Weighted Voting 3

In this lecture, we will explore weighted voting systems further.

- Examples of shortcuts to determining winning coalitions and critical players.
- Determining winning coalitions, critical players, and power from voting rules rather than weights.
- Modeling voting rules with weighted voting systems.
    - Nassau County, NY, Board of Supervisors
    - The United Nations (UN) Security Council.
    - The European Union (EU) Council of Ministers.

A Mathematical Theorem

Theorem. If there are n players, then the total number of possible coalitions is 2^n-1.

Proof. We know that 3 players can form $2^3 - 1 = 7$ coalitions. The formula works for n=3. Now suppose $n \geq 4$.

Assume the formula is true for n-1 players: that is, n-1 players can form $2^{n-1} - 1$ coalitions.

We can use the idea of the preceding example to compute the number of coalitions that n players can form. It will be twice the number that n-1 players can form, plus 1.

$$2*(2^{n-1} - 1) + 1 = 2*2^{n-1} - 2*1 + 1$$

$$= 2^n - 2 + 1$$

$$= 2^n - 1$$

We conclude, by the mathematical principle of *induction*, that the formula is correct for all n.

Banzhaf Power Index Revisited

- Step 1. List all *winning* coalitions.
- Step 2. Determine the critical players in each winning coalition.
- Step 3. Count how many times a particular player P_i is critical and call this number B_i.
- Step 4. Count the total number of times players are critical and call this number T.

The Banzhaf power index of player P_i is the fraction $\frac{B_i}{T}$.

The Banzhaf power distribution is the complete list of all players' Banzhaf power indexes (which always sums to 1).

P_1: P_2: P_3: … P_n:

Example – [7: 3, 3, 1, 1, 1]

Winning Coalitions	Critical Players

B_1 = B_2 = B_3 = B_4 = B_5 = T =

Banzhaf Power distribution:

P_1: P_2: P_3: P_4: P_5:

What is special about players P_1 and P_2?

Example – [6: 3, 3, 1, 1, 1]

Winning Coalitions	Critical Players		Winning Coalitions	Critical Players

B_1 = B_2 = B_3 = B_4 = B_5 = T =

Banzhaf Power distribution:

P_1: P_2: P_3: P_4: P_5:

Do players P_1 and P_2 still have veto power?

Observations

Consider again the previous examples:

[7: 3, 3, 1, 1, 1]

[6: 3, 3, 1, 1, 1]

In which example do the players with 3 votes (the "heavyweights" have more power?

In the first, [7: 3, 3, 1, 1, 1], the Banzhaf power index of player P_1 is

$$7/17 = 0.4117\ldots \approx 41\%$$

In the second, [7: 3, 3, 1, 1, 1], the Banzhaf power index of player P_1 is

$$4/11 = 0.3636\ldots \approx 36\%$$

Compare the numbers. Lowering the quota, while keeping the weights the same, has weakened the power of the heavyweights.

Example – the Committees

Kissinger Committee. The Kissinger Committee consists of five members B, C, D, E, and K who is chairman. Majority rules, except that the chairman votes *only* to break a tie.

Senate Committee. The Senate Committee consists of four members A, B, C, and D with A as chairman. The chairman always votes. Majority rules, except that in case of a 2-2 tie, the coalition containing the chairman A wins.

In each case, how much power does the chairman have compared to the other committee members?

Even without weights, we can figure out what are the winning coalitions and critical players from the voting rules. Thus, we can compute the Banzhaf power indexes.

Example. The Kissinger Committee with members B, C, D, E, chair K, and voting rules: majority rules, except chair votes only to break tie.

Winning Coalitions without chairman	Critical Players		Winning Coalitions with chairman	Critical Players

Times critical: K: B: C: D: E: T:

Banzhaf power distribution:

K: B: C: D: E:

Example. The Senate Committee with chair A, members B, C, and D, and voting rules: majority rules, except chair A always votes, and in case of 2-2 tie, coalition containing A wins.

Winning Coalitions	**Critical Players**		**Winning Coalitions**	**Critical Players**

Times critical: A: B: C: D: T:

Banzhaf power distribution:

A: B: C: D:

Modeling Voting Rules with a Weighted Voting System

Sometimes a voting situation with complicated voting rules can be *modeled* with a weighted voting system.

What we mean by *modeling* voting rules is to find a weighted voting system that has the same wining and losing coalitions and critical players as the voting rules allow.

Example – the Senate Committee revisited

Let us try to find a weighted voting system that models the Senate Committee: four players, one of whom is chair; majority rules, except that in case of a 2-2 tie, the coalition containing the chair wins.

- A model with four players with equal weights won't work, since then only 3-player coalitions would be winning.
- The chair, whom we will name P_1, needs to have more weight.

- The other three players should have equal, lesser weights (why?).
- The simplest weights to use would be 2, 1, 1, 1, where the chair has weight 2 and the others weight 1.
- What should we set the quota to be?

$$[q: 2, 1, 1, 1]$$

- From the quota weight inequality

$$5/2 < q \leq 5$$

- The only possible (whole number) values for q are 3, 4, and 5.
- Which choice gives any 3-player coalition enough votes to meet the quota and any 2-player coalition including player P_1?
  - We have only one choice: q = 3, so the model is

$$[3: 2, 1, 1, 1]$$

Nassau County Board of Supervisors

In 1964, Nassau County, New York, had a Board of Supervisors with 115 votes total divided among 6 districts as shown in the table. A simple majority was required to pass a motion.

In effect, the board operated as a weighted voting system

[58: 31, 31, 28, 21, 2, 2]

In lawsuits beginning in 1965, lawyer John Banzhaf argued that three of the districts had all the power, divided equally, and that the three smaller districts had no power (they were never *critical* to a coalition being winning).

District	**Votes (in 1964)**
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

He introduced what is now known as the Banzhaf Power Index to make this argument.

Observations. [58: 31, 31, 28, 21, 2, 2]

- Any two of the three “heaviest” districts form a winning coalition since their weights meet or exceed the quota.
- The three lightest districts plus any single heavy district does not meet the quota.
- Thus, any winning coalition must contain at least two of the heavy districts.
- But then any further district joining that coalition cannot be critical, since those two heavies already meet the quota.
- So the three lightest districts are never critical to a winning coalition.

Conclusion: the three lightest districts are powerless dummies!

Banzhaf won the court cases, and Nassau County now has a board where power, and not merely votes, is proportional to population.

United Nations Security Council

The current United Nations Security Council is organized as follows;

- 5 *Permanent* members: United States, Russia, United Kingdom (Britain), France, and China (Peoples' Republic).
- 10 rotating, *nonpermanent* members, chosen in rotation from the other nations in the UN.
- To pass a motion, all 5 permanent members plus at least 4 nonpermanent members must vote yes.
- *Veto power*: if just one permanent member votes no, the motion fails.

Our goal is to model the UN Security Council with a weighted voting system.

There will be 15 players, the first 5 being the permanent members. We have to find weights and quota.

Analysis. Let us start with a 15-player model with five players with equal weights, but as yet unknown, so symbolized with x, and 10 players with weight 1.

$$[q: x, x, x, x, x, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

Since we don't know x, we also don't know the quota q.

The voting rules place restrictions on q in terms of the weights of the players.

- Since 5 permanent members plus 4 nonpermanent members are enough to pass a motion, we must have

$$5x + 4 \geq q$$

- Since just one permanent member voting no defeats a motion, even if everyone else votes for it, we must have

$$4x + 10 < q$$

- Putting these inequalities together, we obtain

$$4x + 10 < q \leq 5x + 4$$

Solution Method. We will try to find the smallest whole number solutions for x and q that make the above inequality true.

Value of x	Inequality $4x + 10 < q \leq 5x + 4$	Value of q
x = 5		
x = 6		
x = 7		
x = 8		

Model. The UN Security Council can be modeled by a weighted voting system [___: __, __, __, __, __, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

Current UN Security Council as a weighted voting system

[39: 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

Observations.

- The total number of votes is set at 5*7 + 10 = 45.
- Since 45/2 < 39 $\leq$ 45, this is a reasonable weighted voting system.
- If one permanent member votes no on a motion, then even with all the other members voting for the motion, the total weight of the coalition is only 4*7 + 10 = 38, and the motion fails.
- Each permanent member can thus block any motion. So each permanent member has veto power.
- If all 5 permanent members vote for a motion, and 4 nonpermanent members join them, then the weigh of that coalition is 5*7 + 4 = 39, and the motion passes.
- It is a bit tedious to calculate, but the power distribution is

  Permanent members: 16.7%   Nonpermanent: 1.65%

Do you think the framers intended this 10-fold power difference?

EU Council of Ministers

The European Union Council of Ministers operates as a weighted voting system

[62: 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]

With 87 total votes, note that 62/87 = 0.7126… is a *supermajority*, in this case about 71% of the votes.

What quota would be a 2/3 supermajority? A 3/4 supermajority?

Is every country critical in some coalition (i.e., no dummies)?

Does any country have veto power?

How well does the distribution of votes match the distribution of power?

Banzhaf Power Distribution in the EU

Country	Votes		Banzhaf Power	
	Number	Percent	Index	Percent
France Germany Italy U.K.	10	11.49%	1849/16,565	11.16%
Spain	8	9.20%	1531/16,565	9.24%
Belgium Greece Netherlands Portugal	5	5.75%	973/16,565	5.87%
Austria Sweden	4	4.60%	793/16,565	4.79%
Denmark Finland Ireland	3	3.45%	595/16,565	3.59%
Luxembourg	2	2.30	375/16,565	2.26%

Questions

1. In the EU Council of Ministers, what would be the general effect of lowering the quota? More or fewer winning coalitions? Critical players?

2. What would be the general effect of raising the quota? More or fewer winning coalitions? Critical Players?

3. Could the quota be made high enough so that some members had veto power?

4. What would be the effect of majority rule (i.e, setting the quota at 44 – out of 87)?

5. Why do you think a supermajority was chosen as the quota?