# TOPOLOGY NOTES
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## 2. Real Line

We know a point is connected, and we know an indiscreet space is connected. But that about exhausts our current state of examples. In this section we prove a general theorem about a kind of ordered space from which is follows that the real line and any of its intervals or rays is connected. (See Munkries, Section 24.) The key properties that make this possible are in the following definition.

**Definition 2.1.** A nondegenerate simply ordered space $L$ is called a *linear continuum* iff it satisfies the following:

(1) $L$ has the least upper bound property.
(2) If $x, y$ in $L$, then there exists $z \in L$ such that $x < z < y$.

A subset $C$ of an ordered set is *convex* iff whenever $a, b \in C$, $a \leq b$, then the interval $[a, b] \subset C$.

**Theorem 2.2.** *If $L$ is a linear continuum, then $L$ is connected, and so is any convex set in $L$.* □

**Corollary 2.3.** *The real line $\mathbb{R}$ is connected, and so are rays and intervals in $\mathbb{R}$.* □

**Corollary 2.4** (Intermediate Value Theorem)**.** *Let $f : X \to Y$ be a continuous function of a connected space into an ordered space. If for $a, b \in X$, $r \in Y$ lies between $f(a)$ and $f(b)$ in the order on $Y$, then there is a point $c \in X$ such that $f(c) = r$.* □

**Example 2.5.** *Let $X$ be a well-ordered set. Then $X \times [0, 1)$ in the dictionary order topology is a linear continuum.* □

**Example 2.6.** *In particular, $\mathbb{N} \times [0, 1)$ is homeomorphic to the real ray $[0, \infty)$.*

**Proposition 2.7.** *Let $f : [0, 1] \to [0, 1]$ be continuous. Then there is a* fixed point *of $f$; that is, a point $x \in [0, 1]$ such that $f(x) = x$.* □

Intuitively, two things are connected to each other if you can get from one to the other by a path with no breaks. We make this idea formal.

**Definition 2.8.** Let $X$ be a space and $x, y \in X$. We say a function $f : [a, b] \to X$ is a *path* from $x$ to $y$ iff $f$ is continuous, $f(a) = x$, and $f(b) = y$. If moreover, $f$ is one-to-one, we say $f$ is an *arc*. A space is said to be *path connected* (respectively, *arcwise connected*) iff every two (different) points of $X$ can be connected by a path (arc).

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**Proposition 2.9.** *A path connected space is connected.*

**Example 2.10.** *Connected does not imply path connected, as the following examples illustrate.*

1. *The square* $I \times I$ *(where* $I = [0, 1]$*) with the dictionary order topology is connected but not path connected.*
2. *The* sin 1/x continuum $X = \overline{\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1 \text{ and } y = \sin(\frac{1}{x})\}}$ *is connected but not path connected.*

□

## Assignment 2, part 1

Easy: 1–3: unboxed items above and page 158: 4

Harder: 4–7 : page 152: 8; page 158: 8a, 8c, 10.

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