Weighted Voting 2

Now that we understand the structure of a weighted voting system, we will introduce the ideas that will lead us to a mathematical definition of voting *power*. The new ideas are

- Coalitions
  - Winning and losing coalitions
  - Counting coalitions
- Banzhaf Power Index
  - Banzhaf Power Distribution

Coalitions

- A *coalition* is a group (set) of players joining forces to vote the same way on a motion. (We allow one player alone to be a coalition.)
- The *weight* of a coalition is the sum of the weights of the players in it.
- A coalition is a *winning* coalition if its weight equals or exceeds the quota.
- A coalition is *losing* if its weight is less than the quota.

Our assumptions about weighted voting systems, in particular, the quota-weight inequality, imply the following facts:

- There is at least one winning coalition.
- There is at least one losing coalition.
- Given a winning coalition, the players **not** in it form a **losing** coalition.

Set Notation

We sometimes use set notation, namely curly brackets, to denote the members of a coalition. For example,

$$\{P_1, P_2\}$$

denotes the coalition consisting of players P_1 and P_2.

In the weighted voting system [4: 3, 2, 1], what are the weights of the following coalitions?

$$\{P_1, P_3\}$$

$$\{P_1\}$$

Is $\{P_1, P_3\}$ a winning or a losing coalition?

What about the coalition $\{P_1\}$?

Coalition Table

A table of all the possible coalitions for a given weighted voting system, together with the weight of each coalition, and whether it is winning or losing, will be essential to determining **power**.

Being **systematic** in constructing the coalition table is the only way to make sure you do not forget to list any of the coalitions.

Consider the weighted voting system: [4: 3, 2, 1]

Coalition Table		
Coalition	Weight	Winning or Losing?

How many players?

How many coalitions?

How many winning?

Determine the coalition table for the weighted voting system

[2: 1, 1, 1]

Coalition Table		
Coalition	Weight	Winning or Losing?

How many players?

How many coalitions?

How many winning?

Compare to the preceding example.

Compare to the example [101: 99, 98, 3].

How Many Coalitions?

Before we look at a specific example with four players, we pause to figure out how many coalitions there will be. Consider a weighted voting system with four players P_1, P_2, P_3, and P_4. We can divide the coalitions into two types.

Coalitions without P_4	Coalitions with P_4
$\{P_1\}$	$\{P_1, P_4\}$
$\{P_2\}$	$\{P_2, P_4\}$
$\{P_3\}$	$\{P_3, P_4\}$
$\{P_1, P_2\}$	$\{P_1, P_2, P_4\}$
$\{P_1, P_3\}$	$\{P_1, P_3, P_4\}$
$\{P_2, P_3\}$	$\{P_2, P_3, P_4\}$
$\{P_1, P_2, P_3\}$	$\{P_1, P_2, P_3, P_4\}$
$\{\ \}$	$\{P_4\}$

There are

$$2*(7) + 1 = 15$$

coalitions when there are 4 players.

This can also be written

$$2^4 - 1 = 15$$

In general, for n players:

$$2^n - 1$$

Example – four players

Find the coalition table for the weighted voting system

[5: 3, 2, 2, 1]

Coalition	Weight	Win/Lose?		Coalition	Weight	Win/Lose?

For each of the following examples

- What is the total number of possible coalitions
- Is each weighted voting system a reasonable one?
- Does any have a dictator, dummies, or player(s) with veto power?

1. [5: 6, 2, 1]
2. [10: 4, 3, 2, 1]
3. [7: 3, 3, 1, 1, 1]
4. [6: 3, 3, 1, 1, 1]
5. [25: 9, 8, 7, 6, 5, 4]
6. [58: 31, 31, 28, 21, 2, 2]
7. [13: 4, 4, 2, 2, 1, 1, 1, 1, 1]
8. [39: 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
9. [62: 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]

Recognizing Dictators

How does one recognize a dictator, player with veto power, or a dummy from the weighted voting system?

- A *dictator* has a weight that exceeds or equals the quota.
  - For example: [7: 8, 4, 2, 1]
- A player has *veto power* if the combined weight of all the other players does not meet the quota.
  - A dictator automatically has veto power.
  - For example: [7: 4, 3, 2, 1]
- A player is a *dummy* if he never makes a difference in a coalition being winning or losing. (He is never *critical* – see below.)
  - If there is a dictator, then all the other players are dummies.
  - A player can be a dummy without there being any dictator.
    - For example: [7: 4, 3, 1, 1]

Critical Players

A player whose desertion from a winning coalition makes it a losing coalition is called a *critical player* for that coalition.

Example. Consider the weighted voting system

[5: 3, 2, 1]

Coalition	Weight	W/L?	Critical Players
$\{P_1\}$			
$\{P_2\}$			
$\{P_3\}$			
$\{P_1, P_2\}$			
$\{P_1, P_3\}$			
$\{P_2, P_3\}$			
$\{P_1, P_2, P_3\}$			

Banzhaf Power Index

- Step 1. List all possible coalitions.
- Step 2. Determine the winning coalitions.
- Step 3. Determine the critical players in each winning coalition.
- Step 4. Count how many times a particular player P_i is critical and call this number B_i.
- Step 5. Count the total number of times players are critical and call this number T. Note T is the sum of all the B_i's.

The Banzhaf power index of player P_i is the fraction $\frac{B_i}{T}$.

The Banzhaf power distribution is the complete list of all players' Banzhaf power indexes (which always sums to 1).

P_1: P_2: P_3: … P_n:

Example – one person, one vote

Consider the weighted voting system [2: 1, 1, 1]

Coalition	Weight	W/L?	Critical Players
$\{P_1\}$			
$\{P_2\}$			
$\{P_3\}$			
$\{P_1, P_2\}$			
$\{P_1, P_3\}$			
$\{P_2, P_3\}$			
$\{P_1, P_2, P_3\}$			

B_1 = B_2 = B_3 = T =

Banzhaf Power distribution:

P_1: P_2: P_3:

In a one person, one vote situation, all players have equal power.

Example: [4: 3, 2, 1]

Coalition	Weight	W/L?	Critical Players
$\{P_1\}$			
$\{P_2\}$			
$\{P_3\}$			
$\{P_1, P_2\}$			
$\{P_1, P_3\}$			
$\{P_2, P_3\}$			
$\{P_1, P_2, P_3\}$			

$B_1 =$ $B_2 =$ $B_3 =$ $T =$

Banzhaf Power distribution:

P_1: P_2: P_3:

Veto power? Dictator? Dummies?

Example: [3: 3, 1, 1]

Coalition	Weight	W/L?	Critical Players
$\{P_1\}$			
$\{P_2\}$			
$\{P_3\}$			
$\{P_1, P_2\}$			
$\{P_1, P_3\}$			
$\{P_2, P_3\}$			
$\{P_1, P_2, P_3\}$			

B_1 = B_2 = B_3 = T =

Banzhaf Power distribution:

P_1: P_2: P_3:

Veto power? Dictator? Dummies?

Example: [4: 2, 2, 1]

Coalition	Weight	W/L?	Critical Players
$\{P_1\}$			
$\{P_2\}$			
$\{P_3\}$			
$\{P_1, P_2\}$			
$\{P_1, P_3\}$			
$\{P_2, P_3\}$			
$\{P_1, P_2, P_3\}$			

B_1 = B_2 = B_3 = T =

Banzhaf Power distribution:

P_1: P_2: P_3:

Veto power? Dictator? Dummies?

Example: [5: 3, 2, 2, 1]

Coalition	Weight	W/L	Critical Players		Coalition	Weight	W/L	Critical Players
$\{P_1\}$					$\{P_2, P_4\}$			
$\{P_2\}$					$\{P_3, P_4\}$			
$\{P_3\}$					$\{P_1, P_2, P_3\}$			
$\{P_4\}$					$\{P_1, P_2, P_4\}$			
$\{P_1, P_2\}$					$\{P_1, P_3, P_4\}$			
$\{P_1, P_3\}$					$\{P_2, P_3, P_4\}$			
$\{P_1, P_4\}$					$\{P_1, P_2, P_3, P_4\}$			
$\{P_2, P_3\}$								

B_1 = B_2 = B_3 = B_4 = T =

Banzhaf Power distribution:

P_1: P_2: P_3: P_4:

Example: [6: 4, 3, 2, 1]

This time we will try a shortcut and list only the winning coalitions.

Winning Coalitions	**Critical Players**

B_1 = B_2 = B_3 = B_4 = T =

Banzhaf Power distribution:

P_1: P_2: P_3: P_4:

Banzhaf Power Index Revisited

- Step 1. List all *winning* coalitions.
- Step 2. Determine the critical players in each winning coalition.
- Step 3. Count how many times a particular player P_i is critical and call this number B_i.
- Step 4. Count the total number of times players are critical and call this number T.

The Banzhaf power index of player P_i is the fraction $\dfrac{B_i}{T}$.

The Banzhaf power distribution is the complete list of all players' Banzhaf power indexes (which always sums to 1).

P_1: P_2: P_3: … P_n: