MA 110 Homework 2 ANSWERS

This homework assignment is to be written out, showing all work, with problems numbered and answers clearly indicated. The assignment is due to be handed in by Tuesday, October 10. Turn in all your work, but I suggest you keep a copy of the question sheets to study for the test coming up on October 17. Late assignments will not be accepted after the key is made available.

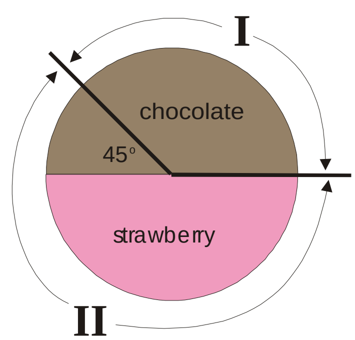
Table 2.1 refers to a situation where four players (A, B, C, and D) agree to divide a cake fairly by the Lone Divider method. The table shows how each player values each of the four slices that have been cut by the divider.

Table 2.1				
Slice / **Player**	s_1	s_2	s_3	s_4
A	32%	20%	24%	24%
B	25%	25%	25%	25%
C	25%	15%	30%	30%
D	26%	26%	26%	22%

1. Assuming all players play honestly and intelligently, which player was the divider?
   Player B was the divider because that player values all pieces equally.

2. Assuming they play honestly and intelligently, what should each chooser's declaration be? [1,2,2]
   A: s_1 C: s_1, s_3, s_4 D: s_1, s_2, s_3
   A piece must be worth at least 25% (1/4) to a player to be acceptable.

3. Ted and Carol buy a half-chocolate, half-strawberry cake for $12.00. They want to divide it by the divider-chooser method. Ted likes chocolate twice as much as strawberry. Carol likes strawberry three times as much as chocolate.



   a. What is the value of the chocolate half of the cake to Ted? What is the value of the strawberry half of the cake to Ted? [1]
      Ted values chocolate 2/3*$12 = $8.
      He values strawberry 1/3*$12 = $4.
   b. What is the value of the chocolate half of the cake to Carol? What is the value of the strawberry half of the cake to Carol? [1]
      Carol values strawberry 3/4*$12 = $9.
      Carol values chocolate ¼*$12 = $3.
   c. Suppose Ted is the divider and cuts the cake as shown into pieces I and II. Show that pieces I and II are of equal value to Ted. [2]
      Value(II)=Value(strawberry)+45/180*Value(Choc)
      Value(II)= $4 + 1/4*$8=$6
      So, Value(I)=$12-Value(II)=$6.
      They are the same value.
   d. Which piece will Carol choose? Explain why. [1]
      Carol values Strawberry more than chocolate, so she will choose piece II which contains all the strawberry.

Tables 2.2 and 2.3 refer to a situation in which five players (one divider and four choosers) are going to divide a cake fairly by the Lone Divider method. The divider has cut the cake into five slices $\{s_1,s_2,s_3,s_4,s_5\}$ and the choosers declarations are as follows

Table 2.2	Declaration
Chooser 1	{s_1, s_3}
Chooser 2	{s_2}
Chooser 3	{s_1, s_5}
Chooser 4	{s_3, s_4}

Table 2.3	Declaration
Chooser 1	{s_1, s_3}
Chooser 2	{s_3}
Chooser 3	{s_2, s_5}
Chooser 4	{s_1, s_3}

4. If the choosers' declarations are as in Table 2.2, describe a fair division of the cake.
   All possible answers to 4 and 5 are given after question 5 below.

5. Referring again to Table 2.2, describe a fair division of the cake different from your answer to #4, or explain why there is only one way to do it fairly.

C1: s_1	C2: s_2	C3: s_5	C4: s_3	D: s_4
C1: s_1	C2: s_2	C3: s_5	C4: s_4	D: s_3
C1: s_3	C2: s_2	C3: s_1	C4: s_4	D: s_5
C1: s_3	C2: s_2	C3: s_5	C4: s_4	D: s_1

6. If the choosers' declarations are as in Table 2.3, describe how to proceed to obtain a fair division of the cake? [Hint: be sure you correctly identify the "picky group."]
   The simplest answer is:
   D: s_4 C3: s_2 Then C1, C2, C4 recombine s_1+s_3+s_5 into a new cake and play again.
   Other answers are possible, but all first satisfy the divider with a piece other than s_1 or s_3, then involve C1, C2, C4 in a picky group, and s_1, s_3 and as many pieces as continuing players in a new cake, and a new game. Another acceptable solution is to give the Divider s_4, and then all Choosers recombine s_1+s_2+s_3+s_5 into a new cake and play again.

7. With reference to Table 2.3, explain what is possibly unfair about the following division of the cake: Divider gets s_4; Chooser 2 gets s_3; Chooser 3 gets s_2; Choosers 1 and 4 recombine pieces s_1 and s_5 into a new cake, and play lone divider again to divide the remaining cake fairly between them.
   The problem is giving s_3 to C2. Suppose C1 and C4 value the pieces as in the table below:

	s_1	s_2	s_3	s_4	s_5
C1	25%	15%	35%	10%	15%
C4	25%	15%	35%	10%	15%

   The above valuations are consistent with the declarations that they made. The new cake consisting of s_1 + s_4 is only worth 35% of the original to each of them. That is not enough to divide fairly and each get at least 20% of the original. So the ultimate division would not be fair.

8. Three players, A, B, C, agree to divide some candy by the method of markers. Assume that you are player A and that you like Reese's Pieces (denoted R, below), but do not particularly like plain M&Ms (denoted M) or peanut M&Ms (denoted P). Place your markers below so that you will be guaranteed to receive a fair share.
   Only Rs below are significant. Player A will mark the array with two markers to get an equal number of Rs in each of three segments. More than one correct answer is possible. One answer is below. The first A marker could correctly be placed anywhere up to the double vertical bar ||. The second A marker is inflexible.

P M M P R M P R ↑A P M P M M || R P M M R ↑A R P M R M P

9. – 10. Four heirs, A, B, C, and D inherit four items from their uncle's estate: a sofa, a wine cellar, a car, and a rare book. They decide to divide the items among themselves by the method of sealed bids. Their bids are as in the bid table below. Complete the bid and allocation tables below and describe the final settlement, including what cash amounts each player has ultimately paid or gained.

Bids	**Players**				
Item	**A**	**B**	**C**	**D**	**High Biddder**
sofa	1,500	2,500	3,500	3,000	C
wine	13,000	11,500	11,000	12,000	A
car	4,500	4,000	5,500	6,000	D
book	2,000	2,000	2,500	3,000	D
Total Value	21,000	20,000	22,500	24,000	
Fair Share	5,250	5,000	5,625	6,000	

Allocation				
Player	**Item(s)**	**Value**	**Share**	**Put in (Take out)**
A	wine	13,000	5,250	Put in 7,750
B	none	0	5,000	Take out 5,000
C	sofa	3,500	5,625	Take out 2,125
D	car, book	9,000	6,000	Put in 3,000
			Surplus	3,625
			Share of Surplus	906.25

Describe the final settlement for each player below (including cash balance).

A: Gets the wine cellar and pays $7,750 – 906.25 = $6,843.75 cash.

B: Gets no item, but gets $5,000 + 906.25 = $5,906.25 cash.

C: Get sofa and also gets $2,125 + 906.25 = $3,031.25 cash.

D: Gets car car and book, and pays $3,000 – 906.25 = $2,093.75 cash.

Note that the cash paid and the cash received balances out to $0. [Accept round-off to $1.]

11. Three players (A, B, and C) are fairly dividing some items by the Method of Markers. They have marked the linear array below as shown. Describe the allocation of items to each player and indicate what items, if any, are leftover.

B C A

1 2 3 4 5 6 7 8 9 10 11 12 13

↑ ↑ ↑ ↑ ↑ ↑

B A C C B A

Leftover: 4, 5, 9, 10

12. Four players (A, B, C, and D) are fairly dividing some items by the Method of Markers. They have marked the linear array below as shown. Describe the allocation of items to each player and indicate what items, if any, are leftover.

D A C B

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

↑ ↑ ↑ ↑↑ ↑ ↑↑ ↑↑ ↑ ↑

D C A AB C BD CA B D

Leftover: 4, 5, 8, 11, 12

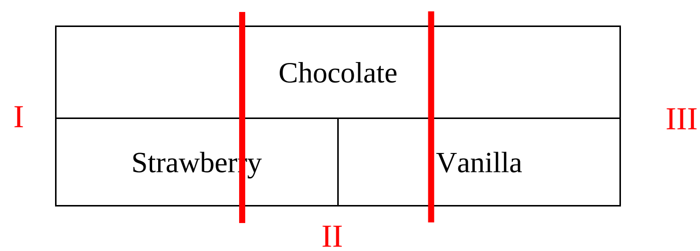
13. Alice and Ted divorce amicably and have to divide the house they share. They decide to use the method of sealed bids. Alice bids $130,000 and Ted bids $160,000. What is the final result of applying the method of sealed bids? Be sure to address any exchange of cash.

Player	Alice	Ted
Bid	130,000	160,000
Share	65,000	80,000

Player	Value	Share	Cash Pot
Alice	0	65,000	Take out 65,000
Ted	160,000	80,000	Put in 80,000
		Surplus	15,000

Ted is the higher bidder, so will get the house. Alice will get only cash. The surplus is split equally between Alice and Ted. <u>Hence, the final result is that Ted gets the house and pays Alice $72,500.</u>

14. You and two friends buy a rectangular cake, shown below. The cake is 1/2 chocolate, 1/4 strawberry, and 1/4 vanilla. You decide to divide it among the three of you by the lone divider method. You are the divider. You like strawberry just as much as vanilla, and you like vanilla twice as much as chocolate. Make cuts of the cake into the appropriate number of pieces so that you will receive a fair share. Label the pieces carefully.



Assume the cake is drawn to scale. One correct answer is to note that each flavor is itself worth 1/3 of the cake to you. So you can make the cuts along the lines between flavors. Another correct answer is pictured above. A third correct answer is to split each flavor into 3 equal pieces: C1, C2, C3, V1, V2, V3, S1, S2, S3. A “piece” then consist of, say, C1, V1, and S1, etc. There are still other correct answers.

15. Four players (A, B, C, and D) agree to divide some candy by the method of markers. Assume that you are player A and that you like Reese’s Pieces (denoted R, below) twice as much as plain M&Ms (denoted M) or peanut M&Ms (denoted P). Place your markers below so that you will be guaranteed to receive a fair share. [Hint: assign a numerical value to each piece.]

1 1 1 1 1 2 1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 2 1 1 1 2 (32)

M P M P M R P M M P R P M P R M M R P M M R M P M R

↑ ↑ ↑

A A A

Each of the four segments is worth 8 units to A.

16. Three players (A, B, and C) agree to divide some candy by the method of markers. Assume that you are player B and that you like plain M&Ms (denoted M, below) three times as much as Reese's Pieces (denoted R) or peanut M&Ms (denoted P). Place your markers below so that you will be guaranteed to receive a fair share.

3 1 1 1 1 1 1 3 1 1 1 1 3 1 1 (21)

M P P P R R P M P P R P M P R

↑ ↑

B B

Each of the three segments is worth 7 units to B.

Table 3.1 describes the small country Pardoxia with four states and populations as shown.

Table 3.1					
State	Ariza	Birma	Culpa	Doma	Total
Population (1,000's)	11,035	8025	845	95	

17. Determine a Hamilton's Method apportionment of the seats of the legislature of Pardoxia with 200 seats.

Table 3.1 → #17					
State	Ariza	Birma	Culpa	Doma	Total
Population (1,000's)	11,035	8025	845	95	20,000
Standard divisor =	20,000/200 = 100				
Standard Quotas	110.35	80.25	8.45	0.95	
Round Down	110	80	8	0	198
Hamilton App't	110	80	9	1	200

18. Suppose that the legislature is increased to 201 seats. What is now the Hamilton apportionment of the Pardoxian legislature? Did anything paradoxical happen? Explain

Table 3.1→ #18					
State	Ariza	Birma	Culpa	Doma	Total
Population (1,000's)	11,035	8025	845	95	20,000
Standard divisor =	20,000/201 = 99.502				
Standard Quotas	110.90	80.65	8.49	0.954	
Round Down	110	80	8	0	198
Hamilton App't	111	81	8	1	201

Culpa lost a seat and Ariza and Birma gained a seat each, even though the populations remained the same and the legislature increased. This is an example of the Alabama Paradox.

19. Suppose that 130,000 people migrate from Ariza to Birma because of a devastating hurricane. The populations of Culpa and Doma remain the same. How does this change the Hamilton apportionment? Did anything paradoxical happen? Explain.

Table 3.1→ #19					
State	Ariza	Birma	Culpa	Doma	Total
Population (1,000's)	10,905	8155	845	95	20,000
Standard divisor =	20,000/200 = 100				
Standard Quotas	109.05	81.55	8.45	0.95	
Round Down	109	81	8	0	198
Hamilton App't	109	82	8	1	200

Since 100,000 people entitle a state to a seat, it is reasonable that Ariza lost a seat to Birma when 130,000 moved. But why did Culpa also lose a seat to Birma when that state was not involved in the migration? This is an example of the Migration Paradox.

20. In each apportionment above, determine which states received their upper quota of seats, and which states received their lower quota. Explain why Hamilton's Method always satisfies the Quota Rule.

	Problem		
State	17	18	19
A	Lower quota	Upper quota	Lower quota
B	Lower quota	Upper quota	Upper quota
C	Upper quota	Lower quota	Lower quota
D	Upper quota	Upper quota	Upper quota

Under Hamilton's method, you first round down, then add at most one surplus seat per state. Hence a state either gets its lower quota (because you rounded down) or gets its upper quota (because it gets a surplus seat). Thus Hamilton's Method always awards to a state either its lower quota or its upper quota of seats.

21. A school district has four schools with student populations in the following Table 4.1. The district must buy busses to transport all students. Assume that a bus holds 45 students, and each bus can be used for just one trip. The district decides to apportion busses to schools by Hamilton's Method. How many busses does each school get? Explain what the standard divisor and standard quota represent in this example. Is this a good solution?

Table 4.1					
School	High	Middle 1	Middle 2	Elementary	Total
Students	999	310	300	776	2,385
Standard Divisor=	45 (given)	Number of busses = 2,385/45 = 53			
Standard quotas	22.20	6.88	6.66	17.24	
Round Down	22	6	6	17	51
Hamilton App't	22	7	7	17	53

Since each bus holds 45 students and there are 2,385 students altogether, the district must purchase 53 busses. Each 45 students requires one bus, so the Standard Divisor is 45 students per bus. The Standard Quotas are the number of busses per school, found by dividing the student population by the Standard Divisor. A better solution might be to buy 55 busses and round all quotas up. That way, no students would be left standing unable to get on the bus!