

TOPOLOGY NOTES

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SYLLABUS

Course: MA 670 Topology 1
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Format of Course. This is a “do-it-yourself” course. You are *absolutely forbidden* to consult any textbook. You may talk to each other informally and to me, but work presented at the board must be your own or properly credited. It is OK to say “The idea for this proof was suggested to me by ...”

Material. The only material for the course will be this set of **Topology Notes** which will be updated and distributed periodically. Occasionally, there may actually be errors in the notes. Please point them out and they will be corrected in the next update.

Most items appearing in italics (theorems, lemmas, propositions, corollaries, problems, and examples) are for students to work and present proofs and explanations at the board. Set out items not in italics (definitions, axioms, remarks, conjectures, and questions) are for your use and reference.

Exercises are also in italics, but are not to be presented at the board. They may count toward your grade (see “Rules” below). You are responsible for knowing how to solve them. You may use true statements from the exercises in your boardwork without proof.

Rules.

- (1) Each item presented at the board and defended correctly counts one (1) point, unless otherwise noted.
- (2) Priority order for presenting is determined by:
 - (a) Persons with lowest board point total.
 - (b) Persons who have not yet presented that day.
 - (c) Random experiments to break ties.
- (3) Exercises are not presented in class, but may be turned in for a written homework grade (see “Grading” below). (Deadline is first Monday a week after we move beyond the subsection in which the exercise appears.) Proof and/or explanation is always required — a bare answer never suffices.
- (4) There are two tests: midterm and final (see “Grading” below).

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Key words and phrases. topology.

Grading. Items will be weighted the flexible amounts indicated below so as to produce the best individual grade. You can (and may) rely entirely on boardwork for your grade.

Boardwork	20–100%	rank-ordered subject to a minimum
Homework	0–30%	rank-ordered subject to a minimum
Tests	0–50%	mid-term and final count equally

You *are not required* to do any written homework unless you want to. It can only help your grade. You *are required* to take the tests, though they cannot hurt you either if your boardwork total is good.

Attendance. Attendance in class is required, as you are there not only to present your own results, but to critique the work of others. Participation is expected. Unexcused absence is 1 point penalty per day on your boardwork total. Lateness of above 20 minutes is 1/2 point penalty. After a warning from me, *your* grace time may be shortened.

Worked Problems and Proofs. As the Notes are re-issued, problems, examples, lemmas, propositions, theorems, etc. that have been completed (proved or explained at the board) are marked \square at the end of the statement. The latest version will be kept up-to-date on my website www.math.uab.edu/mayer.

1. REVIEW OF NAIVE SET THEORY, NAIVE LOGIC, AND NOTATION

It is assumed that you are familiar with elementary logic and set theory. This section is not meant to be exhaustive of the facts about sets and logic that may come up in the course. Most of the student work in this section is in the form of exercises. However, that does not mean that you should not read this section. You are responsible for knowing how to solve the exercises. And there are some non-exercises for board work.

1.1. Logic. Logic begins with *statements* which we won't define, except that they must be capable of being true or false.

Remark 1.1. Note that “iff” is shorthand for “if, and only if.”

Definition 1.2. Let p and q denote statements. The logical connectives, notation, and truth conditions are as follows:

- (1) *Not* p , symbolized $\neg p$, is true iff p is false.
- (2) *If* p , *then* q , symbolized $p \Rightarrow q$, is true, iff it is not the case that p is true and q is false. (Also called “ p implies q .”)
- (3) p *iff* q , symbolized $p \Leftrightarrow q$, is true iff p and q have the same truth value. (Also called “ p is equivalent to q .”)
- (4) p *and* q , symbolized $p \wedge q$, is true iff both p and q are true. (Also called the “conjunction” of p and q .)
- (5) p *or* q , symbolized $p \vee q$ is true, iff at least one of p and q is true. (Also called the “disjunction” of p and q .)

Exercise 1.3. Which of the following statements are always true? We'll call them *theorems of logic* and use them in formulating proofs.

- (1) $p \Rightarrow (q \Rightarrow p)$.
- (2) $p \Leftrightarrow q$ is true iff $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is true.
- (3) $p \Rightarrow q$ is true iff $\neg p \Rightarrow \neg q$ is true.
- (4) $p \Rightarrow q$ is true iff $q \Rightarrow p$ is true.
- (5) $p \Rightarrow q$ is true iff $\neg q \Rightarrow \neg p$ is true.
- (6) $(p \wedge q) \Rightarrow r$ is true iff $p \Rightarrow r$ and $q \Rightarrow r$ are both true.
- (7) $(p \vee q) \Rightarrow r$ is true iff either $p \Rightarrow r$ or $q \Rightarrow r$ is true.
- (8) $(p \vee q) \Rightarrow r$ is true iff $p \Rightarrow r$ and $q \Rightarrow r$ are both true.

1.2. Set Theory. We won't define what a set is either.

Remark 1.4. We use the symbol $:=$ as shorthand for “is defined to be equal to” or “means the same as,” whichever is appropriate.

Definition 1.5. Let A, B, C denote sets and let x, y , and z denote elements which may be members of a set. Set theoretic statements, connectives, and notation are defined as follows:

- (1) x is a member of A , symbolized $x \in A$, is true iff x is an element in the set A . (We also use $A \ni x$ to mean $x \in A$.)
- (2) $x \notin A := \neg(x \in A)$.
- (3) Sets A and B are equal, symbolized $A = B$, is true iff A and B contain the same members.
- (4) A is a subset of B , symbolized $A \subset B$, is true iff every member of A is also a member of B . (Equivalently, $A \subset B$ is false, symbolized $A \not\subset B$, iff there is a member of A which is not a member of B .) (We also use $B \supset A$ to mean $A \subset B$.)

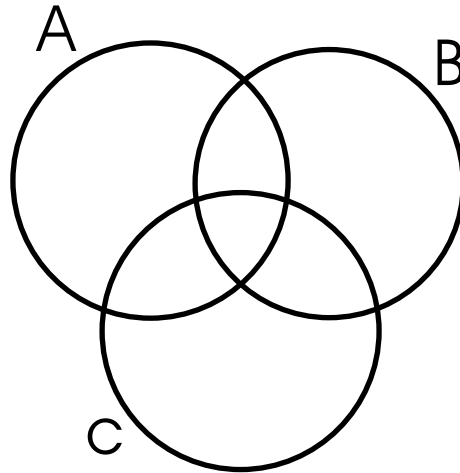


FIGURE 1. Venn diagram for three sets: a way of discovering truths.

- (5) *The union of the sets A and B , symbolized $A \cup B$, is the set $A \cup B := \{x \mid x \in A \vee x \in B\}$.*
- (6) *The intersection of the sets A and B , symbolized $A \cap B$, is the set $A \cap B := \{x \mid x \in A \wedge x \in B\}$.*
- (7) *The difference of the sets A and B , symbolized $A \setminus B$, is the set $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$. We call $A \setminus B$ the *complement* of B (in A).*
- (8) *The empty set is the set with no members and is symbolized \emptyset .*

Exercise 1.6. *Which of the following set theoretic statements are true? If a statement is false, can you find a “related” statement which is true?*

- (1) $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$.
- (2) $(A \subset B \wedge B \subset C) \Rightarrow A \subset C$.
- (3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- (4) *Can you switch the roles of \cap and \cup in the above statement?*
- (5) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
- (6) *Can you replace \cap by \cup in the above statement?*
- (7) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
- (8) *Can you switch the roles of \cap and \cup in the above statement?*

1.3. Quantifiers. “Naive” set theory does not develop set theory axiomatically, and hence is subject to subtle contradictions (called paradoxes). It is unlikely that we will run into a paradox unless we try. “Naive” logic does not develop logic axiomatically, and hence is subject to subtle confusions. We do not carefully distinguish between syntax (the study of well-formed formulas and of what is provable) and semantics (the study of truth). In particular, we assume that “the provable” and “the always true” coincide. These paradoxes and confusions can not arise until we introduce quantifiers. Once quantifiers are defined, and we maintain naiveté, we are subject to paradox and confusion, but don’t let it worry you in this course.

Definition 1.7. Let $P(x)$ denote a statement about the individual x in some set U of individuals. The quantifiers *for all* and *for some* have the following truth conditions:

- (1) *For all $x \in U$, $P(x)$, symbolized $\forall x \in U, P(x)$, is true iff, for every individual x in the set U , P holds true for that individual x .*

- (2) For some $x \in U$, $P(x)$, symbolized $\exists x \in U, P(x)$, is true iff there is at least one individual $y \in U$ for which $P(y)$ holds true.

Remark 1.8. Note that if $U = \emptyset$, then $\forall x \in U, P(x)$ is “vacuously” true, and $\exists x \in U, P(x)$ is necessarily false.

Exercise 1.9. What must be the case if both $\forall x \in U, P(x)$ and $\forall x \in U, \neg P(x)$ are true?

Exercise 1.10. Form the denial of each of the following statements.

- (1) $\forall x \in U, P(x)$.
- (2) $\exists y \in V, Q(y)$.
- (3) $\forall x \in U, (P(x) \Rightarrow Q(x))$.
- (4) $\forall x \in U, \exists y \in V, (P(x) \Rightarrow Q(y))$.
- (5) $(\forall x \in U, (P(x) \Rightarrow (\exists y \in V, Q(y))))$.
- (6) $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in U, \forall y \in U, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.
- (7) $\forall x \in U, \forall y \in U, \forall \varepsilon > 0, \exists \delta > 0, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.
- (8) Explain the difference between the previous two statements.

Remark 1.11. We will use the term “collection” synonymous with the term “set” in order to avoid phrases such as “set of sets... .” For clarity in “levels” of sets, we will use the following conventions:

- (1) Lower case Roman letters a, b, c, \dots denote elements of sets that are typically not themselves sets.
- (2) Upper case Roman letters A, B, C, \dots denote sets.
- (3) Caligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ denote collections of sets.
- (4) Lower case Greek letters $\alpha, \beta, \gamma, \dots$ sometimes denote indices, when we want to index a collection of sets in order to distinguish the members. But i, j, k, \dots may also denote indices, particular if the indices are natural numbers or integers.
- (5) Upper case Greek letters are used as needed (sometimes as a set of indices.)

We can use quantifiers to extend our set theoretic connectives and statements to arbitrary collections of sets.

Definition 1.12. Let \mathcal{A} and \mathcal{B} denote collections of sets. Let $\{A_\alpha\}_{\alpha \in \Gamma}$ denote an indexed collection of sets. Set theoretic connectives and statements may be further extended as below. (This is only a sample.)

- (1) The union of the collection \mathcal{A} , symbolized $\bigcup \mathcal{A}$, is the set $\bigcup \mathcal{A} := \{x \mid \exists A \in \mathcal{A}, x \in A\}$.
- (2) The intersection of the collection \mathcal{A} , symbolized $\bigcap \mathcal{A}$, is the set $\bigcap \mathcal{A} := \{x \mid \forall A \in \mathcal{A}, x \in A\}$.
- (3) The union of the indexed collection $\{A_\alpha\}_{\alpha \in \Gamma}$, symbolized $\bigcup_{\alpha \in \Gamma} A_\alpha$, is the set $\bigcup_{\alpha \in \Gamma} A_\alpha := \{x \mid \exists \alpha \in \Gamma, x \in A_\alpha\}$.
- (4) The intersection of the indexed collection $\{A_\alpha\}_{\alpha \in \Gamma}$, symbolized $\bigcap_{\alpha \in \Gamma} A_\alpha$, is the set $\bigcap_{\alpha \in \Gamma} A_\alpha := \{x \mid \forall \alpha \in \Gamma, x \in A_\alpha\}$.

In case the indexed collection $\{A_i\}_{i \in \Gamma}$ is indexed by a finite or countably infinite set, for example $\Gamma = \{1, 2, \dots, n\}$ or $\Gamma = \mathbb{Z}^+$, we usually write unions (likewise, intersections) as $\bigcup_{i=1}^n A_i$ or $\bigcup_{i=1}^\infty A_i$.

Problem 1.13. Which of the following set theoretic statements are true? If a statement is false, can you find a “related” statement which is true? (Each counts 1 point.)

- (1) $A \cap (\bigcup \mathcal{B}) = \bigcup_{B \in \mathcal{B}} (A \cap B)$. (Note that \mathcal{B} is being used as its own index set.) \square
- (2) $A \setminus (\bigcup \mathcal{B}) = \bigcap_{B \in \mathcal{B}} (A \setminus B)$. \square

- (3) Can you switch the roles of \cap and \cup in the above statements? \square
- (4) Suppose $\mathcal{A} = \emptyset$. What is $\cup\mathcal{A}$? \square
- (5) There is a somewhat counter-intuitive consequence of Definition 1.12(2) above. What is it?

1.4. Proofs. You have all had at least one semester of Advanced Calculus, so it is assumed that you know what a proof is. I remind you that there are generally three methods of proof: (1) direct, (2) by contraposition, and (3) indirect (by contradiction). There is also the technique of proof by induction. A counter-example is essentially a proof that a statement is false, but it requires existence; that is, a counter-example must be specific and name individuals and sets.

Exercise 1.14. Explain how you might go about proving each statement below.

- (1) $\forall x \in U, (P(x) \Rightarrow Q(x))$.
- (2) $\forall x \in U, \exists y \in V, (P(x) \Rightarrow Q(y))$.
- (3) $(\forall x \in U, (P(x)) \Rightarrow (\exists y \in V, Q(y)))$.
- (4) $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in U, \forall y \in U, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.
- (5) $\forall x \in U, \forall y \in U, \forall \varepsilon > 0, \exists \delta > 0, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.

1.5. Functions.

Definition 1.15. Let A and B be sets. A *function* f , symbolized $f : A \rightarrow B$, is an unambiguous rule that assigns to each element of the set A an element of the set B . Set A is called the *domain* of f and set B is called the *range* of f . For each $a \in A$, we use $f(a)$ to denote the member of B assigned to a . Let $C \subset A$. We use the notation $f(C) := \{b \in B \mid \exists a \in C, f(a) = b\}$, and call $f(C)$ the *image* of C . The function f is *onto* iff $f(A) = B$. The function f is *one-to-one* iff $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. Let $D \subset B$. It is convenient to use the notation $f^{-1}(D) := \{a \in A \mid \exists d \in D, f(a) = d\}$. We call $f^{-1}(D)$ the *pre-image* of D . A function $f : A \rightarrow B$ is a *one-to-one correspondence* from A to B iff f is both onto and one-to-one.

Exercise 1.16. Suppose $f : A \rightarrow B$ is a function and $C \subset A$.

- (1) Why do you suppose we do not generally call f^{-1} the inverse of f ?
- (2) Suppose that $f : A \rightarrow B$ is a one-to-one correspondence? Show there is a one-to-one correspondence from B to A . (We call this one-to-one correspondence the inverse of f and symbolize it $f^{-1} : B \rightarrow A$.)
- (3) What can you say about the statement $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$?
- (4) Is $f : A \rightarrow f(A)$ a function? What kind now?
- (5) Is $f|_C : C \rightarrow f(C)$ a function? (Here $f|_C$ is the function defined by $f|_C(a) := f(a)$ for all $a \in C$. We call $f|_C$ the restriction of f to C .)

Exercise 1.17. Let A , B , and C be sets. Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.

- (1) Show that $g \circ f : A \rightarrow C$ defined by $(g \circ f)(a) := g(f(a))$ is a function from A to C . (We call $g \circ f$ the composition of f and g .)
- (2) Show that f and g onto (respectively, one-to-one) implies $g \circ f$ is onto (respectively, one-to-one).
- (3) Show by example that $g \circ f$ is not necessarily the same as $f \circ g$, even when both are defined.

Exercise 1.18. Let A and B be subsets of X , and $f : X \rightarrow Y$ a function. Determine the relationship between the following pairs:

- (1) $f(A \cap B)$ and $f(A) \cap f(B)$.

- (2) $f(A \cup B)$ and $f(A) \cup f(B)$.
- (3) $f(A \setminus B)$ and $f(A) \setminus f(B)$.
- (4) $A \subset B$ and $f(A) \subset f(B)$.

Problem 1.19. Let A and B be subsets of Y , and $f : X \rightarrow Y$ a function. Determine the relationship between the following pairs of sets. (Each counts 1 point.)

- (1) $f^{-1}(A \cap B)$ and $f^{-1}(A) \cap f^{-1}(B)$. \square
- (2) $f^{-1}(A \cup B)$ and $f^{-1}(A) \cup f^{-1}(B)$. \square
- (3) $f^{-1}(A \setminus B)$ and $f^{-1}(A) \setminus f^{-1}(B)$. \square
- (4) $A \subset B$ and $f^{-1}(A) \subset f^{-1}(B)$. \square

Problem 1.20. Suppose $f : A \rightarrow B$ is an onto function. Describe how you might go about defining a one-to-one correspondence with B as range. \square (Must the domain be a subset of A ?) \square

2. TOPOLOGY, THE BEGINNING

In this section we begin the study of topology proper. We will need some sets to work with (and put topologies on). So we have some standard notation for certain sets.

2.1. Finite and Infinite Sets.

Remark 2.1. We will use the following notation for well-known sets of numbers. You may assume the standard “numerical” properties like order, addition, sign, et cetera.

- (1) *The set of natural numbers*, symbolized \mathbb{N} , is the set $\mathbb{N} := \{0, 1, 2, \dots\}$.
- (2) *The set of integers*, symbolized \mathbb{Z} , is the set $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- (3) *The set of positive integers*, symbolized \mathbb{Z}^+ , is the set $\mathbb{Z}^+ := \{n \in \mathbb{Z} \mid n > 0\}$.
- (4) *The set of rational numbers* is symbolized by \mathbb{Q} .
- (5) *The set of real numbers* is symbolized by \mathbb{R} .
- (6) *The set of complex numbers* is symbolized by \mathbb{C} .

Note that $\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N} \supset \mathbb{Z}^+$.

Definition 2.2. A set X is said to be *finite* if $X = \emptyset$ or $\exists N \in \mathbb{Z}^+$ and an onto function $f : \{n \in \mathbb{Z}^+ \mid n \leq N\} \rightarrow X$. A set X is said to be *infinite* iff there is a one-to-one function $f : \mathbb{Z}^+ \rightarrow X$. In the latter case, if f is also onto we call X *countably infinite*. A set which is finite or countably infinite is called *countable*.

Exercise 2.3. Show that an infinite set is not finite.

Problem 2.4. Is there an infinite set which is not countably infinite? \square

2.2. Open and Closed Sets. “Why don’t you start with what we already know?” A good question, and there are different schools of thought on this. I have belonged to more than one school at different times, and still have not made up my mind. Many books on topology start with metric spaces, and the real line in particular, as motivating examples. The definition of a topology on a set is at first glance rather abstract. We will eventually (by Section 3.2) apply it to sets where you have previously seen topological concepts applied, like the real numbers. For the time being, we take the more difficult abstract route so that you will understand these new concepts in themselves. Then you will be able to make the connections to more familiar notions without the familiarity distorting the concept.

Definition 2.5. Let X be a set. A *topology* \mathcal{T} on X is a collection of subsets of X satisfying the following axioms:

- (1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
- (2) If \mathcal{U} is a subcollection of \mathcal{T} , then $\bigcup \mathcal{U} \in \mathcal{T}$.
- (3) If $\{U_1, U_2, \dots, U_n\}$ is a finite subcollection of \mathcal{T} , then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call members of \mathcal{T} the *open* sets of X . We call X with topology \mathcal{T} a *topological space* or *space* for short. The elements of a space X are usually called *points*.

Strictly speaking, a topological space is a pair (X, \mathcal{T}) , but where \mathcal{T} has been given or is understood, we will call X a space.

Exercise 2.6. Let $X = \{a, b, c\}$. Let $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Is \mathcal{T} a topology on X ?

Example 2.7. Let $X = \{a, b, c\}$. Let $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Show \mathcal{T} is a topology on X . \square

Problem 2.8. Let $X = \{a, b, c\}$. Find all the different topologies on X . \square (This can be worth 2 points for a really good answer.) \square

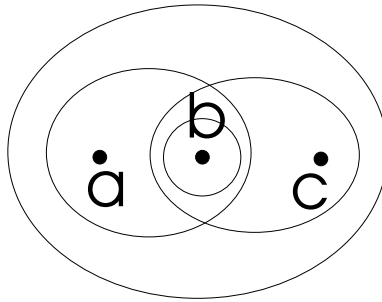


FIGURE 2. A convenient way of illustrating a topology on $X = \{a, b, c\}$.

Definition 2.9. Let X be a space with topology \mathcal{T} . The *closed* sets of X are the sets in the collection $\{C \subset X \mid \exists U \in \mathcal{T}, C = X \setminus U\}$. (\square uniqueness.)

Exercise 2.10. Let $X = \{a, b, c\}$. $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Find all the closed sets of X . Are any sets neither open nor closed?

Problem 2.11. Let $X = \{a, b, c\}$. Let $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Find all the closed sets of X . \square

Exercise 2.12. Let the topology \mathcal{T}_a on \mathbb{N} be the collection of all subsets of \mathbb{N} . What are the closed subsets of \mathbb{N} ?

Proposition 2.13. Let X be a space. Let \mathcal{C} be a collection of closed sets of X . Then $\bigcap \mathcal{C}$ is a closed set in X . \square

Proposition 2.14. Let X be a space. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ be a finite collection of closed sets of X . Then $\bigcup \mathcal{C}$ is a closed set in X . \square

Example 2.15. On \mathbb{N} , let $\{\{0\}, \{1\}, \{2\}, \dots\}$ be a subset of a topology \mathcal{T}_d . What other sets must be in \mathcal{T}_d at a minimum for it to be a topology on \mathbb{N} ? (We call \mathcal{T} the discrete topology on \mathbb{N} .) \square

Example 2.16. On \mathbb{N} , let topology \mathcal{T}_{cf} be the topology consisting of \emptyset and all complements of finite sets. Verify that \mathcal{T}_{cf} is a topology. (We call \mathcal{T}_{cf} the cofinite topology on \mathbb{N} .) What are the closed sets? \square

Example 2.17. On \mathbb{N} , let $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \dots\}$ be a subset of a topology \mathcal{T}_{dd} . What other sets must be in \mathcal{T}_{dd} at a minimum for it to be a topology on \mathbb{N} ?

2.3. Closure of a Set and Limit Points.

Definition 2.18. Let X be a space. Let A be a subset of X . The *closure* of the set A , denoted \bar{A} is the intersection of the collection of all closed sets in X that contain A .

Exercise 2.19. (1) The closure of a set is a closed set. (2) If A is closed then $\bar{A} = A$.

Proposition 2.20. Let X be a space. Let A be a subset of X . Then $\bar{A} = \{x \in X \mid \forall U \text{ open in } X \text{ such that } U \ni x, U \cap A \neq \emptyset\}$. \square

Definition 2.21. Let X be a space. Let A be a subset of X . We say that a point $x \in X$ is a *limit point* of A iff for every open set U in X containing x , $(U \setminus \{x\}) \cap A \neq \emptyset$. The set of limit points of A is denoted A' .

Exercise 2.22. Let $\mathbb{E} \subset \mathbb{N}$ be the set of even natural numbers. Give \mathbb{N} the \mathcal{T}_a topology (see Exercise 2.12). What is \mathbb{E}' ? What is $\bar{\mathbb{E}}$?

Problem 2.23. Let $\mathbb{E} \subset \mathbb{N}$ be the set of even natural numbers. Give \mathbb{N} the cofinite topology. What is \mathbb{E}' ? What is $\bar{\mathbb{E}}$? \square

Proposition 2.24. Let X be a space. Let A be a subset of X . Then $\overline{A} = A \cup A'$. \square

Corollary 2.25. A subset A of a space X is closed iff A contains all its limit points. \square

Problem 2.26. Give an example of a topology on \mathbb{N} and a subset A such that $\overline{A} \neq A$ and $\overline{A} \neq \mathbb{N}$. \square

Definition 2.27. Let A be a nonempty set. A function $x : \mathbb{Z}^+ \rightarrow A$ is called a *sequence*. For convenience, we often write x_i instead of $x(i)$, $i \in \mathbb{Z}^+$, and we write $\{x_i\}_{i=1}^\infty$ for the sequence.

Exercise 2.28. What is the difference between $\{x_i\}_{i=1}^\infty$ as a sequence in A and as a subset of A ? \square

Definition 2.29. Let X be a space. Let A be a subset of X . We say that a point $x \in X$ is a *sequential limit point* of A iff there is a sequence $\{x_i\}_{i=1}^\infty$ in $A \setminus \{x\}$ such that for every open set U in X containing x , there is an $I \in \mathbb{Z}^+$ such that U contains x_i for all $i \geq I$.

Example 2.30. Let \mathcal{T}_s be the topology on \mathbb{N} which contains \emptyset, \mathbb{N} , and for each $N \in \mathbb{N}$ with $N > 0$, the set $\{n \in \mathbb{N} \mid n \geq N\} \cup \{0\}$. Verify that \mathcal{T}_s is a topology. Describe the open sets, closed sets, limit points of subsets, and sequential limit points of subsets of \mathbb{N} with this topology. \square

Exercise 2.31. Let X be a space. Let A be a subset of X . Let x be a sequential limit point of A . Then x is a limit point of A . \square

Problem 2.32. Is there an example of a space X , a subset $Y \subset X$, and a point $x \in X$ such that x is a limit point of Y , but not a sequential limit point of Y ? \square (A good solution, still open, is worth 2 points.) \square

2.4. Subspace Topology. Sometimes we use a given topology on a set X to induce conveniently a topology on a subset of X .

Definition 2.33. Let X be a space with topology \mathcal{T} and $Y \subset X$. \mathcal{T} induces a topology \mathcal{T}_Y on Y , called the *subspace topology* on Y , given by

$$\mathcal{T}_Y := \{V \subset Y \mid \exists U \in \mathcal{T}, V = U \cap Y\}.$$

Problem 2.34. Let $\mathbb{E} \subset \mathbb{N}$ be the set of even natural numbers. Give \mathbb{N} the topology \mathcal{T}_{dd} (see Example 2.17). Describe the subspace topology on \mathbb{E} . (\square , but need to relate to 2.17.) \square

Exercise 2.35. Let X be a space, $Y \subset X$, and $Z \subset Y$. Give Y the subspace topology. Then the subspace topology on Z as a subspace of X is the same as the subspace topology on Z as a subspace of Y . \square

Problem 2.36. Let X be a space and $Y \subset X$. Give Y the subspace topology. Describe (in a useful way) the closed sets in Y . $\square \square$ (But open for a different solution.)

Proposition 2.37. Let X be a space, $Y \subset X$, and $Z \subset Y$. Give Y the subspace topology. The following hold:

- (1) Suppose Y is closed in X . Then Z closed in Y implies Z closed in X .
- (2) Suppose Y is open in X . Then Z open in Y implies Z open in X .

\square

Problem 2.38. Show by example(s) that Proposition 2.37 does not generally hold if Y is not closed (respectively, not open) in X . \square

Problem 2.39. Let X be a space and $Y \subset X$. Give Y the subspace topology. Let $A \subset Y$. The notation \overline{A} is ambiguous. Do we mean the closure of A in the original space X or the closure of A in the subspace Y ? Show by example that this is really ambiguous. Second point: Find a good disambiguating solution. \square

2.5. Basis for a Topology. Sometimes a topology on a set is too complicated to list all the open sets. In that case, we use a smaller collection of open sets, called a *basis* to generate the desired topology.

Exercise 2.40. *You have already seen this idea in a few previous exercises, examples, and problems. Which ones?*

Definition 2.41. A set \mathcal{B} is a *basis* for a topology on a set X provided \mathcal{B} satisfies the following axioms:

- (1) $\bigcup \mathcal{B} = X$.
- (2) $\forall B_1, B_2 \in \mathcal{B}, \forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}, x \in B_3 \subset B_1 \cap B_2$.

The topology \mathcal{T} on X generated by \mathcal{B} is defined by

$$\mathcal{T} := \{U \subset X \mid \exists \mathcal{C} \subset \mathcal{B}, U = \bigcup \mathcal{C}\}.$$

In other words, $U \in \mathcal{T}$ iff U is a union of elements of \mathcal{B} .

Exercise 2.42. (1) *The topology generated by a basis is a topology.* (2) *The sets in the basis are open sets.*

Proposition 2.43. *Let X be a space with topology \mathcal{T} and \mathcal{B} a subcollection of \mathcal{T} such that for all $U \in \mathcal{T}$, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Then \mathcal{B} is a basis that generates the topology \mathcal{T} on X . \square*

We can do it even cheaper!

Definition 2.44. A set \mathcal{S} is a *subbasis* for a topology on a set X provided $\bigcup \mathcal{S} = X$. The topology \mathcal{T} on X generated by \mathcal{S} is defined to be the collection of all unions of finite intersections of elements of \mathcal{S} .

Problem 2.45. *Can you define the basis generated by a subbasis? \square*

Exercise 2.46. (1) *The topology generated by a subbasis is a topology.* (2) *The sets in the subbasis are open sets.*

Theorem 2.47. *Suppose \mathcal{B} is a basis for a topology on X and $Y \subset X$. Then $\mathcal{B}_Y := \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y . \square*

Exercise 2.48. *Why has everything “theorem-like” up to this point been a proposition, and now we have the first theorem?*

Theorem 2.49. *Let X be a space with topology \mathcal{T} generated by a basis \mathcal{B} . Let $A \subset X$. Then*

$$\overline{A} = \{x \in X \mid \forall B \in \mathcal{B} \text{ such that } x \in B, B \cap A \neq \emptyset\}.$$

\square

2.6. Products of Sets and the Product Topology.

Definition 2.50. Let X and Y be sets. The *product* of X and Y , denoted $X \times Y$, is the set of *ordered pairs* given by

$$X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}.$$

Exercise 2.51. *Let $A, C \subset X$ and $B, D \subset Y$.*

- (1) *Show that $A \times B \subset X \times Y$.*
- (2) *Prove or disprove: $(X \setminus A) \times (Y \setminus B) = (X \times Y) \setminus (A \times B)$.*
- (3) *Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.*
- (4) *Can we replace \cap by \cup in the above statement?*

Definition 2.52. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be spaces. The *product topology* on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} := \{U \times V \subset X \times Y \mid U \in \mathcal{T}_X \wedge V \in \mathcal{T}_Y\}.$$

Exercise 2.53. Show that the product topology is well-defined, that is, \mathcal{B} in Definition 2.52 is a basis for a topology.

Problem 2.54. Give an example to show that the basis for the product topology on $X \times Y$ is just a basis, and not generally a topology. \square

Theorem 2.55. Let X and Y be spaces, $A \subset X$, $B \subset Y$, and give $X \times Y$ the product topology. Then $\overline{A \times B} = \overline{A} \times \overline{B}$. \square (But open for a different proof.)

Problem 2.56. Let X and Y be spaces, $A \subset X$, $B \subset Y$, and give $X \times Y$ the product topology. What holds between $A' \times B'$ and $(A \times B)'$? \square

Theorem 2.57. Let X and Y be spaces, $A \subset X$, $B \subset Y$, and give $X \times Y$ the product topology. Give A and B the subspace topologies. Show that the product topology on $A \times B$ is the same as the subspace topology on $A \times B$ as a subspace of $X \times Y$. \square

Theorem 2.58. Let X and Y be spaces. Consider the collection \mathcal{S} of all subsets of $X \times Y$ of the forms: $U \times Y$ where U is open in X , and $X \times V$ where V is open in Y . Show that \mathcal{S} is a subbasis for the product topology on $X \times Y$. \square

Definition 2.59. Let X and Y be sets. The *projection functions* are the functions $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ defined by $\pi_1((x, y)) = x$, respectively, $\pi_2((x, y)) = y$.

Exercise 2.60. Let X and Y be spaces and give $X \times Y$ the product topology. Show that π_1 and π_2 have the properties:

- (1) π_1 and π_2 are onto.
- (2) $\forall U$ open in X , $\pi_1^{-1}(U)$ is open in $X \times Y$.
- (3) $\forall V$ open in Y , $\pi_2^{-1}(V)$ is open in $X \times Y$.

Corollary 2.61. The collection of all sets of the forms $\pi_1^{-1}(U)$, U open in X , and $\pi_2^{-1}(V)$, V open in Y , is a subbasis for the product topology on $X \times Y$. \square

Proposition 2.62. Let X and Y be spaces and give $X \times Y$ the product topology. Projections $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ have the property: $\forall W$ open in $X \times Y$, $\pi_1(W)$ is open in X and $\pi_2(W)$ is open in Y . \square

Theorem 2.63. Let X and Y be spaces with bases \mathcal{B}_X and \mathcal{B}_Y , respectively. Then

$$\mathcal{B}_{X \times Y} := \{A \times B \subset X \times Y \mid A \in \mathcal{B}_X \wedge B \in \mathcal{B}_Y\}$$

is a basis for the product topology on $X \times Y$. \square

3. TOPOLOGY, THE END OF THE BEGINNING

We now end the beginning with some of the topological concepts that are closer to their motivation. In this section, the real line is the paradigm toward which we head.

3.1. Linear Order. Our goal is to define a topology on a set induced by an order on that set. But first we have to make clear what we mean by “order” on a set.

Definition 3.1. Let X be a set. A *relation* R on X is a set of ordered pairs from $X \times X$. For $x, y \in X$, rather than write $(x, y) \in R$, we usually write xRy , putting the symbol of the relation between x and y .

Exercise 3.2. Consider the relation “is a subset of” symbolized \subset on the collection of all subsets of \mathbb{N} . Rather than write $(\{1, 2\}, \{1, 2, 3\}) \in \subset$, we write $\{1, 2\} \subset \{1, 2, 3\}$. For this relation, determine the answer to each of the following questions:

- (1) Given A, B subsets of \mathbb{N} , are they always related by \subset ?
- (2) Given A, B subsets of \mathbb{N} , is it possible that both $A \subset B$ and $B \subset A$?
- (3) Given A, B, C subsets of \mathbb{N} , if $A \subset B$ and $B \subset C$, what may we conclude?
- (4) How does \emptyset stand in the relation?
- (5) How does \mathbb{N} stand in the relation?

3.1.1. Linear Orders. Certain types of relations have served important roles in analysis. Among them are *orders*, and we shall first study *linear orders*. A *nondegenerate* set is a set with more than one element.

Definition 3.3. Let X be a set and $<$ a relation on X . We say $<$ is a *linear order* iff it satisfies the following axioms:

- (1) Antireflexive: $\forall x \in X, \neg(x < x)$.
- (2) Comparable: \forall different $x, y \in X, x < y \vee y < x$.
- (3) Transitive: \forall different $x, y, z \in X, (x < y \wedge y < z) \implies x < z$.

A set with a linear order is a pair $(X, <)$. When we say X is a *linearly ordered set*, we understand a particular linear order $<$ on X . When we say X is an *ordered set* we usually mean X is a linearly ordered set.

Remark 3.4. Given a set X and linear order $<$, for convenience we write $a \leq b$ as shorthand for $a < b \vee a = b$. The reader should be able to figure out the standard meaning of $>$ and \geq .

Remark 3.5. The natural order on the natural numbers \mathbb{N} , conveniently symbolized by $<$, is a linear order. The natural order on the real numbers \mathbb{R} is also a linear order. These examples motivate the definition above.

Exercise 3.6. Prove that a linear order is antisymmetric: \forall different $x, y \in X, \neg(x < y \wedge y < x)$.

Problem 3.7. Why is \subset **not** a linear order on the collection of subsets of \mathbb{N} ? Which axioms fail? \square

Exercise 3.8. Let X be a set with linear order $<$. Let $Y \subset X$. Show that $<$ induces a linear order on Y . (Define it — compare “restricting” a function to a subset of its domain.)

Once we have the concept of a linear order, we can define the concept of an *interval* in that order. Though we use the terms “open” and “closed” below in defining types of intervals, the reader should not jump to the conclusion that these have anything to do with open and closed sets in a topology (yet).

Definition 3.9. Let X be a nondegenerate set with linear order $<$. We define the following subsets of X as types of *intervals*:

- (1) For each $a < b \in X$, define $(a, b) := \{x \in X \mid a < x < b\}$ as the *open interval* between a and b .
- (2) For each $a \leq b \in X$, define $[a, b] := \{x \in X \mid a \leq x \leq b\}$ as the *closed interval* between a and b .
- (3) For each $a < b \in X$, define $[a, b) := \{x \in X \mid a \leq x < b\}$ as the *half-open interval* between a and b , including a .
- (4) For each $a < b \in X$, define $(a, b] := \{x \in X \mid a < x \leq b\}$ as the *half-open interval* between a and b , including b .

Exercise 3.10. Let X be a nondegenerate set with linear order $<$. The following are among the many specific questions one should be able to answer about intervals of X .

- (1) If $a = b$, what can you say about (a, b) ?
- (2) If $a < b$ and if there are no elements of X such that $a < x < b$, then what can you say about (a, b) (example?)
- (3) It is not the case that $[a, b] = \emptyset$, but it can be a singleton (when?)
- (4) Suppose $(a, b) \cap (c, d) \neq \emptyset$. Is there a larger interval that contains them both?
- (5) Is it ever the case that $(a, b) = [c, d]$?

Because orders are so important, a lot of specialized terminology has developed. We define much of it below.

Definition 3.11. Let X be a nondegenerate set with linear order $<$. If there an element $\alpha \in X$ such that $\forall x \in X, \alpha \leq x$, we say α is the *least* (or *smallest*) element of X . If there is an element $\omega \in X$ such that $\forall x \in X, x \leq \omega$, we say ω is the *greatest* (or *largest*) element of X . Without committing ourselves to the existence of least or greatest elements in X , we define the following four types of *rays*:

- (1) For each $a \in X$, the *positive open ray* from a is the set $(a, \infty) := \{x \in X \mid a < x\}$.
- (2) For each $a \in X$, the *negative open ray* from a is the set $(-\infty, a) := \{x \in X \mid x < a\}$.
- (3) For each $a \in X$, the *positive closed ray* from a is the set $[a, \infty) := \{x \in X \mid a \leq x\}$.
- (4) For each $a \in X$, the *negative closed ray* from a is the set $(-\infty, a] := \{x \in X \mid x \leq a\}$.

The positive open ray is sometimes called the set of *successors* of a ; the negative open ray is sometimes called the set of *predecessors* of a .

Proposition 3.12. Let X be a nondegenerate set with linear order $<$. Show that (1) every open interval is an intersection of open rays, (2) every closed interval is an intersection of closed rays, (3) if X has a least element α , then $\forall b \neq \alpha \in X, [\alpha, b) = (-\infty, b)$, (4) if X has a greatest element ω , then $\forall a \neq \omega \in X, (a, \omega] = (a, \infty)$, (5) and if X has a least element α and a greatest element ω , then $X = [\alpha, \omega]$. \square

Proposition 3.13. If X with linear order $<$ has a least (respectively, greatest) element, then it is unique. \square

3.2. Order Topology.

Definition 3.14. Let X be a nondegenerate set with linear order $<$. The *order topology* on X is the topology generated by the subbasis \mathcal{S} consisting of all positive and negative open rays; that is,

$$\mathcal{S} = \{(-\infty, b) \mid b \in X\} \cup \{(a, \infty) \mid a \in X\}.$$

Exercise 3.15. Let X be a nondegenerate ordered set with the order topology. Show the following: (1) every open interval is open, (2) every closed interval is closed, (3) every open ray is open, and (4) every closed ray is closed.

Problem 3.16. Suppose X has no least nor greatest element. Describe the basis \mathcal{B} for the order topology generated by the subbasis \mathcal{S} . Show that $\mathcal{B} \setminus \mathcal{S}$ is also a basis for the order topology on X . \square

Problem 3.17. Suppose $X = [\alpha, \omega]$ has both a least and a greatest element. Describe the basis \mathcal{B} for the order topology generated by the subbasis \mathcal{S} . Is $\mathcal{B} \setminus \mathcal{S}$ also a basis for the order topology on X ? \square

Remark 3.18. For the real numbers \mathbb{R} with the natural order $<$, the basis described in Problem 3.16 is called the *standard basis* for the *standard topology* on \mathbb{R} . For the *unit interval* $[0, 1]$ with the natural order $<$, the basis described in Problem 3.17 is called the *standard basis* for the *standard topology* on $[0, 1]$.

Problem 3.19. Give \mathbb{R} the standard (order) topology. Show that every point of \mathbb{R} is a limit point of \mathbb{R} . \square

Problem 3.20. For $X = \mathbb{N}$ with the natural order $<$, describe the order topology on \mathbb{N} . Have we seen this topology on \mathbb{N} before? Find a minimal basis for the order topology on \mathbb{N} . \square

Problem 3.21. Let X be a nondegenerate ordered set in the order topology. Let $a < b \in X$. Then the closed interval $[a, b]$ has a subspace topology induced by the order topology on X . But $[a, b]$ has the order induced by the order on X . So $[a, b]$ also has an order topology induced directly by the inherited order. Are these two topologies on $[a, b]$ the same? \square

Exercise 3.22. Let $X = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\} \subset \mathbb{R}$. (1) What is the relationship between the subspace topology on X and the order topology on X ? (2) Give X the order topology. What are the limit points of X ?

Problem 3.23. Let $X = [0, 1) \cup [2, 3] \subset \mathbb{R}$. On the one hand, X has a subspace topology \mathcal{T}_1 induced by the standard (order) topology on \mathbb{R} . On the other hand, X has a natural order induced by the natural order on \mathbb{R} . This order induces an order topology \mathcal{T}_2 on X . How are \mathcal{T}_1 and \mathcal{T}_2 related?

Exercise 3.24. Redo Problem 3.23 with $X = [0, 1] \cup [2, 3]$.

3.2.1. *Dictionary Order.* Because \mathbb{R} , $[0, 1]$, and \mathbb{N} have natural orders, and standard order topologies, we have thought about them a lot in the many times they have crossed our intellectual path. So we need some examples (1) of ordered sets with order topologies that are less familiar, and (2) examples of topologies that are defined in terms of order, but are not order topologies. We introduce the first in this subsection.

Definition 3.25. Let X and Y be ordered sets with orders $<$ and \prec , respectively. We define the *dictionary order* \triangleleft on the product set $X \times Y$ as follows: $(a, b) \triangleleft (x, y)$ iff either $a < x$ or $a = x$ and $b \prec y$.

Remark 3.26. It is only to avoid confusion that we use three different symbols of order in Definition 3.25. We could have used the same symbol $<$ throughout and relied on context.

Exercise 3.27. Let $X = \{0, 1\}$ and $Y = [0, 1]$ with their natural orders. Put the dictionary order on $X \times Y$. Put the order topology on $X \times Y$. Answer the following:

- (1) Place in order $(0, 0), (1, 0), (0, 1), (1, 1)$.
- (2) Is $\{0\} \times [0, 1]$ an open set?
- (3) Is $\{0\} \times [0, 1]$ a closed set?
- (4) Is there any point in the order between $(0, 1)$ and $(1, 0)$?
- (5) Does this order have both a least and a greatest element?

Problem 3.28. Let $X = \{0, 1\}$ and $Y = [0, 1]$ with their natural orders. Put the order topologies on X and Y . Put the product topology on $X \times Y$. Now, put the dictionary order on $X \times Y$. Put the order topology on $X \times Y$. Is the order topology on $X \times Y$ the same as the product topology on $X \times Y$? \square

Exercise 3.29. Let $X = \{0, 1\}$ and $Y = [0, 1]$ with their natural orders. Put the dictionary order on $Y \times X$. Put the order topology on $Y \times X$. Answer the following:

- (1) Place in order $(0, 0), (1, 0), (0, 1), (1, 1)$.
- (2) Is $[0, 1] \times \{0\}$ an open set?
- (3) Is $[0, 1] \times \{0\}$ a closed set?
- (4) Is there any point in the order between $(0, 1)$ and $(1, 0)$?
- (5) Does this order have both a least and a greatest element?

Problem 3.30. Let $Y = \mathbb{N} \times \mathbb{N}$ and give Y the dictionary order and the order topology. Determine which points of Y are limit points of Y . \square

Problem 3.31. There is an obvious one-to-one correspondence f between $X = \mathbb{N} \times [0, 1)$ and $Y = [0, \infty) \subset \mathbb{R}$. Define f . Give X the dictionary order topology \mathcal{T}_X , and give Y the order topology \mathcal{T}_Y induced by the natural order on Y . Show that f induces a one-to-one correspondence between \mathcal{T}_X and \mathcal{T}_Y . \square

Problem 3.32. How might you define in a useful way what it means for a set X with topology \mathcal{T}_X and a set Y with topology \mathcal{T}_Y to be “topologically equivalent?” Use your definition to give some examples (finite and infinite) of topologically equivalent spaces, and of spaces which are not topologically equivalent.

3.2.2. *Floor Topology.* We now define a topology on an ordered set using the order (and interval notation), but the topology is not (usually) an order topology. This type of topology is a rich source of examples.

Definition 3.33. Let X be a non-degenerate ordered set. The *floor topology* \mathcal{T}_f on X is the topology generated by the basis $\mathcal{B}_f := \{[a, b) \mid a < b \in X\}$.

Exercise 3.34. Show that standard (order) topology on \mathbb{N} and the floor topology are the same.

Definition 3.35. Let X be a set and \mathcal{T}_1 and \mathcal{T}_2 be topologies on X . If $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that \mathcal{T}_1 is *coarser* than \mathcal{T}_2 and that \mathcal{T}_2 is *finer* than \mathcal{T}_1 . If $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_1 \neq \mathcal{T}_2$, we say that \mathcal{T}_1 is *strictly coarser* than \mathcal{T}_2 and that \mathcal{T}_2 is *strictly finer* than \mathcal{T}_1 .

Exercise 3.36. It is obvious that if \mathcal{T}_1 is both coarser and finer than \mathcal{T}_2 on X , then the topologies are equal (the same, equivalent, take your pick). Prove it anyway. Then give me a reason why we might have chosen the “coarser/finer” language.

Problem 3.37. Consider the following five topologies on \mathbb{N} : the trivial $\{\emptyset, \mathbb{N}\}$ topology, the discrete topology (see Exercise 2.12 and Example 2.15), the standard (order) topology, the floor topology, and the cofinite topology (see Example 2.16). How are they related? \square

Problem 3.38. Give \mathbb{R} the floor topology. Is every point of \mathbb{R} a limit point of \mathbb{R} ? \square

Problem 3.39. What is the relationship between the standard topology on \mathbb{R} and the floor topology? \square

3.3. Hausdorff Spaces.

Definition 3.40. A space X with topology \mathcal{T} is said to be a *Hausdorff space* iff it satisfies the following axiom: (H) \forall different $x, y \in X, \exists U, V \in \mathcal{T}$, such that $x \in U \wedge y \in V \wedge U \cap V = \emptyset$.

Exercise 3.41. Give examples of topologies on the set $X = \{0, 1, 2\}$ such that X (1) is and (2) is not a Hausdorff space.

Problem 3.42. Give \mathbb{N} the cofinite topology (see Example 2.16). Is \mathbb{N} a Hausdorff space in the cofinite topology? \square

Theorem 3.43. Let X be a nondegenerate ordered set with the order topology. Then X is a Hausdorff space. \square

Remark 3.44. In particular, in view of Theorem 3.43, \mathbb{R} with the standard topology is a Hausdorff space.

Problem 3.45. Is \mathbb{R} with the floor topology a Hausdorff space? \square

Proposition 3.46. Let X be a Hausdorff space and $Y \subset X$. Then Y with the subspace topology is a Hausdorff space. \square

Proposition 3.47. Let X be a Hausdorff space. Then for all $x \in X$, the set $\{x\}$ is closed. Moreover, every finite subset of X is closed. \square

Remark 3.48. The statement “for all $x \in X$, $\{x\}$ is closed” is often expressed by saying “points are closed in X .”

Problem 3.49. Is the converse of Theorem 3.47 true? That is, does “points are closed in X ” imply “ X is Hausdorff?” \square

Theorem 3.50. Let X and Y be Hausdorff spaces. Give $X \times Y$ the product topology. Then $X \times Y$ is a Hausdorff space. \square

Problem 3.51. Does the converse of Theorem 3.50 hold? \square

Theorem 3.52. Let X be an infinite Hausdorff space. Suppose x is a limit point of X . Then for every open set $U \ni x$, U contains infinitely many points of X . \square

Theorem 3.53. Let X be a Hausdorff space. If x and y are sequential limit points of X with respect to the same sequence, then $x = y$. \square

Problem 3.54. Show (by example) that the assumption that X is a Hausdorff space in Theorem 3.53 is necessary. \square

The following problem is worth 2 points for a good solution. (Compare Problem 2.32.)

Problem 3.55. Is there an example of a Hausdorff space X , a subset $Y \subset X$, and a point $x \in X$ such that x is a limit point of Y , but not a sequential limit point of Y ?

3.4. Dense Subsets and Countable Bases.

Definition 3.56. Let X be a space. A subset $D \subset X$ is said to be *dense* in X iff $\overline{D} = X$.

Proposition 3.57. Give \mathbb{R} the standard (order) topology. Then the rationals \mathbb{Q} are dense in \mathbb{R} . \square

Theorem 3.58. The rationals \mathbb{Q} are a countable set. \square

Remark 3.59. Putting Theorem 3.58 and Proposition 3.57 together, we observe that \mathbb{R} has a countable dense subset.

Now that we have seen several examples of parts of this lemma in various proofs, we can draw the elements together and state some general ways of showing one topology is coarser than another.

Lemma 3.60 (Basis Lemma). Let X be a space with bases \mathcal{B}_1 and \mathcal{B}_2 , and topologies \mathcal{T}_1 and \mathcal{T}_2 generated by these bases, respectively. The following statements are equivalent:

- (1) $\forall B_1 \in \mathcal{B}_1, \forall x \in B_1, \exists B_2 \in \mathcal{T}_2, x \in B_2 \subset B_1.$
- (2) $\forall B \in \mathcal{B}_1, \exists \mathcal{C}_B \subset \mathcal{B}_2, B = \bigcup \mathcal{C}_B.$
- (3) $\forall U \in \mathcal{T}_1, \forall x \in U, \exists B \in \mathcal{T}_2, x \in B \subset U.$
- (4) $\mathcal{T}_1 \subset \mathcal{T}_2.$

□ (Replace \mathcal{T}_2 by \mathcal{B}_2 in (1) and (3).) □

Problem 3.61. Give \mathbb{R} the standard (order) topology. Can you define a countable basis for \mathbb{R} ? □

Problem 3.62. Give \mathbb{R} the floor topology. Does \mathbb{R} have a countable dense subset in the floor topology? □

Problem 3.63. Give \mathbb{R} the floor topology. Can you define a countable basis for \mathbb{R} in the floor topology? □

3.5. Connected Spaces. One of the big ideas in topology, motivated by its usefulness in analysis (think “intermediate value theorem”), is the idea of *connectedness*. We are trying to capture the idea that X is/is not broken into “pieces.”

Definition 3.64. Let X be a space. We say that X is *separated* iff there exist nonempty open sets U and V in X such that $X = U \cup V$ and $U \cap V = \emptyset$. If there are such open sets, then we call $X = U \cup V$ a *separation* of X . We say X is *connected* if there does not exist a separation of X .

Exercise 3.65. On $X = \{0, 1, 2\}$, (1) give an example of a nontrivial topology such that X is connected, and (2) give an example such that X is not connected.

Proposition 3.66. Let X be a space. Then X is connected iff the only subsets of X that are both open and closed are \emptyset and X . □

Problem 3.67. Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on the set X . Suppose $\mathcal{T}_1 \subset \mathcal{T}_2$. What does connectedness of X in one topology imply about connectedness of X in the other topology? □

We also want to be able to talk about *connected subsets* of a space.

Definition 3.68. Let X be a space and $Y \subset X$. We say that Y is a *connected subset* of X iff Y is connected in the subspace topology.

Proposition 3.69. Let X be a space and $Y \subset X$. Then Y is separated iff there exist nonempty sets A and B in Y such that $Y = A \cup B$, $A \cap B = \emptyset$, and neither A nor B contains a limit point of the other. □

Example 3.70. Give \mathbb{N} the natural order and the order topology. Show that \mathbb{N} is not connected. What are the connected subsets of \mathbb{N} ? □

Problem 3.71. Give \mathbb{N} the cofinite topology (see Example 2.16). Is \mathbb{N} connected in the cofinite topology? □

Example 3.72. Let $X = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\} \subset \mathbb{R}$. Give X the natural order (as a subset of \mathbb{R}), and the order topology. Show that X is not connected. What are the connected subsets of X ? □

Proposition 3.73. Let $X = U \cup V$ be a separation of X . Suppose Y is a connected subset of X . Then either $Y \subset U$ or $Y \subset V$. □

Proposition 3.74. Let X be a space and \mathcal{C} a collection of connected subsets of X with the property that $\exists x \in X$ such that $\forall C \in \mathcal{C}, x \in C$. Then $\bigcup \mathcal{C}$ is connected. □

Proposition 3.75. *Let X be a space and \mathcal{C} a collection of connected subsets of X with the property that $\exists D \subset X$ such that D is connected and $\forall C \in \mathcal{C}, D \cap C \neq \emptyset$. Then $(\cup \mathcal{C}) \cup D$ is connected. \square*

Proposition 3.76. *Let X be a space and $\{C_n\}_{n=1}^{\infty}$ be a collection of connected subsets of X such that $\forall n \in \mathbb{Z}^+, C_n \cap C_{n+1} \neq \emptyset$. Then $\bigcup_{n=1}^{\infty} C_n$ is connected. \square*

Proposition 3.77. *Let X be a space and C a connected subset of X . Suppose $C \subset D \subset \overline{C}$. Then D is connected. \square*

Theorem 3.78. *Let X and Y be connected spaces. Give $X \times Y$ the product topology. Then $X \times Y$ is connected. \square*

Problem 3.79. *Does the converse of Theorem 3.78 hold? \square*

3.6. Linear Continua. So far we have few examples of connected spaces. Our goal is to show that \mathbb{R} , and any interval or ray in \mathbb{R} , is connected. We shall show more.

Definition 3.80. Let X be a linearly ordered set with order $<$. A set $A \subset X$ is said to be *bounded above* iff $\exists u \in X, \forall a \in A, a \leq u$. We call u an *upper bound* for A . A set $A \subset X$ is said to be *bounded below* iff $\exists l \in X, \forall a \in A, l \leq a$. We call l a *lower bound* for A .

Definition 3.81. Let X be a linearly ordered set with order $<$. Let $A \subset X$. Let \mathcal{U} denote the collection of all upper bounds of A . If \mathcal{U} has a least element u_0 , then we say that u_0 is a *least upper bound* (abbreviated lub) for A . Similarly, let \mathcal{L} denote the collection of all lower bounds of A . If \mathcal{L} has a greatest element l_0 , then we say that l_0 is a *greatest lower bound* (abbreviated glb) for A .

Exercise 3.82. *Let X be a linearly ordered set with order $<$. Suppose $A \subset X$ is a nonempty set. Show the following:*

- (1) *If A has a least upper bound, then it is unique.*
- (2) *If A has a greatest lower bound, then it is unique.*
- (3) *Suppose A has a least upper bound u_0 . If u_0 is in A , then it is the greatest element of A .*
- (4) *Suppose A has a greatest lower bound l_0 . If l_0 is in A , then it is the least element of A .*
- (5) *Using subsets of \mathbb{R} , give two examples of each of the above if-then statements, one satisfying, and one not satisfying, the antecedent of the statement.*

Definition 3.83. Let X be a linearly ordered set with order $<$. We say that X has the *least upper bound property* iff every nonempty subset $A \subset X$ that is bounded above has a least upper bound. Similarly, we say that X has the *greatest lower bound property* iff every nonempty subset $A \subset X$ that is bounded below has a greatest lower bound.

Proposition 3.84. *Let X be a linearly ordered set with order $<$. Then X has the least upper bound property iff X has the greatest lower bound property. \square*

Definition 3.85. Let X be a linearly ordered set with order $<$. We say that X is a *linear continuum* iff it satisfies the following two axioms:

- (1) **LUB:** X has the least upper bound property.
- (2) **Betweenness:** $\forall x < y \in X, \exists z \in X, x < z < y$.

Exercise 3.86. *Give examples of the following:*

- (1) *A linearly ordered set which satisfies LUB and Betweenness.*
- (2) *A linearly ordered set which satisfies LUB, but not Betweenness.*
- (3) *A linearly ordered set which satisfies Betweenness but not LUB.*

Theorem 3.87. *Let X be a nondegenerate ordered set and give X the order topology. If X is connected, then X is a linear continuum. \square*

A good proof of the following theorem (converse to Theorem 3.87 above) is worth 2 points. Be clear in your proof about where you use each axiom of a linear continuum. Remember that you are forbidden to consult any textbook.

Theorem 3.88. *Let X be a linear continuum and give X the order topology. Then X is connected.*

Corollary 3.89. *Let X be a linear continuum and give X the order topology. Each interval and ray in X is connected.*

Corollary 3.90. *The real line \mathbb{R} , and each interval and ray in \mathbb{R} , is connected.*

Exercise 3.91. *Is there a closed subset P of \mathbb{R} (in the standard topology) such that $\mathbb{R} \setminus P$ is not connected, but no proper closed subset of P has this property?*

3.7. More on the Reals.

Theorem 3.92. *Give \mathbb{R} the standard (order) topology. Then there is not a collection of open subsets of \mathbb{R} that is both pairwise disjoint and uncountable. \square*

Problem 3.93. *Is Theorem 3.92 true in the floor topology on \mathbb{R} ?*

Problem 3.94. *Is Theorem 3.92 true in the cofinite topology on \mathbb{R} ? \square*

Theorem 3.95. *Give \mathbb{R} the standard (order) topology. Let U be an open subset of \mathbb{R} . Then U can be written as a union of a pairwise disjoint collection of open intervals and rays.*

Example 3.96. *Give an example of a linear continuum which is not the real line \mathbb{R} , nor topologically equivalent to a subspace of \mathbb{R} .*

While the following theorem does not depend on the topology one places on \mathbb{R} , topology can be used to prove it. The more interesting question is “Must topology be used to prove it?”

Theorem 3.97. *The set of real numbers \mathbb{R} is uncountable.*

3.8. Totally Disconnected Spaces.

Definition 3.98. A space X is *totally disconnected* iff the only connected subsets of X are singletons.

Proposition 3.99. *Give \mathbb{R} the standard (order) topology. Show that \mathbb{R} has at least two disjoint totally disconnected subsets, each dense in \mathbb{R} .*

Problem 3.100. *How many pairwise disjoint, totally disconnected, dense subsets does \mathbb{R} have (in the standard topology)?*

Proposition 3.101. *Give \mathbb{R} the floor topology. Then \mathbb{R} is totally disconnected.*

4. COMPACT SPACES

Compactness of a space has been defined in several different ways. The following development is the most general and useful.

Definition 4.1. Let X be a space. A *cover* of X is a collection \mathcal{A} of subsets of X such that $\bigcup \mathcal{A} = X$. If the sets in \mathcal{A} are open (respectively, closed), we say that \mathcal{A} is an *open cover* (respectively, *closed cover*) of X . Now suppose that \mathcal{A} is a cover of X and that $\mathcal{B} \subset \mathcal{A}$ is a subcollection of \mathcal{A} . We say that \mathcal{B} is a *subcover* iff $\bigcup \mathcal{B} = X$.

We will be concerned primarily with open covers.

Definition 4.2. A space X is *compact* iff every open cover \mathcal{U} of X has a finite subcover $\mathcal{V} \subset \mathcal{U}$.

Exercise 4.3. Show that a finite set X is compact in any topology on X .

Example 4.4. Show that the set of natural numbers \mathbb{N} in the cofinite topology is compact.

Example 4.5. Show that \mathbb{N} in the standard (order) topology is not compact. (That is, find an open cover \mathcal{U} of \mathbb{N} that has no finite subcover.)

Example 4.6. Show that \mathbb{R} is not compact in the standard (order) topology.

Problem 4.7. Is \mathbb{R} compact in the floor topology?

Example 4.8. Let $X = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ with the order topology. Show that X is compact.

Theorem 4.9. Let X be a space with basis \mathcal{B} for the topology on X . Then X is compact iff every cover of X by basis elements has a finite subcover.

Theorem 4.10. Let X be a space with subbasis \mathcal{S} for the topology on X . Then X is compact iff every cover of X by subbasis elements has a finite subcover.

Analogous to the concept of connectedness, we want to be able to refer to compact subsets of a space.

Definition 4.11. Let X be a space and $Y \subset X$ a subspace. We say that Y is a *compact subset* of X iff Y is compact in the subspace topology. Moreover, if \mathcal{U} is a collection of sets open in X such that $\bigcup \mathcal{U} \supset Y$, we call \mathcal{U} an *open cover* of Y (by sets open in X).

Proposition 4.12. Let X be a space and $Y \subset X$ a subspace. Then Y is compact iff given any cover \mathcal{U} of Y by sets open in X , there is a finite subcover $\mathcal{V} \subset \mathcal{U}$ (in the sense that $\bigcup \mathcal{V} \supset Y$).

Proposition 4.13. A closed subset of a compact space is compact.

Exercise 4.14. Show by example that a proper closed subset of \mathbb{R} in the standard (order) topology need not be compact.

Proposition 4.15. A compact subset of a Hausdorff space is closed.

Problem 4.16. Show by example that the assumption that the space is Hausdorff is needed in Proposition 4.15.

Proposition 4.17. Let X be a space and $\{Y_i\}_{i=1}^n$ a finite collection of compact subsets of X . Then $\bigcup_{i=1}^n Y_i$ is compact.

Problem 4.18. Does Proposition 4.17 hold for infinite collections of compact subsets?

Theorem 4.19. Let X and Y be compact spaces. Then $X \times Y$ is compact in the product topology.

Problem 4.20. Is the converse of Theorem 4.19 true?

Problem 4.21. Let X and Y be compact ordered spaces. Is $X \times Y$ compact in the dictionary order topology?

Theorem 4.22. Let X be a compact space. Then every infinite subset of X has a limit point.

Problem 4.23. Is the converse of Theorem 4.22 true?

4.1. Compactness and the Reals. We have already seen that the real numbers in the order topology are not compact. In this section we shall “characterize” the compact subsets of the reals. Each of Theorems 4.24 and 4.27 is worth two points.

Theorem 4.24. *Let $[a, b] \subset \mathbb{R}$ be a closed interval in \mathbb{R} with the standard (order) topology. Then $[a, b]$ is a compact subset of \mathbb{R} .*

Problem 4.25. *Does Theorem 4.24 extend to all linear continua?*

Definition 4.26. Let X be an ordered set. A subset $A \subset X$ is said to be *bounded* iff A has both an upper and a lower bound in X .

Theorem 4.27. *Let A be a closed and bounded subset of \mathbb{R} . Then A is a compact subset of \mathbb{R} .*

Problem 4.28. *Is the converse of Theorem 4.27 true?*

Problem 4.29. *Does Theorem 4.27 extend to all linear continua?*

Problem 4.30. *Does Theorem 4.27 hold in the floor topology on \mathbb{R} ?*

4.2. The Plane \mathbb{R}^2 .

Definition 4.31. The *plane* \mathbb{R}^2 is the product $\mathbb{R} \times \mathbb{R}$ with the product topology.

Problem 4.32. *Does the plane have a countable dense subset?*

Problem 4.33. *Does the plane have a countable basis?*

Recall that $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the *projection* functions from the product to the factors, defined by $\pi_1((a, b)) = a$ and $\pi_2((a, b)) = b$.

Problem 4.34. *Suppose that A is a closed subset of the plane. Does it follow that $\pi_1(A)$ is closed in \mathbb{R} ?*

Definition 4.35. A subset $A \subset \mathbb{R}^2$ is said to be *bounded* iff both $\pi_1(A)$ and $\pi_2(A)$ are bounded as subsets of \mathbb{R} .

That is, $A \subset \mathbb{R}^2$ is bounded iff you can put A in a “box.”

Exercise 4.36. *Find a closed subset of the plane that is not bounded. Find a bounded subset of the plane that is not closed.*

Theorem 4.37. *A closed and bounded subset of the plane is compact.*

Problem 4.38. *Does the converse of Theorem 4.37 hold?*

Problem 4.39. *Is there a collection of open subsets of the plane that is both uncountable and pairwise disjoint?*

Problem 4.40. *Can every open subset of the plane be written as a pairwise disjoint union of sets of the form $A \times B$, where each of A and B is an open interval in \mathbb{R} ?*

Problem 4.41. *Is there a compact subset S of the plane such that $\mathbb{R}^2 \setminus S$ is not connected, but no proper closed subset of S has this property?*

Problem 4.42. *Is there a closed noncompact subset S of the plane such that $\mathbb{R}^2 \setminus S$ is not connected, but no proper closed subset of S has this property?*

4.3. Perfect Spaces.

Definition 4.43. Let X be a space. A subset $A \subset X$ is *perfect* iff every point of A is a limit point of A .

Exercise 4.44. Let \mathbb{R} have the standard (order) topology. Show that \mathbb{R} , and every interval and ray in \mathbb{R} , is perfect.

Example 4.45. Let \mathbb{R} have the standard (order) topology. Give an example of a compact perfect subset of \mathbb{R} that is not connected.

A good solution to the following problem is worth three points.

Problem 4.46. Let \mathbb{R} have the standard (order) topology. Is there a totally disconnected, perfect, compact subset of \mathbb{R} ?

Theorem 4.47. Let X be a nonempty compact perfect Hausdorff space. Then X is uncountable.

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