Probability 1

The formal study of the mathematical laws of chance.

What is the probability that

- A coin when flipped will come up heads?
- Today's lecture will end early?
- You will earn a grade of "A" or "B" in this course?

Example: The Birthday Problem

A HY 101 Class at UAB has 45 students. How likely do you think it is that at least two of them have the same birthday?

(a) Not very likely.
(b) Somewhat likely.
(c) Quite likely.
(d) Extremely likely.

Think about it a minute. Then we'll have you vote.

We are going to need some definitions and a way to quantify probabilities before we can handle situations like the Birthday Problem. We'll return to this problem in a later class.

Basic Definitions

Random experiment – the process of observing a chance event and noting its outcome.

Outcomes – all possible results of the random experiment.

Sample space – the set of all outcomes $S = \{O_1, O_2, O_3, \ldots, O_n\}$, where O_k denotes the kth outcome.

The set of outcomes in the sample space must be:

- *Exhaustive*—that is, include all possible outcomes of the experiment.
- *Mutually exclusive*—that is, no two outcomes can both occur at the same time.

Size of Sample Space – the number N of outcomes in the sample space.

Example. Rolling a Die

Suppose a single die is rolled. We are interested in the number that lands on top. The normal die has 6 sides and 6 numbers. Hence,

$$S = \{1, 2, 3, 4, 5, 6\} \qquad\qquad N = 6$$

The way we handle random experiments mathematically is by making a *probability assignment* to the outcomes in a random experiment.

Probability assignment — the assignment to each outcome O_k of a numerical weight $p(O_k)$, called the *probability* of the outcome, which measures the likelihood of its occurring.

Properties of a probability assignment. Assume that the sample space for a given random experiment is $S = \{O_1, O_2, \ldots O_n\}$. Then

- For each outcome O_k, the probability is number between 0 and 1, inclusive.

$$0 \le p(O_k) \le 1$$

- The sum of the probabilities of all the outcomes in S is 1.

$$p(O_1) + p(O_2) + \ldots p(O_n) = 1$$

Example. Rolling a Fair 6-Sided Die

We saw that $S = \{1, 2, 3, 4, 5, 6\}$ and $N = 6$. "Fair" means what here?

Probability Assignment:

$p(1) =$	$p(2) =$	$p(3) =$	$p(4) =$	$p(5) =$	$p(6) =$

Example. Tossing a Coin

One fair coin is tossed. We observe whether “heads” or “tails” lands face up.

Sample space:

{heads, tails}

or {H, T}

Probability assignment:

$p(\mathrm{H})$ =
$P(\mathrm{T})$ =

Note. The method of assigning probabilities in this example is what we refer to as the “classical approach.” We define this next.

How do we assign probabilities to outcomes?

- Classical Approach.
  - -- Based on gambling ideas.
  - -- The game is fair and (often) all "elementary" outcomes are equally likely. "Elementary" here means "broken down into the smallest pieces."

- Relative Frequency Approach.
  - -- Based on repeatable experiments
  - -- An outcome's probability is the proportion of times the event occurs in the long run.

- Personal Approach.
  - -- Most of life's events are *not* repeatable.
  - -- An outcome's probability is an individual's personal assessment of its likelihood.

Example. Tossing Two Coins

Two fair coins are tossed simultaneously. We observe whether "heads" or "tails" lands face up on the two coins.

We distinguish between the coins. We list one coin first and the other second in the outcomes below.

Sample space:

Probability assignment:

$p(\mathrm{HH}) =$	$p(\mathrm{TT}) =$
$p(\mathrm{HT}) =$	$p(\mathrm{TH}) =$

Example. Drawing Marbles from a Jar I

A jar contains five marbles: two are colored green and three are white. One marble is drawn by hand from the jar *without looking*. We observe the color of the marble drawn.

Sample space:

{green, white}   or   {G, W}

Note that the *observation* determined the sample space, not the numbers of green and white marbles in the jar.

Probability assignment:

p(G) =	p(W) =

Explain why the probability assignment above is correct.
Since there are two colors, why isn't the probability assignment to each color ½ ?

Example. Drawing Marbles from a Jar II

An opaque jar contains an *unknown* number of marbles, some green and some white. We repeat the following experiment many times.

- Draw a marble from the jar without looking, record its color, replace the marble in the jar; shake the jar thoroughly.

The first 100 times,	76 of the marbles are white.
The second 100 times,	85 of the marbles are white.
The next 100 times,	81 of the marbles are white.
The next 200 times,	156 of the marbles are white.
The next 500 times,	394 of the marbles are white.

Based on this series of experiments, we make the following **probability assignment** to the outcomes of the random experiment of drawing one marble from the jar:

$p(\mathrm{G}) =$	$p(\mathrm{W}) =$

I bet you $1 that the next marble drawn from the jar is white. (If it is white, I win—if it is green, you win.) Do you take my bet?

Example. Tennis Match

An expert tennis coach assigns the following probabilities of winning to each of the four players in the semi-final matches of a tennis tournament.

Match	Player	Probability of Winning Match
1	Alan	1/3
1	Bob	2/3
2	Cal	1/4
2	Dave	3/4

On what do you suppose the expert bases his probability assignment?

Assigning Probabilities in Large Sample Spaces

Sometimes the size of the sample space prevents our listing all the outcomes. However, we may still be able to assign probabilities as the next two examples show.

Example. Tossing 10 Coins

10 fair coins are tossed at one time. Distinguishing the coins, we observe the sequence of heads and rails that comes up. (Our analysis would be the same if we had tossed one coin 10 times and distinguished the tosses.)

Size of the sample space: each coin can come up 2 ways; there are 10 stages (the 10 coins); by the multiplication principle, the size of the sample space is:

Sample space: too long to list, but includes all sequences of 10 heads or tails; for example,

Probability assignment: the probability of each outcome is:

Example. Tossing a Die 5 Times

One fair 6-sided die is tossed 5 times. We observe the sequence of numbers that comes up.
(Our analysis would be the same if this were 5 fair 6-sided dice all being tossed at once).

Size of the sample space:

Sample space:

Probability assignment:

Events

An **event** is any set of outcomes of a random experiment; that is, any subset of the sample space of the experiment.

- The *probability* of a given *event* is the *sum* of the probabilities of the outcomes in the event.
- If each outcome is equally likely, then the probability of an event is the event size divided by the sample size.

Random Experiment. A pair of fair 6-sided dice (one white, one black) is rolled, and we observe the faces which show up. We use (2,3) to indicate there is a 2 on the white die and a 3 on the black die.

Sample Space:

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Example. Some Events for Tossing 2 Dice

The dice were fair in the random experiment above. Hence, each of the outcomes is equally likely with **probability**:

Some of the **events** in the previous experiment include:

- Faces add up to 11.
- Faces add up to 4.
- White die shows 1.

Event	Outcomes	Probability
Faces add up to 11	{(5,6), (6,5)}	
Faces add up to 4	{(1,3), (2,2), (3,1)}	
White die shows 1	{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)}	