TOPOLOGY NOTES
SUMMER, 2006

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1. Connected Spaces

One of the big ideas in topology, motivated by its usefulness in analysis (think "intermediate value theorem"), is the idea of *connectedness*. We are trying to capture the idea that X is/is not broken into "pieces." See Munkries, Section 23.

Definition 1.1. Let X be a space. We say that X is *separated* iff there exist nonempty open sets U and V in X such that $X = U \cup V$ and $U \cap V = \emptyset$. If there are such open sets, then we call $X = U \cup V$ a *separation* of X. We say X is *connected* if there does not exist a separation of X.

Exercise 1.2. *On $X = \{0, 1, 2\}$, (1) give an example of a nontrivial topology such that X is connected, and (2) give an example such that X is not connected.*

Proposition 1.3. *Let X be a space. Then X is connected iff the only subsets of X that are both open and closed are $\emptyset$ and X.*

Problem 1.4. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on the set X. Suppose $\mathcal{T}_1 \subset \mathcal{T}_2$. What does connectedness of X in one topology imply about connectedness of X in the other topology?*

We also want to be able to talk about *connected subsets* of a space.

Definition 1.5. Let X be a space and $Y \subset X$. We say that Y is a *connected subset* of X iff Y is connected in the subspace topology.

Proposition 1.6. *Let X be a space and $Y \subset X$. Then Y is separated iff there exist nonempty sets A and B in Y such that $Y = A \cup B$, $A \cap B = \emptyset$, and neither A nor B contains a limit point of the other.* □

Example 1.7. *Give $\mathbb{N}$ the natural order and the order topology. Show that $\mathbb{N}$ is not connected. What are the connected subsets of $\mathbb{N}$?*

Problem 1.8. *Give $\mathbb{N}$ the cofinite topology. Is $\mathbb{N}$ connected in the cofinite topology?*

Proposition 1.9. *Let $X = U \cup V$ be a separation of X. Suppose Y is a connected subset of X. Then either $Y \subset U$ or $Y \subset V$.* □

Proposition 1.10. *Let X be a space and $\mathcal{C}$ a collection of connected subsets of X with the property that $\exists x \in X$ such that $\forall C \in \mathcal{C}, x \in C$. Then $\cup\mathcal{C}$ is connected.* □

Proposition 1.11. *Let X be a space and C a connected subset of X. Suppose $C \subset D \subset \overline{C}$. Then D is connected.* □

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Theorem 1.12. *The continuous image of a connected set is connected.* □

Theorem 1.13. *The finite Cartesian product of connected spaces is connected.* □

Example 1.14. $\mathbb{R}^\omega$ *is connected in the product toplogy, but not connected in the box topology.* □

Assignment 1. This assignment is due Friday, June 9.

Easy - 1–5: The "unboxed" items above.

Harder - 6–9: Page 152: 2, 6, 7, 11

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