Laser Physics I

PH481/581-VT (Mirov)

Lecture 6. Review of chapters 1-2

Fall 2013
C. Davis, “Lasers and Electro-optics”
Problem 1.6

Estimate the total force produced by the photon pressure of the sun on an aluminum sheet of area $10^6 \text{ m}^2$ situated on the surface of the earth. Use sensible values for the parameters of the problem in order to produce a numerical result.

1. Calculate energy density emitted by the sun from UV $\lambda = 200 \mu \text{m}$ to IR $\lambda = 20,000 \mu \text{m}$.

For $\lambda_1 = 200 \mu \text{m}$:

$$\gamma_1 = \frac{c}{\lambda_1} = \frac{3 \times 10^8}{200 \times 10^{-6}} = 1.5 \times 10^{13} \text{ Hz}$$

For $\lambda_2 = 20,000 \mu \text{m}$:

$$\gamma_2 = \frac{c}{\lambda_2} = \frac{3 \times 10^8}{20000 \times 10^{-6}} = 1.5 \times 10^{11} \text{ Hz}$$

$$\gamma_{UV} \approx 6000 \text{ K}$$

$$\rho(y) = \frac{8 \pi \hbar y^3}{c^3} \left( \frac{1}{e^{\frac{\hbar y}{kT}} - 1} \right) \frac{y_2}{y_1} \int_{y_1}^{y_2} \frac{y^3}{e^{\frac{\hbar y}{kT}} - 1} \, dy$$

$$\rho = \int_{y_1}^{y_2} \frac{8 \pi \hbar y^3}{c^3} \left( \frac{1}{e^{\frac{\hbar y}{kT}} - 1} \right) \, dy = \frac{8 \pi \hbar}{c^3} \int_{y_1}^{y_2} \frac{y^3}{e^{\frac{\hbar y}{kT}} - 1} \, dy$$

$$e^{\frac{\hbar y}{kT}} - 1 = e^{-1}$$

For small values of $x$:

$$\frac{x}{e^x - 1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow e^x - 1 = x$$

$$\frac{\hbar y}{kT} = 1$$

$$\frac{\hbar y}{kT} = \frac{1.4 \times 10^{-3}}{6000} = \frac{1.4 \times 10^{-3}}{3 \times 10^8} = 1.8 \times 10^{-15} \text{ m}^2$$

$$\rho = \frac{8 \pi \hbar}{c^3} \int_{y_1}^{y_2} \frac{y^3}{e^{y_2} - 1} \, dy$$

$$= \frac{8 \pi \hbar}{c^3} \frac{y_2^4}{4} \left[ \frac{y^3}{3} \right]_{y_1}^{y_2}$$

$$= \frac{8 \pi \hbar}{c^3} \left( \frac{y_2^4}{4} \left[ \frac{y^3}{3} \right]_{y_1}^{y_2} \right)$$

$$= \frac{94}{12} \times \frac{(1.5 \times 10^{13})^3}{3} = 86.6 \times 10^8 \text{ m}^2$$
2) Estimation of radiation pressure.

3) perfectly absorbing surface. If \( \Phi \) - photon flux \( \rightarrow \) total momentum transferred to the surface \( S \) in the time interval \( \Delta t \) is

\[
Q = \Phi \Delta t S \Delta t
\]

where \( \frac{\hbar}{c} = \frac{h}{c} \)

Force \( F \) can be calculated from impulse-momentum theorem:

\[
F \Delta t = Q_2 - Q_1
\]

\( \Rightarrow \) radiation pressure \( P = \frac{F}{S} = \frac{Q}{\Delta t S} = \frac{\Phi \Delta t S}{\Delta t S} = \frac{\hbar}{c} \)

Intensity of photon flux \( I = \Phi \frac{h}{c} \Rightarrow \)

\[ P = \frac{I}{c} \]

8) for perfectly reflecting surface (our case)

\[ Q_2 - (-Q_1) = 2Q \]

\[ \Rightarrow \left( P = \frac{2I}{c} \right) = 2P(r) \]
3) Calculate fraction of photons that will reach our aluminum sheet.

\[ \frac{8}{7 \times 10^3} \]

\[ \frac{1.5 \times 10^3}{1000} \]

If total energy that deviates emitter is so that energy reaches our aluminum sheet is

\[ S_S = \frac{500}{4\pi} = \frac{500}{1.5 \times 10^3} = 5.3 \times 10^{-7} \]

4) We calculated pressure from monochromatic wave \( P = 2P_S(v) \)

for the sun emitting from \( \gamma_1 \) to \( \gamma_2 \)

\[ P_{sun} = 2 \int S_S(v) dv = 2.53 \times 10^{-7} \int P(v) dv = \]

\[ = 2.53 \times 10^{-7} \times 86.6 \frac{m}{s} = 9 \times 10^{-5} \frac{m}{s^2} = 10^{-4} \frac{N}{m^2} = 10^{-4} \rho \]

5) \[ F = P_S \times S = \left( 9 \times 10^{-5} \frac{N}{m^2} \right) (10^6 \frac{m^2}{s^2}) = 90 \text{N} \]
Problem 1.7

Calculate the total stored energy in a 1 m³ box that lies between the wavelengths 10.5 μm and 10.7 μm at a temperature of 3000 K.

1) The energy density of the radiation within the cavity per volume, per frequency interval is

\[ P(\nu) = \frac{8\pi\hbar^2 k^4}{c^3} \left( \frac{1}{e^{\frac{\hbar\nu}{kT}} - 1} \right) \]

2) Total energy stored is

\[ E = V \times \int P(\nu) d\nu \quad \text{since} \quad V = 1 \text{ m}^3 \]

\[ E = \int_{\nu_1}^{\nu_2} P(\nu) d\nu \]

where

\[ \nu_1 = \frac{c}{\lambda_1} = \frac{3 \times 10^8 \text{ m/s}}{10.7 \times 10^{-6} \text{ m}} = 2.8 \times 10^{13} \]

\[ \nu_2 = \frac{c}{\lambda_2} = \frac{3 \times 10^8 \text{ m/s}}{10.5 \times 10^{-6} \text{ m}} = 2.86 \times 10^{13} \]

3) Let us estimate \( \frac{\hbar\nu}{kT} = \frac{6.62 \times 10^{-34} \text{ Js} \times 2.8 \times 10^{13}}{1.38 \times 10^{-23} \text{ J/K} \cdot 3000 \text{ K}} = 0.45 \)

\[ e^{\frac{\hbar\nu}{kT}} - 1 \approx 0.56 \]

approximation \( e^x \approx 1 + ax \) where \( a = \frac{\hbar\nu}{kT} \)

4) \[ E = \int_{\nu_1}^{\nu_2} P(\nu) d\nu = \frac{8\pi kT}{c^3} \int_{\nu_1}^{\nu_2} \nu^2 d\nu = \frac{8\pi kT}{c^3} \left[ \frac{\nu^3}{3} \right]_{\nu_1}^{\nu_2} = \frac{2.86 \times 10^{13}}{2.8 \times 10^{13}} \]

\[ = 3.00 \times 10^{-6} \text{ J} - 2.86 \times 10^{-7} \text{ J} = 1.85 \times 10^{-5} \text{ J} = 1.9 \mu\text{J} \]
Problem. Find the relation between spontaneous emission lifetime and cross section for a simple atomic transition.

Cross-section of emission at frequency $\nu$

$$\sigma(\nu) = \frac{c^2 A_{21}}{8\pi \nu_o^2} g(\nu, \nu_o) = \frac{\lambda_o^2}{8\pi n^2} \frac{g(\nu, \nu_o)}{\tau_{sp}} \quad (1)$$

where $\lambda_o = c / \nu_o$ is the wavelength (in vacuum) of an e.m. wave whose frequency corresponds to the center of the line.

Equation (1) can be used either to obtain the value of $\sigma$, when $\tau_{sp}$ is known, or the value of $\tau_{sp}$ when $\sigma$ is known.

If $\sigma(\nu)$ is known, to calculate $\tau_{sp}$ from (1) we multiply both sides by $d\nu$ and integrate. Since $\int g(\nu) d\nu = 1$ we get:

$$\tau_{sp} = \frac{\lambda_o^2}{8\pi n^2} \frac{1}{\int \sigma(\nu) d\nu}$$
Problem. Radiative lifetime and quantum yield of the ruby laser transition.
The R$_1$ laser transition of ruby has to a good approximation a Lorentzian shape of width (FWHM) 330 GHz at room temperature. The measured peak transition cross-section is $\sigma=2.5\times10^{-20}$ cm$^2$. Calculate the radiative lifetime (the refractive index is $n=1.76$). Since the observed room temperature lifetime is 3 ms, what is the fluorescence quantum yield.

From $\sigma(\nu) = \frac{c^2 A_{21}}{8\pi \nu_o^2} g(\nu, \nu_o) = \frac{\lambda_o^2}{8\pi n^2} \frac{g(\nu, \nu_o)}{\tau_{sp}}$ (1) we have

$$\tau_{sp} = \frac{\lambda_o^2}{8\pi n^2} \frac{g(\nu, \nu_o)}{\sigma} \quad (2)$$

For a Lorentzian lineshape at $\nu = \nu_o$, we have

$$g(0) = \frac{2}{\pi \Delta \nu_o}$$

For $\lambda_o = 694nm$, we obtain from eq. (2) the spontaneous emission lifetime (i.e. the radiative lifetime)

$$\tau_{sp} = \frac{\lambda_o^2}{8\pi n^2} \frac{2}{\pi \Delta \nu_o \sigma} = 4.78ms$$

Fluorescence quantum yield $\phi$ is defined as the ratio of the number of emitted photons to the number of atoms initially raised to the level 2

$$\phi = \frac{\tau}{\tau_r} = 0.625$$

where observed lifetime $\tau$ given by

$$\frac{1}{\tau} = \frac{1}{\tau_r} + \frac{1}{\tau_{nr}}$$
1. For a cavity volume $V = 1 \text{ cm}^3$ calculate the number of modes that fall within a bandwidth $\Delta \lambda = 10 \text{ nm}$ centered at $\lambda = 600 \text{ nm}$.

   - Number of modes per unit volume per frequency range is
     $$n = \frac{1}{V} \Delta \nu = \frac{8 \pi \nu^2 c}{c^3} (\text{for an empty cavity})$$

   - Number of modes for a cavity volume $V$ that falls within a frequency bandwidth $\Delta \nu$ is:
     $$N = n \cdot V / \Delta \nu$$

   - Assume $n = \text{const}$ over $\Delta \nu$

   - To solve the problem find relationship between $\Delta \nu$ and $\Delta \lambda$.
     - Since $V = \frac{c}{\lambda}$, $\Delta \nu = -\frac{c}{\lambda^2} \Delta \lambda$

   - Using (1) and (3) in Eq. (2)
     $$N = \frac{8 \pi \nu^2 c}{c^3} \frac{V \cdot \Delta \lambda}{\lambda^2} = \frac{8 \pi \nu \Delta \lambda V}{\lambda^4}$$

     $$= \frac{8 \pi \cdot (10 \times 10^{-9}) \cdot (1 \times 10^{-6} \text{ m}^3)}{(600 \times 10^{-9})^4} = 1.9 \times 10^{12} \text{ modes}$$
A He-Ne laser, operating at 632.8 nm has an output power of $P=1.0$ mW with a 1-mm beam diameter. Power in the cavity is $99P$ since the output mirror has 1% transmission. The beam diameter is also 1 mm inside the laser cavity and the power is uniform over the beam cross-section. The laser linewidth is $1.5 \times 10^8$ Hz.

a) What is the ratio of stimulated and spontaneous emission rates $\frac{B_{21}P(\nu)/A_{21}}{A_{21}}$? (Hint: $I(\nu)=\text{power}/(\text{beam cross-sectional area}) \times (\text{frequency width of the beam}); \rho(\nu)=I(\nu)/c$

b) What is the effective blackbody temperature of the laser beam near the output mirror in the cavity.

\begin{align*}
1) & \quad \nu = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{6.328 \times 10^{-4} \text{ m}} = 4.74 \times 10^{14} \text{ Hz} \\
2) & \quad \frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} = \frac{(8\pi) \left(6.63 \times 10^{-34} \text{ J s}\right) \left(4.74 \times 10^{14} \text{ Hz}\right)^3}{(3 \times 10^8 \text{ m/s})^3} = 6.57 \times 10^{-37} \text{ m}^3 \\
\Rightarrow & \quad \frac{B_{21}}{A_{21}} = \frac{1.52 \times 10^{13} \text{ m}^3}{9.5} \\
3) & \quad S_0(\nu) = \frac{P(\nu)}{A_{21} \Delta \nu \cdot c} = \frac{9.9 \times 10^{-12} \text{ W}}{5 \times 10^{-7} \text{ m}^2} = 2.8 \times 10^{-12} \frac{\text{W}}{\text{m}^2} \\
\Rightarrow & \quad \frac{B_{21}}{A_{21}} \cdot S_0(\nu) = \left(1.52 \times 10^{13} \frac{\text{m}^3}{9.5}\right) \cdot \left(2.8 \times 10^{-12} \frac{\text{W}}{\text{m}^2}\right) = 42.6 \\
6) & \quad \frac{B_{21}}{A_{21}} \cdot S_0(\nu) = \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{e^{\frac{h \nu}{k T}} - 1} = \frac{1}{e^{\frac{h \nu}{k T}} - 1} \\
\Rightarrow & \quad \frac{1}{e^{\frac{h \nu}{k T}} - 1} = 42.6 \\
\Rightarrow & \quad \frac{h \nu}{k T} - 1 = \frac{1}{42.6} = 0.0235 \\
\Rightarrow & \quad \frac{h \nu}{k T} = \ln \left(1.0235\right) = 2.32 \times 10^{-2} \\
\Rightarrow & \quad \frac{h \nu}{k T} = \ln \left(1.0235\right) = 2.32 \times 10^{-2} \\
\Rightarrow & \quad T = \frac{h \nu}{(2.32 \times 10^{-2}) \cdot k} = \frac{\left(6.63 \times 10^{-34} \text{ J s}\right) \left(4.74 \times 10^{14} \text{ Hz}\right)}{(2.32 \times 10^{-2}) \left(1.38 \times 10^{-23} \text{ J/K}\right)} = 9.8 \times 10^5 \text{ K}
\end{align*}
3. Consider the ideal laser medium shown below. The pump excites the atoms to state 2 at a rate $R_2$, which then decays to state 1 at a rate $\tau_{21}^{-1}$ and back to state 0 at a rate $\tau_{20}^{-1}$. State 1 decays back to state 0 so fast that the approximation $N_1 \approx 0$ is appropriate. The radiative rate for the 2→1 transition is $6 \times 10^6 \text{sec}^{-1}$, and its width is 10 GHz. (Assume Lorentzian profile and steady state.)

(a) What is the stimulated emission cross section for the 2→1 transition?

[Hint: $\sigma_2(v_o,v) = \gamma(v_o,v)/\Delta N$]

(b) What must be the pump rate $R_2$ in order to obtain a small-signal gain coefficient of 0.01 cm$^{-1}$?

(c) What is the saturation intensity for the 2→1 transition?

$$A_{21} = 6 \times 10^6 \text{sec}^{-1}$$

$$\tau_{21} = 100 \text{ ns}$$

$$\tau_{20} = 200 \text{ ns}$$

Hint: $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$

$$\frac{1}{\tau_2} = \frac{1}{\tau_{21}} + \frac{1}{\tau_{20}} = \frac{1}{100} + \frac{1}{200} = \frac{3}{200}$$

$$\Rightarrow \tau_2 = 6.7 \text{ ns}$$

$$\varrho = \text{NA} = 67 \text{ nm}$$

4) $\gamma = \frac{(5.5 - 3.2) \times 10^{-19}}{6.62 \times 10^{-24}} = 5.56 \times 10^{14} \text{ Hz}$

$$g(Y_0, v) = \frac{2}{\pi \Delta v} \left[ 1 + \left( \frac{2(v - v_0)}{\Delta v} \right)^2 \right]$$

$$g(Y_0, v) = \frac{2}{\pi \Delta v} \left[ 1 + \left( \frac{2(v - v_0)}{\Delta v} \right)^2 \right]$$

$$64 \times 10^{-14} \text{ cm}^2$$

5) $N_2 - N_1 = \frac{g(Y_0, v)}{\Delta p} = 2.27 \times 10^{11} \text{ cm}^{-3}$

$$N_2 - N_1 = \frac{R_2 \tau_2 - R_2 \tau_2^2 (1 - 1 - A_{21} \tau_2)}{1 + \frac{1 + \tau_2}{\tau_2}} = \frac{R_2 \tau_2}{1 + \frac{1 + \tau_2}{\tau_2}} \text{, where } W = \frac{A_{21} c^2 I_s}{8 \pi^2 \sqrt{3}} \gamma(y, v)$$

$$I_s = \frac{8 \pi \hbar v^3}{c^2 \varphi \gamma(y, v)} = \frac{8 \pi \cdot (6.62 \times 10^{-34}) \cdot (5.56 \times 10^{14})^3}{(3 \times 10^2)^2 (0.4) (6.4 \times 10^{-11})} = 124 \frac{W}{\text{cm}^2} = \left(1 \times 10^2 \right)^2 \frac{W}{\text{cm}^2}$$

$$\varphi = A_{21} \tau_2 \left[ 1 + \left( 1 - A_{21} \tau_2 \right) \frac{2}{\tau_2} \right] = (6 \times 10^{-6}) (6.7 \times 10^{-9}) = 0.4$$

6) Since we are speaking about a small signal gain, $W = 0$

$$= \Rightarrow R_2 = \frac{N_2 - N_1}{2} = \frac{2.27 \times 10^{11}}{6.7 \times 10^{-9}} = 3.4 \times 10^{18} \text{ atoms}$$
4. A homogeneously broadened laser transition at $\lambda=10.6 \, \mu m$ (CO$_2$) has the following characteristics: $A_{21}=0.34 \, s^{-1}$; $J_2=21$; $J_1=20$; $\Delta v_b=1\, GHz$.

a) What is the stimulated emission cross section (gain coefficient/population inversion) at line center?

b) What must be the population inversion $N_2-(g_2/g_1)N_1$ to obtain a gain coefficient of 2 cm$^{-1}$?

c) If the lifetime of the upper state is 10$\mu$s and that of the lower state 0.1$\mu$s, what is the saturation intensity?

(Hint: For a simple atom degeneracy is related to the total angular momentum quantum number "J" by $g=2J+1$)

\[
\sigma(\nu) = \frac{\mathcal{J}(\nu)}{\mathcal{N}_2 - \frac{\mathcal{g}_2}{\mathcal{g}_1} \mathcal{N}_1} = \frac{c^2 A_{21} \nu^2}{8\pi^2 \nu^2} \frac{2/\nu \Delta \nu}{1 + [2(\nu - \nu_0)/\Delta \nu]^2}
\]

\[
\sigma(\nu) = \frac{\mathcal{J}(\nu)}{\mathcal{N}_2 - \frac{\mathcal{g}_2}{\mathcal{g}_1} \mathcal{N}_1} = \frac{c^2 A_{21} \nu^2}{8\pi^2 \nu^2} \frac{2}{\nu \Delta \nu} = \frac{(3 \times 10^5 \, MHz)^2 (0.34 s^{-1})}{8\pi (2.83 \times 10^{-13} \, Hz)^2 \nu \times 10^{12} \, Hz}
\]

\[
\nu_0 = \frac{c}{\lambda_0} = \frac{3 \times 10^8 \, m}{10.6 \times 10^{-6} \, m} = 2.83 \times 10^{13} \, Hz
\]

\[
\sigma(\nu_0) = \frac{2 \, cm^{-1}}{9.7 \times 10^{-18} \, cm^2} = 2 \times 10^{17} \, cm^{-3}
\]

\[
I_s(\nu_0) = \frac{8\pi h \nu^3}{c^2 \phi g(\nu_0, \nu)} =
\]

\[
\phi = A_{21} \nu^2 \left[1 + (1 - A_{21} \nu^2) \frac{\lambda_2}{\nu_2} \right] = (0.34 s^{-1}) (6 \times 10^{-6}) \left[1 + (1 - 0.34 \times 10^{-6}) \frac{1}{10} \right]
\]

\[
= 3.4 \times 10^{-6} \left[1 + (1 - 3.4 \times 10^{-6}) \frac{1}{10} \right] \approx 3.74 \times 10^{-6}
\]

\[
I_s(\nu_0) = \frac{8\pi (6.62 \times 10^{-34} \, Js) (2.83 \times 10^{13} \, Hz)^3}{(3 \times 10^8 \, m/s)^2 \cdot 2.83 \times 10^{-6} \cdot \frac{2}{\nu \times 10^3}} = 75 \times 10^6 \, W/m^2 = \frac{75 \times 10^2 \, W}{cm^2}
\]
4. An experiment involving a homogeneously broadened amplifier is depicted in the diagram below. For an input intensity of 1 W/cm², the gain (output/input) is 10 dB (i.e., \(G_o(dB)=10\log(I_{out}/I_{in})\)). If the input intensity is doubled to 2W/cm², the gain is reduced to 9 dB.

![Amplifier Diagram](image.png)

(a) What is the small-signal gain (i.e. \(I_{in}\to0\)) of this amplifier (in dB)?

(b) What is the saturation intensity?

\[
\begin{align*}
\text{Gain} &= 10 \log \frac{I_{out}}{I_{in}} = 10 \log 1 = 0 \\
\text{Gain} &= 10(0.9) = 9 \text{ dB}
\end{align*}
\]

\[
\begin{align*}
G_{so} &= 9 \text{ dB} = 10 \log \frac{I_{out}}{2} \\
\Rightarrow I_{out} &= 2 \times (10^{0.9}) = 15.89 \text{ W}
\end{align*}
\]

\[
\begin{align*}
\text{Use} \\
I_{out} &= I_{in} e^{G_{so} - \frac{I_{out}}{I_{s}}} \\
10 &= 1 e^{G_{so} - \frac{15.89 - 2}{I_{s}}} \\
15.89 &= 2 e^{G_{so} - \frac{15.89 - 2}{I_{s}}} \\
7.945 &= e^{G_{so} - \frac{13.89}{I_{s}}}
\end{align*}
\]

\[
\begin{align*}
2.3 &= G_{so} - \frac{9}{I_{s}} \\
2073 &= G_{so} - \frac{13.89}{I_{s}} \\
0.227 &= \frac{13.89 - 21.5}{I_{s}} \\
\Rightarrow I_{s} &= 21.5 \text{ W/cm²}
\end{align*}
\]

Substitute \(I_{s}\) in (1) \(G_{so} = 2.3 + \frac{9}{21.5} = 2.72\)

Convert \(G_{so}\) in dB

\[
G_{so}(dB) = 10 \log e^{G_{so}} = 10 \log 2.72 = 11.9 \text{ dB}
\]
5. The following question refer to an atomic system with $J_2=1$ and $J_1=2$.

(a) What is the ratio $B_{12}/B_{21}$?

(b) If the lineshape function could be approximated by the graph shown below, $A_{21}=10^6$ s$^{-1}$, $\lambda=640.1$ nm and $N_2=N_1=10^{12}$ cm$^{-3}$, what is the small-signal gain coefficient for the $2 \rightarrow 1$ transition at $v=v_0$?

\[ \frac{B_{12}}{B_{21}} = \frac{\frac{3}{2} - \frac{1}{2}}{\frac{3}{2} - \frac{1}{2}} = \frac{2.1 + 1}{2.2 + 1} = \frac{3}{5} = 0.6 \]

\[ N_2 - \frac{9}{2} N_1 = 10^{12} - 0.6 \times 10^{12} = (4 \times 10^{11} \text{ cm}^{-3}) \]

\[ \int g(v_0, v) dv = 1 \]

\[ \int g(v_0, v) dv = \frac{1}{2} g(v_0)(v_0 - v_0 + 2) + \frac{1}{2} g(v_0)(v_0 + 1 - v_0) = \]

\[ = \frac{1}{2} g(v_0) \cdot 3(\text{cm}) = 1 \]

\[ \Rightarrow g(v_0) = \frac{2}{3} (\text{GHz})^{-1} = \frac{2}{3} \times 10^{-9} \text{s} \]

\[ g(v_0) = (4 \times 10^{-11} \text{ cm}^{-3}) \cdot \left( \frac{640.1 \times 10^{-7} \text{ cm}^2}{\text{s}} \cdot (10^6 \text{ s}^{-1}) \right) \times \frac{2}{3} \times 10^{-9} = \]

\[ = 4.3 \times 10^{-2} \text{ cm}^{-1} \]