Apportionment 1

In the next few lectures, we consider the problem of dividing indivisible things (say, seats in a legislature) among players (say, states) in proportion to something (say, population).

- If we have 50 identical pieces of candy, how do we divide them fairly among 5 children?
    - Now suppose the candy is in payment for time spent doing household chores, and the children have worked different lengths of time.
- How do we divide a legislature of 435 members among 50 states in proportion to the state populations?

Example – Mother's M&Ms

Mother has 50 M&Ms to divide among her 5 children in proportion to the amount of time each spent on household chores. The time spent on chores is as follows.

	Al	Bet	Con	Doug	El	Total
Minutes Worked	150	78	173	204	295	900

How many M&Ms should each child receive?

$$\frac{\text{900 minutes worked}}{\text{50 pieces of candy}} = 18\,\frac{\text{minutes}}{\text{piece}}$$

Of course, no child conveniently worked a whole multiple of 18 minutes, as the following table shows.

	Al	Bet	Con	Doug	El	Total
Minutes Worked	150	78	173	204	295	900
$\div$ 18						

Assume that we do not want to cut an M&M up into fractional pieces. Then we must somehow "round off" the fractional parts to whole numbers to determine who gets how many pieces.

	Al	Bet	Con	Doug	El	Total
Minutes Worked	150	78	173	204	295	900
$\div$ 18	8.33	4.33	9.61	11.33	16.39	50
Round conventionally						
Round up						
Round down						

What is wrong with each attempt at rounding?

Apportionment Problems

- **Problem.** How do we distribute a fixed number of ***identical*** and ***indivisible*** items among several players, if each player is entitled to a different ***proportion*** of the total?
- **Solution.** Find an ***apportionment method*** that does the fairest job of apportioning the items to the players.

Proportion of the total – a fractional part, where the sum of all parts is 1 (equivalently, a percentage, where the total is 100%).

Apportionment method – any systematic procedure for solving an apportionment problem.

Example – Legislature of Parador

The 250 seats in the legislature of Parador are to be apportioned among the 6 states in Parador in proportion to the population of each state. The populations are as in the table below.

State	Ala	Bam	Cana	Da	Ele	Fant	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500

How many seats should each state receive?

Standard Divisor

$$D = \frac{\text{Total Population}}{\text{Number of Seats}}$$

Standard Quota (for each state)

$$\text{State's Standard Quota} = \frac{\text{State's Population}}{D}$$

For Parador's legislature, the ***Standard Divisor*** is

$$D = \frac{12{,}500{,}000}{250} = 50{,}000$$

Parador							
State	A	B	C	D	E	F	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500
House: M = 250		Standard Divisor D = 50 (in 1000s)					
Standard Quota							
Round Down							
Round Up							
Round Conventionally							

What is the problem which each way of rounding?

Some U.S. History

“Representatives and direct Taxes shall be apportioned among the several States which may be included in this Union, according to their respective Numbers. … The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at least one Representative; …”

--- Article 1, Section 2, Constitution of the United States

Observations

- Seats in the House of Representatives are to be ***apportioned*** to the states on the basis of their respective populations.
- Congress must ***enumerate*** the population of the states every 10 years.
- Congress must establish by law the ***number of representatives*** and the ***apportionment method***.
- Each state must get at least one seat.

Some familiar names from U.S. history who have suggested apportionment methods:

Alexander Hamilton
Thomas Jefferson
William Lowndes
John Quincy Adams
Daniel Webster

Hamilton’s Method

- **Step 1.** Calculate the ***Standard Divisor*** D.
- **Step 2.** Calculate each state’s ***Standard Quota***.
- **Step 3.** Round each quota down to the integer part.
- **Step 4.** Distribute ***Surplus Seats*** in the order of larger ***fractional parts*** of the quotas.
- **Step 5.** List the final Hamilton apportionment.

History. Hamilton’s method is the third method actually to be adopted by Congress (in 1852). It was the first proposed (1791), and was passed by Congress, but President George Washington vetoed it (the first Presidential veto in U.S. history!). Congress then adopted Jefferson’s method (1791), and later Webster’s method (1842).

We study Hamilton’s method first because it is the simplest to use and understand.

Parador – Hamilton's method

State	A	B	C	D	E	F	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500
House: M = 250		Standard Divisor D = 50 (in 1000s)					
Standard Quota	32.92	138.72	3.08	41.82	13.70	19.76	250
Round Down	32	138	3	41	13	19	246
Fractional Part							
Surplus Seats							
Hamilton App't							

Who got the last surplus seat handed out, and who just missed the cut?

Is Hamilton's method fair?

If state **E** does not get a surplus seat, then each of its districts have to be larger:

0.70/13 = ___________________

of the standard district size of 50,000, or _______________ extra people per district.

If state **B** does not get a surplus seat, then each of its districts have to be just a little larger:

0.72/138 = ___________________

of the standard district size of 50,000, or _______________ extra people per district.

It appears that the people of state E individually have a bit less influence in the legislature than the people of state B. (This is what President Washington objected to.)

We see that Hamilton's method penalizes small states more than large states in handing out surplus seats.

Against Hamilton's Method

- President Washington agreed with Jefferson's argument that the "unrepresented fractions" were damaging to democracy, so he vetoed Hamilton's method.

- Representative William Lowndes from South Carolina (a small state) proposed in 1822 a modification of Hamilton's method which used ***relative*** fractional parts to hand out surplus seats.

  - The relative fractional parts for states E and B were computed above: E: 0.05384… B: 0.005217…

  - By Lowndes method, E would get a surplus seat before B.

  - Lowndes method was never adopted by Congress.

Historical research topic: New York and Virginia were large states in 1791. Did Washington realize that both New Yorker Hamilton's and Virginian Jefferson's methods favored large states? (We'll see Jefferson's method later.)

Example – Distributing Mother's M&Ms

Compare Hamilton's and Lowndes' methods.

	Al	Bet	Con	Doug	El	Total
Minutes Worked	150	78	173	204	295	900
Candy Pieces: M = 50		Standard Divisor: D = 900/50 = 18				
Standard Quota	8.33	4.33	9.61	11.33	16.39	50
Round down						
Fractional part						
Relative frac. part						
Surplus seats						
Hamilton App't						
Lowndes App't						

Example – School-buses

The Southside School District has to apportion 25 school-buses to its four schools (Elem1, Elem2, Mid, and High) according to the school populations. They use Hamilton's method to apportion the buses to the schools.

	Elem1	Elem2	Mid	High	Total
Student Pop.	320	228	265	187	1000
Buses: M = 25		Standard Divisor: D =			
Standard Quota					
Round Down					
Fractional Part					
Surplus Buses					
Hamilton App't					

The Quota Rule

The integer part of a state's standard quota is called its ***lower quota***.

The integer immediately above a state's standard quota is called its ***upper quota***.

If a state's standard quota is an integer, then that integer is both the lower and upper quota for that state.

Quota Rule – An apportionment method should apportion to a state either its lower quota or its upper quota.

A specific apportionment that gives a state more than its upper quota is said to ***violate the upper quota***.

A specific apportionment that gives a state less than its lower quota is said to ***violate the lower quota***.

An apportionment method that never violates the quota is said to satisfy the quota rule.

Hamilton's method satisfies the quota rule. **Why?**

Alabama Paradox

History. In the apportionment debate of 1880, among several apportionment options, Congress considered the following:

- House size M = 299; use Hamilton's method.
- House size M = 300; use Hamilton's method.

What was expected.

First, do the apportionment with M = 299 using Hamilton's method, apportioning to each state its fair share of the 299 seats.

If M is increased to 300, one expects that one lucky state will get the additional seat, **and no other changes**.

What actually happened.

Option	Texas	Illinois	Alabama
M = 299	9	18	8
M = 300	10	19	7

How can this happen?

Example – Midland

To understand the mathematics behind the Alabama Paradox, we consider the small country of Midland with its three states: TX, IL, and AL.

Suppose Midland has a house size of M = 200 and population as shown in the table below. Apply Hamilton's method.

	TX	IL	AL	Total
Population	10,030	9,030	940	20,000
House: M = 200		Standard Divisor: D =		
Standard Quota				
Round Down				
Surplus seats				
Hamilton App't				

Now suppose the house size is increased to M = 201 and we again apply Hamilton's method.

Note that the standard divisor must be re-calculated because of the change in house size.

	TX	IL	AL	Total
Population	10,030	9,030	940	20,000
House: M = 200		Standard Divisor: D =		
Standard Quota				
Round Down				
Surplus seats				
Hamilton App't				

Mathematically, what is the cause of the crying in Alabama?

More History

Despite the Alabama Paradox, Hamilton's method was adopted in 1880 with a house size of M = 325.

Hamilton's method was adopted again in 1890.

In the apportionment debate of 1901, the House census committee (Chairman: Albert J. Hopkins of Illinois) considered all house sizes from 350 to 400 using Hamilton's method.

For all but M = 357 Colorado gets 3 seats.

For M = 357, Colorado gets 2 seats.

Hopkins, who perhaps dislikes the politics of Coloradans, pushes M = 357 through the committee.

In outrage, Congress defeats Hopkins' bill. The method is changed (to Webster's method – more later on it) with M = 386.

(M = 386 was chosen so that no state would lose a seat from the previous 1890 apportionment.)

Summary of Quota Methods

We have studied two apportionment methods. They are called ***quota methods*** because they satisfy the Quota Rule.

Hamilton's Method

- Distribute surplus seats in order of (absolute) fractional parts

Lowndes Method

- Distribute surplus seats in order of relative fractional parts

Fractional parts can be re-arranged by changes in standard divisor.

- May lead to paradoxes
- More on paradoxes later.

Paradoxes and Hamilton’s Method

Several paradoxes afflict Hamilton’s method.

- **Alabama Paradox**. A state may lose representation when the house size increases, even though the number of states and their populations remain unchanged.
- **New States Paradox**. Suppose a new state A enters the union, and the house size is increased by A’s fair share of new seats. A second state B may lose representation to a third state C, though there is no change in the population of B or C.
    - First noticed when Oklahoma joined the union is 1907, between censuses.

New States Paradox				
Year	M	Oklahoma	New York	Maine
1901	386		38	3
1907	391	5	37	4

- **Population Paradox**. Suppose the size of the house is fixed, but the populations of the states increase. A state A may lose representation to a second state B, even though A's population is increasing at a faster rate than B's.