Fair Division 2

- We have a 50 pieces of candy. How do we divide them fairly among 5 children?
  - Now suppose that there are 4 or 5 different kinds of candy, and some children dislike some of the kinds of candy.
- Consider an inheritance consisting of 5 pieces of antique furniture. How do we divide the inheritance in "equal shares" among 3 heirs?
  - Obviously, we cannot cut the furniture up like a cake!

Method of Markers

The preceding fair division problems are both of the kind we call ***discrete***, that is, there are several indivisible items to be divided fairly among several players.

The two kinds of discrete fair division schemes we shall study are

- The method of ***markers***.
  - Suitable for many objects, comparatively few players, and no objects of great value.
- The method of ***sealed bids***.
  - Suitable for few objects (compared to number of players), objects possibly differing widely in value.

Assumptions for the Method of Markers

The method of markers is applicable to situations where the following conditions hold.

- The m items to be divided are more numerous than the n players. (Simply, and mathematically, put, $m > n$.)
- No one item is worth much more than any other. (No gold watches among the candy!)
- The items are not identical. (If they were identical, why go to all this trouble?)

Procedure for Method of Markers

Assume that there are n players ($n \geq 2$) and m items with $m > n$.

First, the items are laid out at random in a ***linear array*** (a straight line).

The method of markers consists of three ***moves***.

1. **Bidding.** Each player independently and secretly ***marks*** the array, dividing it into n consecutive ***segments***, each of which he views as worth 1/n of the total value of the goods.
2. **Allocation**. This move consists of n ***submoves***.
   First the markers are all revealed.
   - Assign to the player owning the ***first*** marker (left to right) in the ***first set*** of markers that player's ***first*** segment. Remove his markers.

- Assign to the player owning the ***first*** marker in the ***second set*** of markers that player's ***second*** segment. Remove her markers.
- Continue this process until each player has received a segment that she views as a fair share.
- The ***last*** player receives his ***last*** segment.

3. **Leftovers.** Any leftover items can be divided among the players by some fair method.
    - If there are enough leftovers, the method of markers can be used again.
    - If there are too few leftovers, another fair division method must be used.

Marking Rationally

One does not simply mark the linear array into consecutive segments of equal ***length***, unless one views all the items as equal in value.

Example. George, Henry, and Ida are to divide the linear array of candy below by the method of markers. (R = Reese's Pieces, M = plain M&Ms, P = peanut M&Ms.)

Suppose that George is indifferent toward any kind of M&Ms, but particularly likes Reese's Pieces. George is greedy and marks the array as follows. What is likely to happen?

P M P R R P M R M P M

Where should George place his markers to guarantee that he will get a fair share?

P M P R R P M R M P M

Moves

After Henry and Ida have placed their markers (without knowing where George's or each other's are), the markers are revealed and the array appears as below.

What is the allocation of pieces of candy to the 3 players?

← **Allocations**

P M P R R P M R M P M

↑ ↑ ↑ ↑ ↑ ↑

I G H G I H

← **Bids**

Leftover. Are there any leftover pieces?

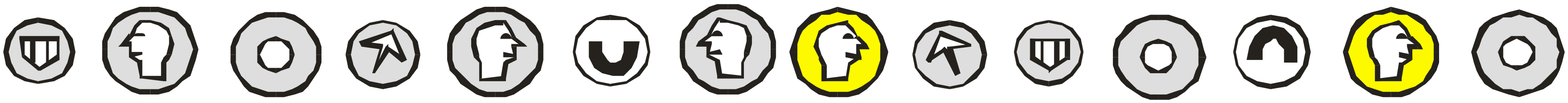
Example – Uncle's George's Coins

Uncle George has died and left his collection of 14 rare coins, none exceptionally more valuable than any of the others, ***in equal shares*** to his nephews Abe (A) and Cal (C) and his niece Bev (B). They decide to divide the collection by the method of markers.

We summarize the moves below, and will carry them out before your very eyes.

- **Linear Array.** Lay the coins out at random in a line.
- **Bidding.** Each player secretly records ***2 markers*** on a copy of the array, dividing the collection into ***3 segments*** that he deems equal in value.
- **Allocation.** Each player receives a segment which she regards as a fair share of the coins.
- **Leftovers.** Any leftover coins can be distributed fairly by some fair division scheme satisfactory to all.

Linear Array



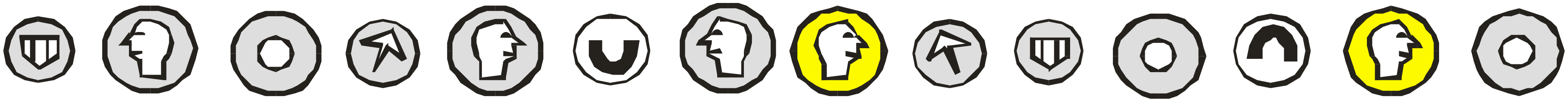
Bidding

Player A – You are player A. You believe coins with ***heads facing right*** are most valuable, so you want to make sure that you get a fair share of them. Place your markers **A**↑ accordingly.

Player B – You are player B. You believe coins with ***round holes in them*** are most valuable, so you want to make sure that you get a fair share of them. Place your markers **B**↑ accordingly.

Player C – You are player C. You believe ***small*** coins are the most desirable, so you want to make sure that you get a fair share of them. Place your markers **C**↑ accordingly.

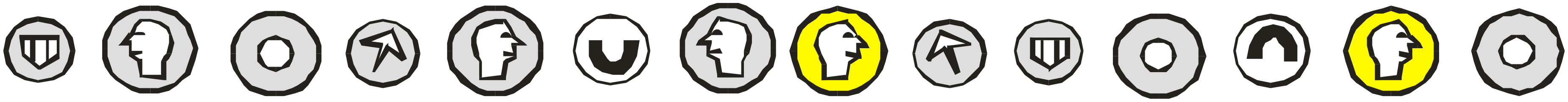
Allocations



Bids

Leftover

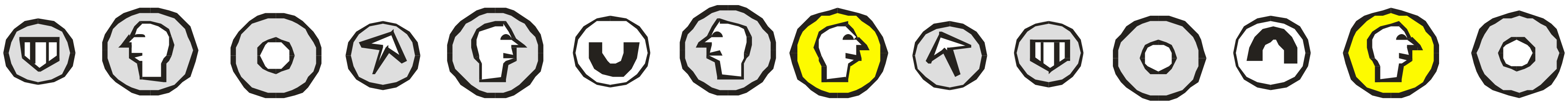
Allocations



Bids

Leftover

Allocations



Bids

Leftover

Example – Finding the sets of markers

Suppose that an array of markers has been marked by George (G), Henry (H), Ida (I) as follows.

P M P R R P M R M P M R M M

I G G I H H

Be careful finding the sets of markers. What is tricky here?

Describe the allocations and any leftovers.