Fair Division 1

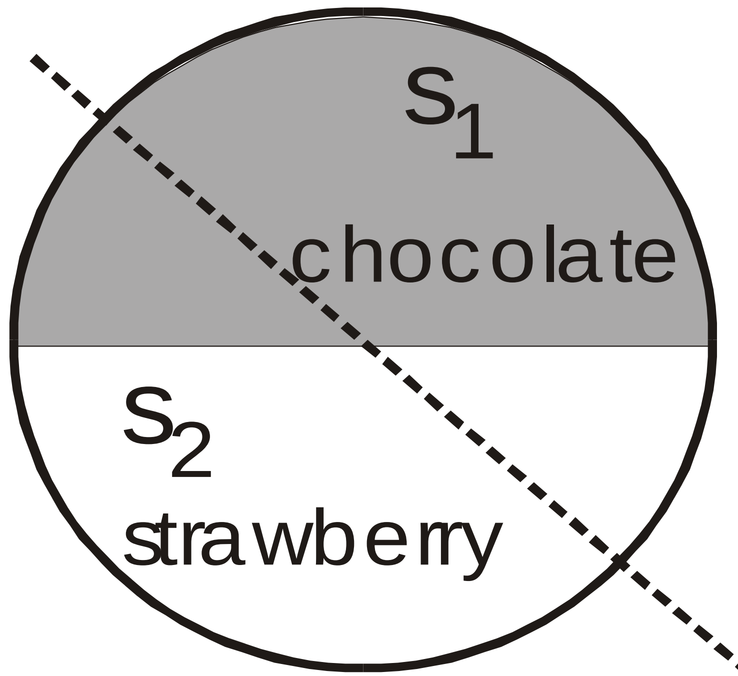
In the next few lectures, we consider the problem of dividing some good among a number of equally deserving people. Some specific examples:

- We have a large pizza. How do we divide it fairly among 4 students?
    - Now suppose that different regions of the pizza are covered with different toppings: anchovies, green peppers, sausage, and pineapple.
- Papa is dead. How do we divide the family farmland fairly among his 3 sons?
- Momma's 5 children have done their chores. How does she divide the candy (as a reward) fairly among them?
    - Suppose there are many types of candy in the mix.

You Cut --- I Choose!

Bob and Rachel win a cake in a raffle and must split it between them. Neither knows anything useful about the other's cake preferences. They toss a coin and Bob becomes the one who cuts.

The cake is half chocolate, half strawberry. Bob likes them equally well, and divides the cake into 2 pieces of equal size without paying attention to how much strawberry or chocolate is in each piece.



Rachel will choose a piece. She hates chocolate. In her value system, the chocolate has no value. Given Bob's cut, which piece does she choose.

Are they **both** satisfied they have a <u>fair share</u>?

Fair Division Problems

- **The elements**:
  - A set of n *players*: $P_1, P_2, \ldots, P_n$
  - A set of *goods* S.
- **The problem**: divide S into shares

$$s_1, s_2, \ldots, s_n$$

one share to each player.

- **The solution**: come up with a fair division scheme for dividing S into shares.

Fair share – any share that, in the opinion of the player receiving it, is worth at least 1/n part of the total value of the goods S.

Fair division scheme – any systematic procedure for solving a fair division problem.

Types of Fair Division Problems

Discrete – the set of goods S is made up of several indivisible objects

- Pieces of candy.
- Houses, cars, furniture, paintings (objects in an inheritance).
- A business, medical practice, factory.

Continuous – the set of goods S is divisible in infinitely many ways

- A cake, pizza, or bottle of wine.
- A piece of land, body of water (fishing fights).
- Money (in a sufficiently large quantity, pennies don't matter).

Mixed – some parts of the goods S are continuous, and some parts are discrete.

Fair Division Schemes

Conditions – we expect a fair division scheme to satisfy the following conditions:

- The procedure is ***decisive***. If the rules are followed, a fair division of the goods S is guaranteed.
- The procedure is ***internal*** to the players. No outside intervention is required to carry out the procedure.
- The players have ***no useful knowledge*** of each other's value system.
- The players are ***rational***. They base their actions on logic, not emotion.

Warning – a fair division scheme does not guarantee that each player **will** receive a fair share. What it guarantees is that no other player can **deprive** a player of his fair share. A player may **deprive himself** through greed or stupidity!

Divider-Chooser Method

We begin with the simplest of ***continuous*** fair division schemes, that for 2 players: the "you cut, I pick" or divider-chooser method. Suppose the goods are a cake.

1. One player, to be fairest, chosen at random (flip a coin) is the ***divider***; the other player is the ***chooser***.
2. The divider cuts the cake into 2 pieces s_1 and s_2 that she deems are each worth ½ the total value of the cake to her.
3. The chooser chooses which piece he wants, s_1 or s_2.
4. The divider receives the remaining piece.

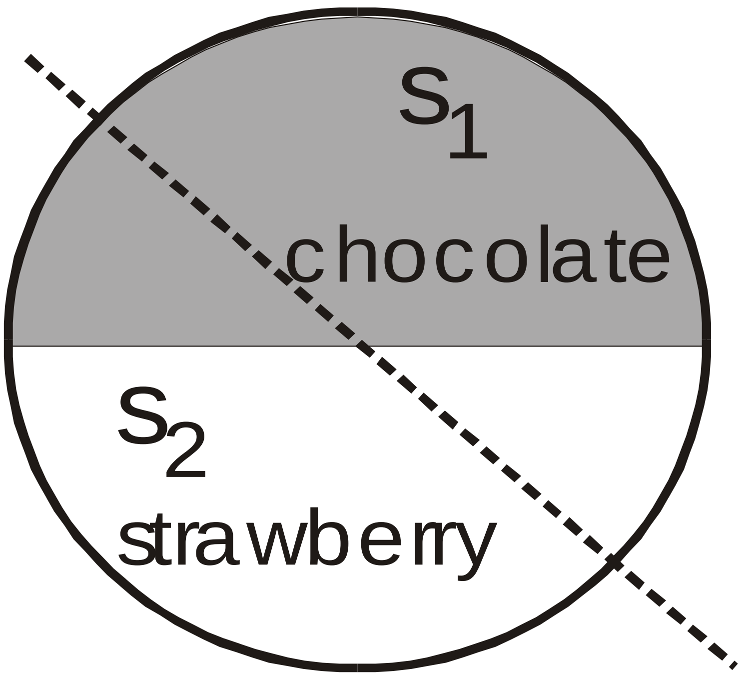
The divider is satisfied she received a fair share because she cut the cake into 2 pieces each worth half the value of the cake to her.

The chooser is satisfied because whenever anything is divided into 2 parts, one part must be worth at least half the total.

Example – Bob and Rachel's cake

Bob and Rachel win a cake in a raffle and must split it between them. Neither knows anything useful about the other's cake preferences. They toss a coin and Bob becomes the divider.

The cake is half chocolate, half strawberry. Bob likes them equally well, and divides the cake into 2 pieces of equal size without paying attention to how much strawberry or chocolate is in each piece.



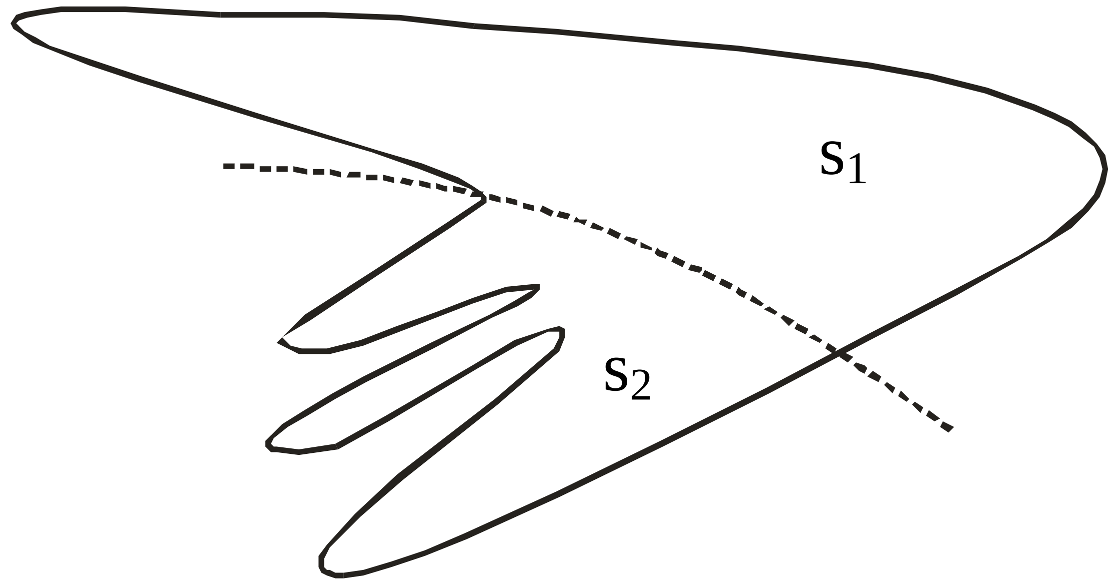
Rachel hates chocolate. In her value system, the chocolate has no value. Given Bob's cut, which piece does she choose.

Why are they both satisfied they have a fair share?

Example – Joe and Jack's fishing

Joe and Jack fish in the same small lake from their similar small boats, but they can't get along. To avoid further argument, they decide to divide the lake fairly between them, with each to fish only in his own part.

Joe is selected as divider by a coin toss, and marks the map of the lake as pictured, dividing it into two parts he deems equal in value.



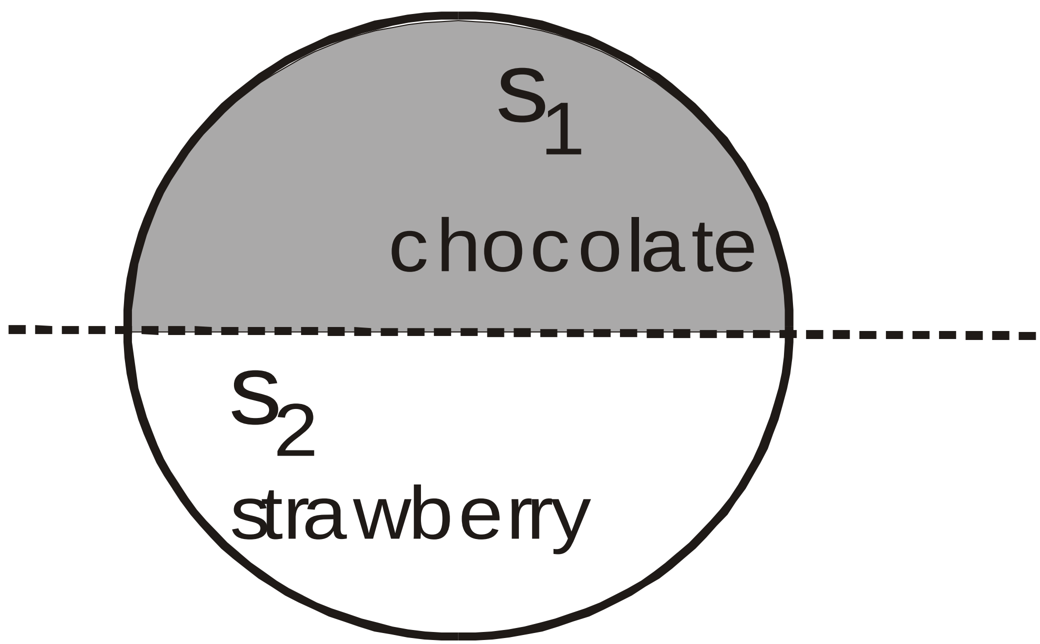
Why do you think one part might be smaller in area than the other? Is surface area the only criterion of good fishing?

As chooser, Jack picks the piece he prefers, namely s_2, so Joe gets s_1. Explain why each has a fair share.

Example – Bob and Rachel revisited

Bob and Rachel win a cake in a raffle and must split it between them. Neither knows anything useful about the other's cake preferences. They toss a coin and Rachel becomes the divider.

Rachel hates chocolate. In her value system, the chocolate half has no value. She divides the cake into 2 pieces of equal size along the diameter between chocolate and strawberry.



Bob likes chocolate and strawberry equally well. After a little thought, he picks the strawberry half, leaving the chocolate half for Rachel (GASP!).

What happened? Why didn't the fair division scheme work so as to satisfy both?

Lone Divider Method

There are several generalizations of the divider-chooser method to 3 or more players.

- The lone divider method.
- The lone chooser method.
- The last diminisher method.

We will study in detail only the lone divider method. The others are described in the text.

Assumptions

- The value of an object does not decrease when it is cut (the "no crumbs" condition).
- Any subset of pieces into which an object is cut can be recombined into a new whole.

Lone Divider Procedure

Assume that there are n players ($n \geq 2$), and that a *divider* and n-1 *choosers* have been selected.

To be fairest, usually the divider is chosen at random (for example, drawing straws).

Moves. The game proceeds through three *moves*.

1. Division
2. Declaration.
3. Distribution.

The last move, in the worst cases, may involve recombining some pieces into a new whole, and playing lone divider again, but with a smaller group of players.

(Remember the child's game of musical chairs?)

Lone Divider Moves

1. **Division** – the divider cuts the goods S into n pieces,

   $$s_1, s_2, \ldots, s_n,$$

   each of which she deems worth 1/n of the total value of the goods.

2. **Declaration** – each chooser declares independently which pieces he deems acceptable, that is, worth at least 1/n of the total value of the goods to him.

3. **Distribution**

   - **No conflict** – distribute the n pieces to the n players so that every player gets a piece acceptable to him. (Often there is more than one way top do this.)

   - **Stand-off** – if $j \geq 2$ or more *picky* players find only j-1 or fewer pieces acceptable among them, then make a *partial distribution* to the divider and any choosers who can be satisfied fairly and play again.

Partial Distribution – more details

Suppose there are j picky players who among them find j-1 (or fewer) pieces acceptable. Then play proceeds as follows:

a. Make a **partial distribution** to the divider and to each of the choosers who can be given a piece **not** in the set of pieces acceptable to players in the picky group, but acceptable to the non-picky chooser.

b. **Recombine** the remaining pieces (which includes all the pieces acceptable to players in the picky group) into a new and smaller whole.

c. **Apply the lone divider method** with the remaining players, who now number at most n-1.

The reason that there are at most n-1 players left is that we can always satisfy the divider. Hence the game will eventually end.

Example 1 – Gotham City

The Joker, Riddler, and Penguin, tired of fighting among themselves for control of organized crime in Gotham City, decide to divide Gotham up into 3 non-overlapping crime empires!

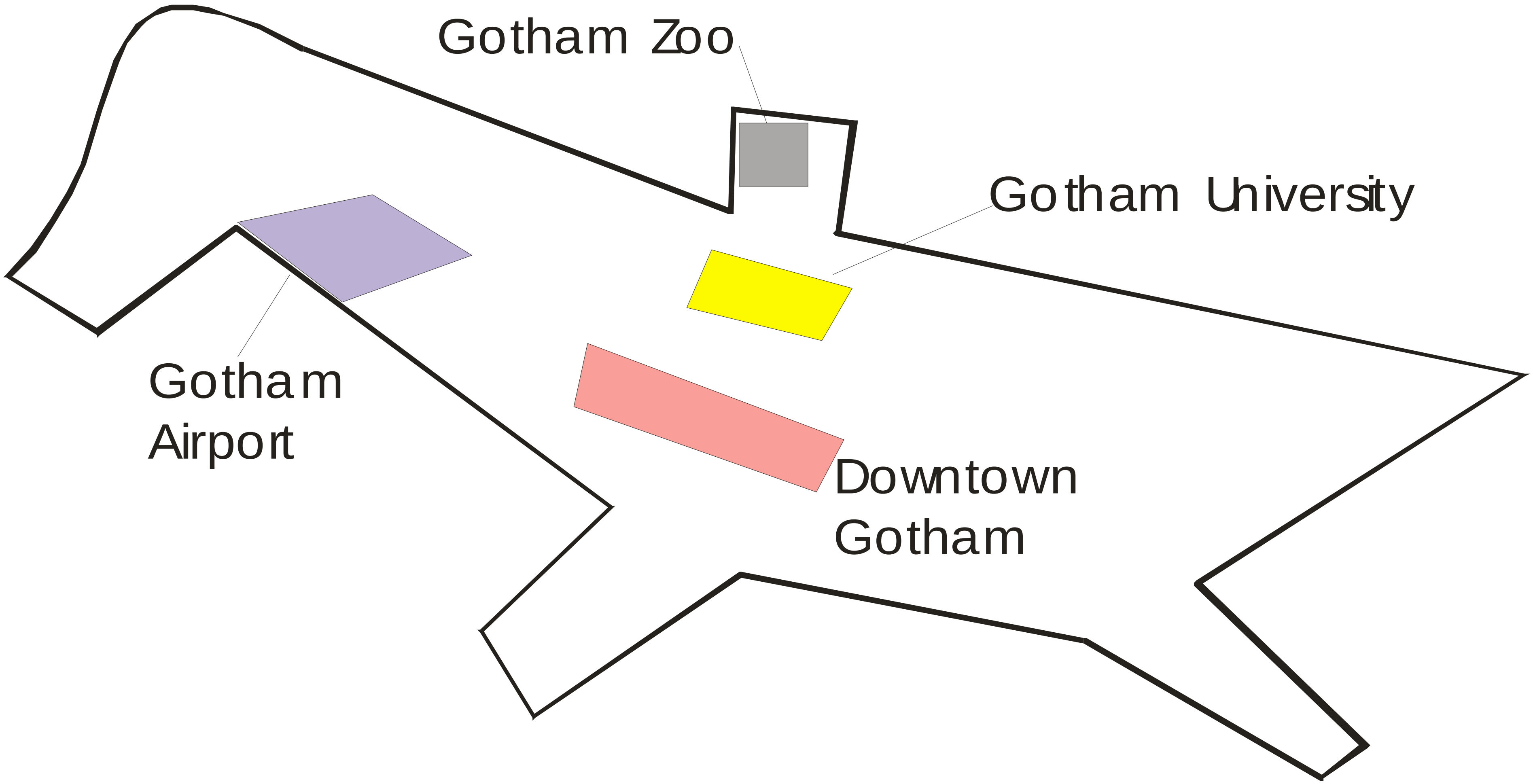
They wire London for the advice of that famed criminal mastermind, Dr. Moriarty (incidentally a mathematician – the maths tutor of Sherlock Homes). He suggests that they use the ***lone divider method*** to split up Gotham City.

They use one of the Joker's honest dice (he has a few) to select a divider. The result is:

Divider: Penguin

Choosers: Joker, Riddler

Division. With broad strokes of his umbrella-pen, Penguin carves the map of Gotham City into three pieces, s_1, s_2, and s_3 that he believes are fair shares in crime value (each worth 1/3 the crime value of Gotham in his opinion).



Map of Gotham City

Case 1 – No conflict

Declaration. Independently, each chooser declares which pieces of Gotham he finds acceptable, that is, worth at least 1/3 to him.

Chooser	Acceptable Pieces
Joker	s_1, s_2
Riddler	s_1, s_3

Distribution. As there is no conflict, the pieces of Gotham can be immediately distributed to the arch-criminals in such a way that each is satisfied he has a share worth at least 1/3 the total value of Gotham.

Arch-Criminal	Piece Received	Alt. 1	Alt. 2
Joker			
Riddler			
Penguin			

Is there another fair division of the cake?

Case 2 – Standoff

Declaration. Independently, each chooser declares which pieces of Gotham he finds acceptable, that is, worth at least 1/3 to him.

Chooser	Acceptable Pieces
Joker	s_1
Riddler	s_1

There is a standoff on piece s_1, so no *fair* full distribution at this point.

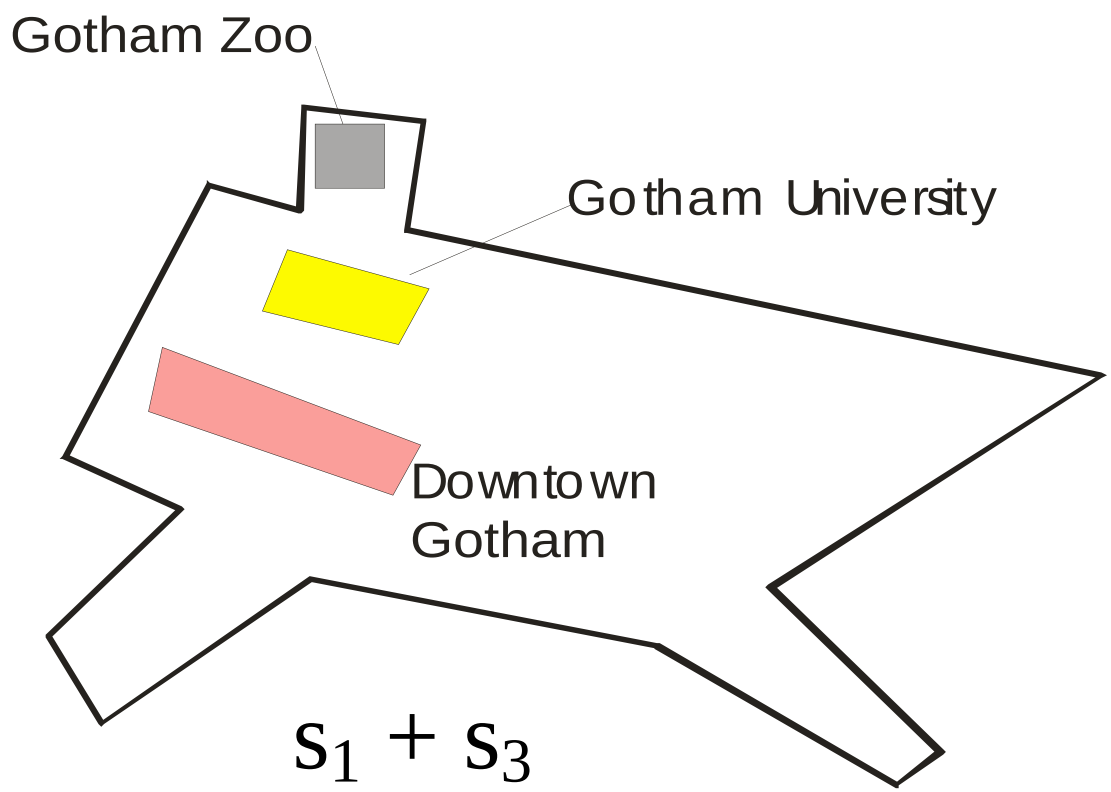
Partial Distribution. As the divider, penguin would be satisfied with any of the three pieces, so we can make a partial distribution.

Arch-Criminal	Piece Received	Alternate
Penguin		
Joker		
Riddler		

Conflict Resolution

To resolve the conflict between Joker and Riddler over piece s_1, we proceed as follows:

- Recombine s_1 and s_3 into a new whole we will call $s_1 + s_3$.
- Joker and Riddler fairly divide $s_1 + s_3$ by the divider-chooser method.



Do the Joker and Riddler get fair shares?

Proposition. The Joker and the Riddler each believe $s_1 + s_3$ is worth more than 2/3 the value of Gotham.

Proof. Joker thinks s_2 is worth less than 1/3 the value of Gotham. Otherwise, he would have listed it as an acceptable piece. Hence, in the Joker's value system:

$$\text{Value}(s_1 + s_3) = 1 - \text{Value}(s_2) > 1 - 1/3 = 2/3$$

The argument from the Riddler's viewpoint is the same.

Corollary. If the Joker and Riddler fairly divide $s_1 + s_3$, each will receive a share that is worth at least 1/3 of the value of Gotham to him.

Proof.

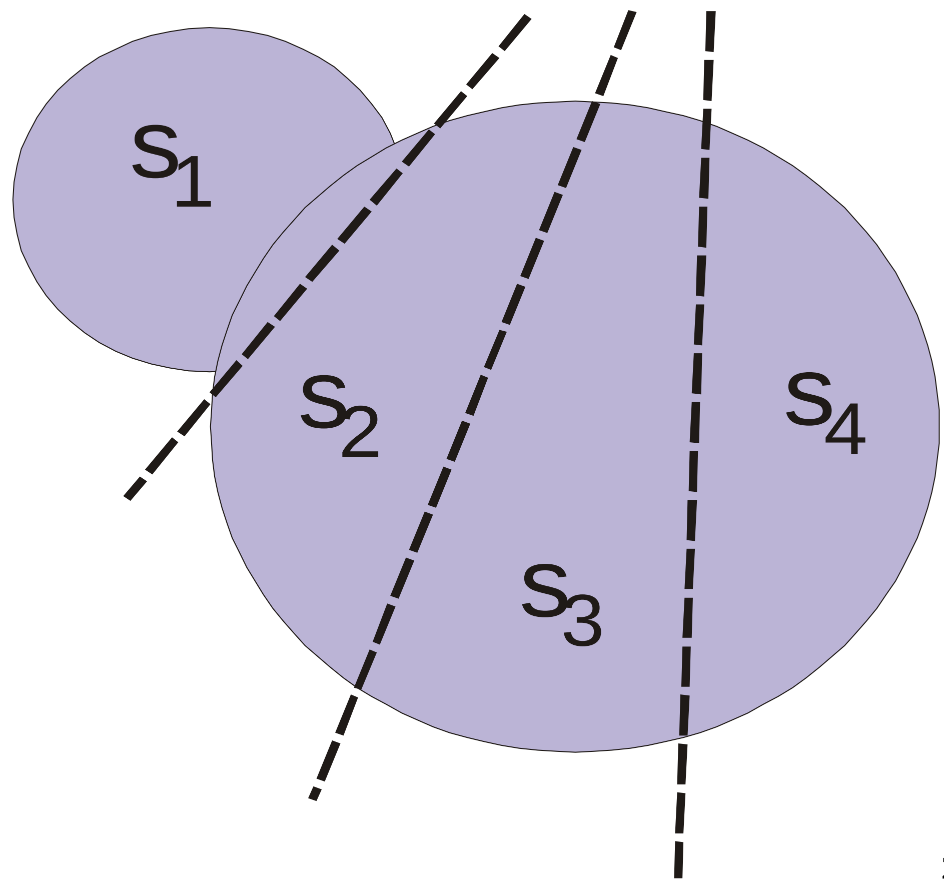
Example 2 – Mickey Mouse

Four friends have just returned from Disney World where they won a Mickey Mouse cake. They ate an ear before they got back to Birmingham. They wish to divide the remaining cake fairly among themselves and decide to use the lone divider method. They draw straws to determine the divider:

Divider: Demi

Choosers: Chea, Chet, Chuck

Division. Demi cuts the cake into four slices s_1, s_2, s_3, and s_4 that she deems equal in value (each worth ¼ the value of the cake to her).



Case 1 – No conflict

Declaration. Independently, the three choosers declares which slices of cake each finds acceptable (worth at least 1/4 the value of the cake to him).

Chooser	Acceptable slices
Chea	s_1, s_2
Chet	s_2, s_3
Chuck	s_1, s_4

Distribution. When there is no conflict in the declarations, the distribution of pieces can immediately follow.

Person	Slice Received	Alt. 1	Alt. 2	Alt. 3
Chea				
Chet				
Chuck				
Demi				

Case 2 – Standoff

Declaration.

Chooser	Acceptable slices
Chea	s_2, s_4
Chet	s_1
Chuck	s_1

Here there is a standoff between Chet and Chuck, two "picky" players who find only 1 piece acceptable between them.

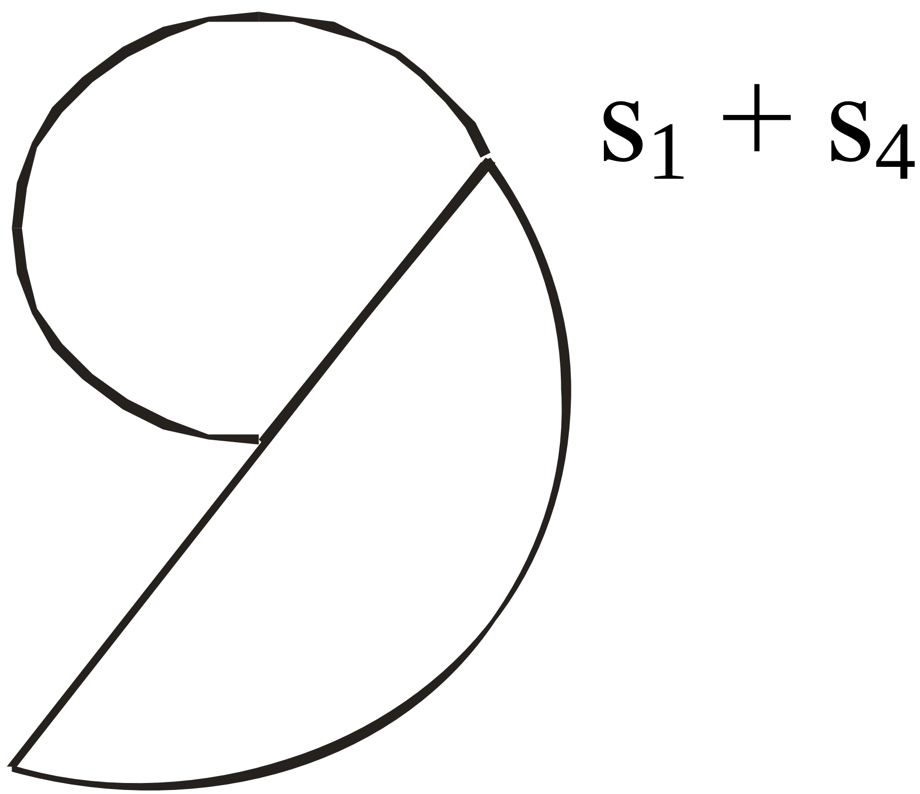
Partial Distribution.

Person	Slice Received	Alternate
Demi		
Chea		
Chet		
Chuck		

Conflict Resolution

The standoff between Chet and Chuck on slice s_1 is resolved as follows.

- Recombine s_1 and s_4 into a new whole called $s_1 + s_4$.
- Chet and Chuck will divide $s_1 + s_4$ fairly by the divider-chooser method (the 2-person version of the lone divider method).



Why is this fair to Chet and Chuck?

Proposition. Chet and Chuck each think $s_1 + s_4$ is worth more than half the value of the cake.

Proof. Chet thinks that s_2 and s_3 are each worth less than ¼ the value of the cake. Otherwise, he would have listed them as acceptable slices. Thus, we know

$$\text{Value}(s_2 + s_3) < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

Therefore, what is left, namely $s_1 + s_4$, must be worth more than half the value of the cake to Chet. The argument from Chuck's viewpoint is the same.

Corollary. If Chet and Chuck divide $s_1 + s_4$ fairly, each will get a share worth a bit more than ¼ the value of the cake to him.

Case 3 – 3-Way Standoff

Declaration.

Chooser	Acceptable slices
Chea	s_1
Chet	s_1
Chuck	s_1

The 3 choosers form a very “picky” group, finding only 1 slice acceptable among them.

Partial Distribution.

Person	Slice Received
Demi	
Chea	
Chet	
Chuck	

Conflict Resolution

There are multiple solutions to this standoff. Why?

Case 4 – 3-Way Standoff, Variant 2

Declaration.

Chooser	Acceptable slices
Chea	s_1, s_2
Chet	s_1
Chuck	s_2

The 3 choosers form a "picky" group, finding only 2 slices acceptable among them.

Partial Distribution.

Person	Slice Received
Demi	
Chea	
Chet	
Chuck	

Conflict Resolution

Case 4 revisited – 3-Way Standoff

Declaration.

Chooser	Acceptable slices
Chea	s_1, s_2
Chet	s_1
Chuck	s_2

Explain what is wrong with the following partial distribution and resolution of the conflict.

Partial Distribution.

Person	Slice Received
Demi	s_3
Chea	divide $s_1 + s_4$
Chet	
Chuck	s_2

Conflict Resolution

Chea and Chet recombine $s_1 + s_4$ into a new cake, and then they divide it fairly between them by using the divider-chooser method.

Case 5 – 3-Way Standoff, Variant 3

Declaration.

Chooser	Acceptable slices
Chea	s_1, s_2
Chet	s_1, s_2
Chuck	s_2

The 3 choosers form a "picky" group, finding only 2 slices acceptable among them.

Partial Distribution.

Person	Slice Received
Demi	
Chea	
Chet	
Chuck	

Conflict Resolution