TOPOLOGY NOTES
SUMMER, 2006

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3. Components

We will define three relations on a space, show that they are each equivalence relations, and how they are related. This will allow us to define several useful subsets of a space that exploit its degree of "connectedness." Recall that an equivalence relation on a set X satisfies the three properties of *reflevivity*, *symmetry*, and *transitivity*. (See Munkries Sections 3 and 25.)

Definition 3.1. Let X be a space. Define the following relations on X.

(1) $x \sim y$ iff there is no separation $X = A \cup B$ with $x \in A$ and $y \in B$.
(2) $x \simeq y$ iff there is a connected subset $C \subset X$ with $x, y \in C$.
(3) $x \approx y$ iff there is a path $f : [a, b] \to X$ with $f(a) = x$ and $f(b) = y$.

Proposition 3.2. *Each of $\sim$, $\simeq$, and $\approx$ is an equivalence relation on X.* $\square$

Proposition 3.3. $x \approx y \implies x \simeq y \implies x \sim y$. $\square$

Example 3.4. *No implication in Proposition 3.3 is reversible.*

Assignment 2, Part 2. Hard: 8: unboxed Example 3.4 above.

Assignment 2, Parts 1 and 2, is due Friday, June 16.

Definition 3.5. For an equivalence relation $\sim$ on X, we use $[x]_\sim$ to denote the $\sim$-equivalence class of $x \in X$, defined by
$$[x]_\sim = \{y \in X \mid x \sim y\}.$$
(Similarly, define $[x]_\simeq$ and $[x]_\approx$ for the relations $\simeq$ and $\approx$, respectively.)

You should recall from previous work that equivalence classes are pairwise disjoint and their union is the whole set X; that is, they form a *partition* of X. In particular, if two equivalence classes meet, then they must be equal.

Definition 3.6. Let $x \in X$. We define the following special terminology for equivalence classes of the relations defined above in Definition 3.1.

(1) $\sim$-equivalence classes are called *quasicomponents*.
(2) $\simeq$-equivalence classes are called *components*.
(3) $\approx$-equivalence classes are called *path components*.

Proposition 3.7. *Let X be a space. The following hold.*

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(1) *Each subset $Q \subset X$ which cannot be separated by a separation of X is contained in a quasicomponent of X.*
(2) *Each connected subset $C \subset X$ is contained in a component of X.*
(3) *Each path connected subset $P \subset X$ is contained in a path component of X.*

Proposition 3.8. *Let X be a space. The following hold.*

(1) *Each component of X is connected, and is maximal with respect to that property.*
(2) *Each path component of X is path connected, and is maximal with respect to that property.*

□

Proposition 3.9. *Let X be a space. The following hold.*

(1) *Each quasicomponent of X is a union of components of X.*
(2) *Each component of X is a union of path components of X.*

Proposition 3.10. *Let X be a space. Quasicomponents and components are closed in X. Path components need not be closed in X.* □

Definition 3.11. A space X is *locally connected at $x \in X$* iff for every neighborhood U of x, there is a connected neighborhood V of x with $V \subset U$. If X is locally connected at every point, we say X is *locally connected*. A space X is *locally path connected at $x \in X$* iff for every neighborhood U of x, there is a path connected neighborhood V of x with $V \subset U$. If X is locally path connected at every point, we say X is *locally path connected*.

Theorem 3.12. *A space is locally connected iff for every open set U in X, each component of U is open in X.* □

Theorem 3.13. *A space is locally path connected iff for every open set U in X, each path component of U is open in X.*

Theorem 3.14. *If a space X is locally path connected, then the components and the path components of X are the same.* □

Theorem 3.15. *If a space X is locally connected, then the components and the quasicomponents of X are the same.*

Assignment 3, Part 1. Easy: 1–4: unboxed items above.

Hard: 5–7: page 162: 4, 8, 10c.

Assignment 3 may be added to on Thursday, June 15, and is due Friday, June 23.

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