma^{[2n]} - y(z - \gamma^T \Phi)^{[2n]}$$

$$= \int_a^b \left\{ (z - \gamma^T \Phi) y^{[2n]} - y(z - \gamma^T \Phi)^{[2n]} + y \gamma^T \Phi \right\}$$

$$= \left[ D((z - \gamma^T \Phi), y) - D(y, (z - \gamma^T \Phi)) - D(y, \gamma^T \Phi) \right]_a^b$$

$$+ (y, \gamma^T \Phi)_D$$

$$= \left[ D((z - \gamma^T \Phi), y) - D(y, z) \right]_a^b + (y, \gamma^T \Phi)_D. \quad \square$$
Lemma A.5. If $z \in D(S^*)$ then $z$ satisfies the parametric endpoint boundary conditions

\[
\tau^+_k z(a) = \sum_{i=1}^{2n} \xi_i \eta_{k+i}^{(i)}, \quad 0 \leq k \leq n - 1, \\
\tau^+_k z(a) = -\sum_{i=1}^{2n} \xi_i \mu_{2n-k}^{(i)}, \quad n \leq k \leq 2n - 1, \\
\tau^+_k z(b) = \sum_{i=1}^{2n} \xi_i \rho_{k+i}^{(i)}, \quad 0 \leq k \leq n - 1, \\
\tau^+_k z(b) = \sum_{i=1}^{2n} \xi_i \psi_{2n-k}^{(i)}, \quad n \leq k \leq 2n - 1,
\]

where $\xi = (\xi_1, ..., \xi_{2n})^T$ is any vector equivalent to $\gamma$ in the sense that $\xi^T \Phi = \frac{1}{2} \gamma^T \Phi$. If $\{\phi^{(i)}: 1 \leq i \leq 2n\}$ is linearly independent then $\xi = \frac{1}{2} \gamma$.

Proof. By Lemma (A.3) there exists a vector $\gamma$ such that $\tau^+_k z$, $0 \leq k \leq 2n - 1$, are absolutely continuous on $[a, b]$, and $(1/\omega) \tau^+_k z \in L^2_\omega[a, b]$. Hence by Green's formula (Lemma A.4) for fixed $z$, the functional $\lambda: \mathcal{D} \to \mathbb{C}$ defined by

\[
\lambda(y) = [D((z - \gamma^T \Phi), y) - D(y, \bar{z})]_\mathcal{D} + (y, \gamma^T \Phi)_\mathcal{D}
\]

has the property that its kernel contains $\mathcal{D}(S)$. By Lemma A.2, there exists a vector $\xi \in \mathbb{C}^{2n}$ such that $\lambda(y) = \sum_{i=1}^{2n} \xi_i U_i(y)$. Hence

\[
\begin{align*}
\lambda(y) & = \sum_{i=1}^{2n} \xi_i \{ -\langle y_2(b), \psi^{(i)} \rangle - \langle y_1(b), \psi^{(i)} \rangle + \langle y_2(a), \eta^{(i)} \rangle \\
& \quad + \langle y_1(a), \mu^{(i)} \rangle + 2(y, \phi^{(i)})_\mathcal{D} \}.
\end{align*}
\]

Consequently,

\[
(y, (\gamma - 2\xi^T \phi))_\mathcal{D} = \left\langle y_2(b), z_1(b) - \sum_{i=1}^{2n} \xi_i \psi^{(i)} \right\rangle \\
+ \left\langle y_1(b), -(z - \gamma^T \phi)_2(b) - \sum_{i=1}^{2n} \xi_i \psi^{(i)} \right\rangle \\
+ \left\langle y_2(a), -z_1(a) + \sum_{i=1}^{2n} \xi_i \eta^{(i)} \right\rangle \\
+ \left\langle y_1(a), (z - \gamma^T \phi)_2(a) + \sum_{i=1}^{2n} \xi_i \mu^{(i)} \right\rangle.
\]

(A.9)
As the right-hand side of the last equation depends only on the values of \( y \) at \( a \) and \( b \), it follows that each side is identically zero for all \( y \in \mathcal{D} \). Clearly \( \gamma^T \Phi = 2z^T \Phi \). Furthermore from [20, Lemma 2, p. 63], one can choose \( y \in \mathcal{D} \), such that \( y_1(a), y_1(b), y_2(a), y_2(b) \) take arbitrary prescribed values. Thus (A.8) follows.

We now have

**Theorem A.6.** The adjoint, \( S^* \), of the restriction \( S \) of \( T_1 \) defined by the boundary conditions (A.1) is defined by

\[
B = \{ z \in L^2_w[a,b] : \text{there exists } \eta \in \mathbb{C}^{2n} \text{ such that } \tau^+_k z, 0 \leq k \leq 2n - 1, \text{ defined by (A.4) are absolutely continuous on } [a,b], \\
(1/\omega) \tau^-_{2n} z \in L^2_w[a,b] \text{ and } z \text{ satisfies the boundary conditions (A.8)} \},
\]

\[
S^*z = (1/\omega) \tau^-_{2n} z, z \in B = \mathcal{D}(S^*).
\]

**Proof.** By Lemmas A.3 and A.5, it is clear that \( \mathcal{D}(S^*) \subseteq B \), and \( S^*z = \tau^+_k z \) for all \( z \) in \( \mathcal{D}(S^*) \). Conversely, if \( z \in B \), it follows from Green's identity (Lemma A.4) that for all \( y \in \mathcal{D}(S) \)

\[
(y, \tau^+_k z) = (Sy, z).
\]

Consequently, \( z \in \mathcal{D}(S^*) \) and the proof is complete.

Finally, we consider the singular case when \( I = [a, b) \) is half-open, and \( \tau \) is regular at \( a \) and singular at \( b \). Following the method outlined in [5] we consider a restriction \( S \) of the maximal operator \( T_1 \) defined by

\[
D(S) = \{ y \in D : U_i y = 0, i = 1, \ldots, n \}, \quad (A.10)
\]

where

\[
U_i y = \langle y_1(a), \mu^{(i)} \rangle + \langle y_2(a), \eta^{(i)} \rangle + 2(y, \phi^{(i)})_D, \quad 1 \leq i \leq n \quad (A.11)
\]

and \( \phi^{(i)} \in H_n, \mu^{(i)}, \eta^{(i)} \in \mathbb{C}^n, 1 \leq i \leq n \). Assume that \( D(S) \) is dense in \( L^2_w[a,b] \). As usual we denote the minimal operator for \( \tau \) on \([a,b)\) by \( T_0 \) (see (3.2)). Observe that as the graph of \( S \) is an orthogonal complement, \( S \) is a closed operator. By analogy with (A.4), and for \( \gamma \in \mathbb{C}^n \) (depending on \( z \)), we define the partial adjoint expressions for \( \tau \) by

\[
\begin{align*}
\tau^-_k z &= z^{[k]}, \\
\tau^+_k z &= (z - \gamma^T \Phi)^{[k]},
\end{align*} \quad 0 \leq k \leq n - 1, \quad (A.12)
\]

Let \( A = [c,d] \) be an arbitrary finite interval in \([a,b)\), and define \( S_A \) as
the restriction of $T_{1,d}$ (the maximal operator for $\tau$ on $A$) with boundary conditions

$$U_{i,d} y = \langle y_1(c), \mu^{(i)} \rangle + \langle y_2(c), \eta^{(i)} \rangle + 2(y, \phi^{(i)} )_{D,d} = 0, \quad 1 \leq i \leq n,$$

where $(\cdot, \cdot)_{D,d}$ and $(\cdot, \cdot)_{A}$ denote the inner products on $H_n(A)$ and $L^2_{\omega}(A)$ respectively. Further, let $S_{o,d}$ be the restriction of $S_d$ defined by the boundary conditions

$$(y, \phi^{(i)})_{D,d} = 0, \quad 1 \leq i \leq n,$$

(i.e., $\mathcal{D}(S_{o,d}) = \mathcal{D}(T_{0,d}) \cap \mathcal{D}(S_d)$); also define $S_d^+$ on $L^2_{\omega}(A)$ by

$$\mathcal{D}(S_d^+) = \left\{ z \in L^2_{\omega}(A) : \tau^+_i z, 0 \leq i \leq 2n - 1, \text{is absolutely continuous on } A, \text{ and } \frac{1}{{\omega}} \tau^+_i z \in L^2_{\omega}(A) \right\}.$$

By Lemma A.4, suitably modified, for $y \in \mathcal{D}(T_{1,d})$ and $z \in \mathcal{D}(S_d^+)$,

$$(T_{1,d} y, z)_d - (y, S_d^+ z)_d = [y, z]_1(d) - [y, z]_1(c) + (y, \gamma^{T}\phi)_{D,d}, \quad (A.13)$$

where

$$[y, z]_1(t) = D((z - \gamma^{T}\phi), y)(t) - D(y, z)(t). \quad (A.14)$$

Let $S^+$ be defined in $L^2_{\omega}[a, b]$ by

$$\mathcal{D}(S^+) = \left\{ z \in L^2_{\omega}[a, b] : \tau^+_i z, 0 \leq i \leq 2n - 1, \text{is locally absolutely continuous on } [a, b], \text{ and } \frac{1}{{\omega}} \tau^+_i z \in L^2_{\omega}[a, b] \right\}.$$

$$S^+ z = \frac{1}{{\omega}} \tau^+_i z, z \in \mathcal{D}(S^+).$$

Also, let $S_0$ be the restriction of $S$ to the domain $\mathcal{D}(S_0) = \mathcal{D}(T_0) \cap \mathcal{D}(S)$, where $\mathcal{D}(T_0)$ is defined in (3.2). It follows from (A.13) that for $y \in \mathcal{D}$ and $z \in \mathcal{D}(S^+)$, $[y, z]_1(b) = \lim_{t \rightarrow b} [y, z]_1(t)$ exists and

$$(T_1 y, z) - (y, S^+ z) = [y, z]_1(b) - [y, z]_1(a) + (y, \gamma^{T}\phi)_{D}. \quad (A.15)$$
Lemma A.7. Assume that (A2) holds, and that \( \tau \) is of limit-point type at \( b \). Then

\[
L^2_\omega(a, b) = \mathcal{R}(S_0) \oplus \mathcal{G},
\]

where \( \mathcal{G} \) denotes the subspace spanned by the set \( \{u_i\} \cup \{\phi^{(i)}: 1 \leq i \leq n\} \), where \( \{u_i\} \) is a basis for \( \mathcal{N}(T_1) \), the null-space of the maximal operator \( T_1 \).

Proof. Observe first that as \( T_0 \) has an invertible extension (cf. the proof of Lemma A.3), \( \mathcal{R}(T_0) \) is closed; as \( T_0^* = T_1^* \), it follows that

\[
\mathcal{R}(T_0) = \mathcal{N}(T_1)^\perp.
\]  

(A.16)

Let \( z \in \mathcal{R}(S_0) \). Then \( z = \tau y \) for some \( y \) in \( \mathcal{D}(S_0) \). Consequently, for \( u_i \) in \( \mathcal{N}(T_1) \)

\[
(z, u_i) = (\tau y, u_i) \\
= (y, \tau u_i) \\
= 0.
\]

Also, as \( T_0 = \overline{T_0} \), where \( T_0 \) is the restriction of \( \tau \) to functions in \( \mathcal{D} \) with compact support contained in \( (a, b) \), one can choose \( \{y_n\} \subset \mathcal{D}(T_0) \) so that \( y_n \to y \) and \( \tau y_n \to \tau y \) in \( L^2_\omega(a, b) \). In addition, it follows from (A.2) and (2.9) that \( y_n \to y \) in the norm \( \| \cdot \|_D \). Thus for \( 1 \leq i \leq n \) we have

\[
(z, \phi^{(i)}) = (\tau y, \phi^{(i)}) \\
= \lim_{n \to \infty} (\tau y_n, \phi^{(i)}) \\
= \lim_{n \to \infty} (y_n, \phi^{(i)})_D \quad \text{by (2.9)}, \\
= (y, \phi^{(i)})_D \\
= 0 \quad \text{as } y \in \mathcal{D}(S_0).
\]  

(A.17)

This means that \( z \in \mathcal{G}^\perp \). Conversely, if \( z \in \mathcal{G}^\perp \), then certainly \( z \in \mathcal{N}(T_1)^\perp \), and hence \( z \in \mathcal{R}(T_0) \) from (A.16). Thus, \( z = \tau y \), where \( y \in \mathcal{D}(T_0) \). Finally, from (A.17), it follows that \( (y, \phi^{(i)})_D = 0, 1 \leq i \leq n \), and hence that \( y \in \mathcal{D}(S_0) \), as required. 

Theorem A.8. If \( T_0 \) has deficiency indices \( (n, n) \) (i.e., \( \tau \) is of limit-point type at \( b \)) and (A.2) holds, then \( S^* \) is the restriction of \( S^+ \) to functions \( z \) satisfying
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$$\tau_k^+ z(a) = \sum_{i=1}^n \xi_i \eta_k^{(i)} + 1, \quad 0 \leq k \leq n - 1,$$

$$\tau_k^+ z(a) = - \sum_{i=1}^n \xi_i \mu_{2n-k}^{(i)}, \quad n \leq k \leq 2n - 1,$$

where $\xi = (\xi_1, ..., \xi_n)^T$ is any vector equivalent to $\gamma$ in the sense that $\xi^T \gamma = \frac{1}{2} \gamma^T \Phi$. If $\{\phi^{(i)}: 1 \leq i \leq n\}$ is linearly independent, then $\xi = \frac{1}{2} \gamma$.

**Proof.** Set $B = \{z \in \mathcal{D}(S^+): z \text{ satisfies } (A.18)\}$. Observe that, by Lemma A.7, and the method of Lemma A.3, we have $S^+ \subset S$. Observe also that for $y \in \mathcal{D}(S)$, $z \in \mathcal{D}(S^+)$,

$$[y, z]_1(b) = [y, z - \gamma^T \Phi](b) - D(y, \gamma^T \Phi)(b)$$

$$= - D(y, \gamma^T \Phi)(b)$$

by [20, p. 78] as $y$ and $z - \gamma^T \Phi$ lie in $\mathcal{D}$. For $A = [a, d]$, let $S_{2,a}$ denote the restriction of $S_d$ satisfying $y^{(i)}(d) = 0, 0 \leq i \leq 2n - 1$, and $(y, \gamma^T \Phi) = 0$. Then $S_{2,a}$ is a $2n$-dimensional extension of $S_{0,a}$. Let $v_1, v_2, ..., v_{2n}$ be a basis for $\mathcal{D}(S_{2,a})/\mathcal{D}(S_{0,a})$. If these functions are defined to be zero in $[c, b)$, then as dim $\mathcal{D}(S)/\mathcal{D}(S_0) = 2n$, we have

$$\mathcal{D}(S) = \mathcal{D}(S_0) + \{v_i: 1 \leq i \leq 2n\}.$$  \hspace{1cm} (A.20)

Now, for $y \in \mathcal{D}(S_0)$ we have from Lemma A.7 and (2.9) that

$$D(y, \gamma^T \Phi)(b) = -(\tau y, \gamma^T \Phi) + (y, \gamma^T \Phi)_D = 0,$$

and thus, from (A.19) and (A.20) that for $y \in \mathcal{D}(S)$ and $z \in \mathcal{D}(S^+)$,

$$[y, z]_1(b) = 0.$$  \hspace{1cm} (A.21)

Consequently, from (A.15), and a similar argument to that used in the proof of Lemma A.5 we have $\mathcal{D}(S^*) \subset B$. Conversely, let $z \in B$. For $y \in \mathcal{D}(S_0)$ it is clear from (A.15) and (A.21) that

$$(Sy, z) = (y, S^+ z).$$  \hspace{1cm} (A.22)

As (A.22) also holds for $y = v_i$, for any $i, 1 \leq i \leq 2n$, it follows from (A.20) and (A.22) that $z \in \mathcal{D}(S^*)$, as required.

**References**


5. R. C. Brown, Private Communication.


