Laser Physics I

PH481/581-VT (Mirov)

Spontaneous and Stimulated Transitions

Lectures 1-2

Fall 2015

C. Davis, “Lasers and Electro-optics”
A laser is an oscillator of optical frequencies that concentrates light energy in spatial, spectral and temporal domains.

**LASER** - Light Amplification by Stimulated Emission of Radiation

**Stimulated emission** - Stimulated emission occurs when a traveling photon interacts with an exited atom. During the interaction, the atom will become de-excited and release a photon of the same frequency and direction of the incident photon.

**Lasers have 3 parts:**

1. **Gain medium** – a place for stimulated emission to occur (crystal, gas, etc.)
2. **Positive feedback** – means for oscillation (mirrors, diffraction grating, etc.)
3. **Source of energy** – an incoming energy source which keeps more atoms in the excited state than in the ground state.
First Solid State Ruby Laser
(Dr. Ted Maiman, Hughes Aircraft 1960)
A laser is an oscillator that operates at *optical* frequencies. These frequencies of operation lie within a spectral region that extends from the very far infrared to the *vacuum ultraviolet* (VUV) or soft-X-ray region. At the lowest frequencies at which they operate, lasers overlap with the frequency coverage of masers (mm scale).

UV – Middle infrared part of electromagnetic spectrum and tuning ranges of the most common solid state lasers
A laser is an oscillator of optical frequencies that concentrates light energy in spatial, spectral and temporal domains.

In 1917 Albert Einstein, in his paper *On the Quantum Theory of Radiation*, laid the foundation for the invention of the laser and its predecessor, the maser, by introducing the concepts of probability coefficients (later to be termed 'Einstein coefficients') for the absorption, spontaneous emission, and stimulated emission of electromagnetic radiation.
A little bit of history

- 1917 Albert Einstein. Basic physics of light emission and absorption by atoms and molecules.
- 1928 Rudolph Walther Landenbourg confirmed the existence of stimulated emission and negative absorption.
- 1939, Valentin Fabrikant predicted the actual use of stimulated emission in gas discharges to amplify light. Observation and patenting of negative absorption in 1944.
- 1954 James Gordon, Herbert Zeigel and Charles Townes proposed and developed maser, a microwave amplifier using stimulated emission.
- 1954 Aleksandr Prokhorov and Nikolai Basov independently proposed and developed maser.

- 1964, Nikolai Basov, Charles Townes, and Aleksandr Prokhorov received the Nobel Prize for "fundamental work in the field of quantum electronics, which has led to the construction of oscillators and amplifiers based on the maser-laser principle."
Periodic pulsed tunable laser using an LiF crystal with F/2+/I-centers, excited by second-harmonic radiation from an Nd/3+/garnet laser.

Academician T. Basiev

Academician T. Basiev, Yu. K. Voron'ko, S. B. Mirov, V. V. Osiko, and A. M. Prokhorov
P. N. Lebedev Physics Institute, USSR Academy of Sciences

(Submitted 10 October 1979)

In this paper we...

In this paper we...

Lebedev Physics Institute

Study of picosecond centers in LiF crystals
P. N. Lebedev Physics Institute, USSR Academy of Sciences

Pis'ma Zh. Eksp. Teor. Fiz. 30, No. 11, 696–700 (5 June 1979)

In this paper we...

Academician A. Prokhorov explains how laser works

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Academician V. Osiko inventor of Cubic Zirconia Gems

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Light and Electromagnetic Waves

- Light is one form of electromagnetic radiation.
- Electromagnetic radiation, which transports energy from point to point at the velocity of light, can be described in terms of both wave and particle "pictures" or "models." This is the famous "wave-particle" duality of all fields or particles in our model of the Universe.

- In the electromagnetic-wave picture, waves are characterized by their frequency $\nu$, wavelength $\lambda$, and the velocity of light $c$, which are inter-related by $c = \nu \lambda$.
- A propagating electromagnetic wave is characterized by a number of field vectors, which vary in time and space. These include the electric field $E$ (volts/m), the magnetic field $H$ (amps/m), the displacement vector $D$ (coulombs/m²), and the magnetic flux density $B$ (tesla).
- For a complete description the polarization state of the wave must also be specified.
- Linearly polarized waves have fixed directions for their field vectors, which do not re-orient themselves as the wave propagates.
- Circularly or elliptically polarized waves have field vectors that trace out circular, or elliptical, helical paths as the wave travels along.
In the particle picture, electromagnetic energy is carried from point to point as quantized packets of energy called *photons*.

The energy of a photon of frequency $\nu$ is $h \nu$, where $h$ is Planck's constant, namely $6.626 \times 10^{-34}$ J s.

Photons have zero mass, and travel at the velocity of light, but carry both linear and angular momentum.

The linear momentum of a photon of wavelength $\lambda$ is $p = h / \lambda$, and the angular momentum depends on the equivalent polarization state of the corresponding wave.

Circularly polarized photons have angular momentum $h / (2\pi) = \hbar$.

Our everyday experience of "light" generally encompasses only the small part of the electromagnetic spectrum to which the human eye is sensitive, a wavelength range running roughly from 400 nm to 700 nm. The full electromagnetic spectrum, going from low to high frequencies, is divided into radiowaves (0-1 GHz), microwaves (1-300 GHz), infrared waves (of wavelength $\lambda = 0.7-1000$ μm; 300 GHz to 430 THz), visible light ($\lambda = 400-700$ nm), ultraviolet light ($\lambda = 10-400$ nm), X-rays ($\lambda = 0.1-10$ nm), and γ-rays ($\lambda < 0.1$ nm).
Some basic electromagnetic theory

In this book we will frequently discuss the electromagnetic wave properties of laser radiation, so we begin with a brief summary of some key concepts that are important in later chapters. A linearly polarized electromagnetic wave of angular frequency $\omega$ that is polarized in the $x$ direction and is traveling in the $z$ direction has electric and magnetic fields

$$E_x = E_0 e^{j(\omega t - kz)} \quad (1.1)$$

$$H_y = H_0 e^{j(\omega t - kz)}.$$  \quad (1.2)

A linearly polarized wave polarized orthogonally to this one would have field pairs $(E_y, H_x)$. The parameter $k$ is called the propagation constant of the wave. The corresponding vector $\mathbf{k}$, the wave vector, represents the direction of propagation of the wave and has magnitude $|\mathbf{k}| = k = 2\pi/\lambda$. The phase velocity of the wave is $c = \omega/k$. The important relations between the $\mathbf{E}$ and $\mathbf{H}$ fields and related vectors of the wave are

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (1.3)$$

and

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (1.4)$$

where $\varepsilon_0 = 8.854 \times 10^{-12}$ F m$^{-1}$ (farads/m) is the permittivity of free space, $\mathbf{P}$ is the polarization produced in the medium through which the wave is traveling, $\mu_0 = 4\pi \times 10^{-12}$ H m$^{-1}$ (henrys/m) is the permeability of free space, and $\mathbf{M}$ is the magnetization produced in the medium through which the wave is traveling. The polarization is produced by the electric field of the wave according to

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}, \quad (1.5)$$
where $\chi$ is the (electric) susceptibility of the medium. The magnetization produced by the magnetic field of the wave is

$$M = \chi_m H,$$  \hspace{1cm} (1.6)

where $\chi_m$ is the magnetic susceptibility of the medium. Using Eqs. (1.5) and (1.6), Eqs. (1.3) and (1.4) can also be written, respectively, as

$$D = \varepsilon_0 \varepsilon_r E$$  \hspace{1cm} (1.7)

and

$$B = \mu_0 \mu_r H,$$  \hspace{1cm} (1.8)

where $\varepsilon_r$ is the relative permittivity and $\mu_r$ is the relative permeability of the medium. Clearly $\varepsilon_r = 1 + \chi$, and $\mu_r = 1 + \chi_m$. One of the triumphs of James Clerk Maxwell in his 1873 formulation of electromagnetic theory [10] in terms of the famous Maxwell equations was to show that the velocity of light was related to the fundamental permittivity and permeability quantities of a medium by

$$c = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}},$$  \hspace{1cm} (1.9)

which was first verified experimentally by Weber and Kohlrausch in 1857 [11]. In 1907 Rosa and Dorsey [12] obtained a value for the velocity of light of 299,788 km s$^{-1}$ in this way. This was the most accurately known value at that time. The propagation constant is $k = \omega \sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}$.

An electromagnetic wave transports energy at the velocity of light. The local energy flux is described in terms of the Poynting vector $S = E \times H$. The time-averaged value of the Poynting vector at a point in space is frequently called the intensity and is measured in W m$^{-2}$ (watts/m$^2$). The direction of the Poynting vector at a point in space is the direction of the light ray. If all the field components of an electromagnetic wave are the same in planes perpendicular to the $k$ vector then this is a plane wave. An ideal spherical wave has equal field components on spherical surfaces, whose centers correspond to the point source of the waves. The Poynting vector of a spherical wave is directed radially outwards at any point on the surface of these spheres. A beam wave, of which a laser beam is an example, has its energy flux localized in a cone around some axial direction, and has field components that vary with the radial distance from this axis.
The polarization state of an electromagnetic wave

Description of the polarization state of an electromagnetic wave, whether this be a plane wave, spherical wave, or beam wave, involves specification of the magnitude, and phase, of orthogonal field components tranverse to the propagation direction $\mathbf{k}$. For propagation in the $z$ direction linearly polarized light propagates with a fixed $\mathbf{D}$ direction, and circularly polarized light has orthogonal $D_x$ and $D_y$ components of equal amplitude, but $90^\circ$ out of phase. The resultant $\mathbf{D}$ traces out a helical path of circular cross-section as the wave propagates. If the $\mathbf{D}$ vector rotates in a clockwise direction when viewed in the propagation direction, the light is said to be left-hand circularly polarized.\(^7\) Elliptically polarized light has orthogonal $D_x$ and $D_y$ components of unequal amplitude and arbitrary phase difference. The polarization state can be specified by the use of the $\textit{Stokes}^8$ parameters, $I$, $Q$, $U$, and $V$, which four parameters are often also written as $S_0$, $S_1$, $S_2$, and $S_3$. The parameter $I$ is related to the total intensity of the wave and can be written, in the case of a right-handed coordinate system with the wave propagating in the $+z$ direction, as

$$I = \langle E_x^2 \rangle + \langle E_y^2 \rangle,$$

where the $\langle \rangle$ indicate time-averaging. The $Q$ parameter is

$$Q = \langle E_x^2 \rangle - \langle E_y^2 \rangle,$$

so $Q$ describes the extent of linear polarization, since for pure $x$ or $y$ polarization $Q = I$. If we write the time-harmonic behavior of the electric-field components as

$$E_x = E_{0x}(t)\cos(\omega t + \phi_x)$$

and

$$E_y = E_{0y}(t)\cos(\omega t + \phi_y),$$

then

$$U = 2\langle E_{0x} E_{0y} \cos(\phi_x - \phi_y) \rangle$$

and

$$V = 2\langle E_{0x} E_{0y} \sin(\phi_x - \phi_y) \rangle.$$

$U$ also describes the linearly polarized character of the wave, while $V$ describes its circular polarization character, since $V$ is maximum for a phase difference of $90^\circ$ between $E_x$ and $E_y$. The polarization state of an electromagnetic wave is often represented on a diagram called the $\textit{Poincaré sphere}$.\(^9\) Figure 1.1 shows a representation of the Poincaré sphere. The Stokes parameters $U$, $V$, and $Q$ are the Cartesian coordinates of a point on the sphere. At any point $P$ on the surface of the sphere the intensity is $I = \sqrt{U^2 + V^2 + Q^2}$. 

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Fig. 1.1 The Poincaré sphere.
1.5 Electromagnetic waves and their propagation through matter

The values of the relative permittivity and relative permeability characterize a medium in terms of its difference from a vacuum. Most materials that are important in a discussion of lasers and optical devices are not strongly magnetic, and it is generally legitimate to assume that for such materials $\mu_r = 1$. It is important to note that the dielectric properties of a medium depend on frequency, even though this has not been explicitly stated in our preceding brief discussion. In addition, to include the properties of real, rather than ideal, matter in our discussion we allow the possibility that the dielectric can be represented by a complex number, so the frequency-dependent dielectric constant can be written as

$$\varepsilon_r(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega). \quad (1.10)$$

Along with this goes a corresponding definition for the complex susceptibility, which is

$$\chi(\omega) = \chi'(\omega) - j\chi''(\omega). \quad (1.11)$$

Clearly, $\chi'(\omega) = \varepsilon'(\omega) - 1$ and $\chi''(\omega) = \varepsilon''(\omega)$. The minus sign in Eqs. (1.10) and (1.11) is a sign convention. We shall see that, with this sign convention, when $\chi''(\omega)$ is positive the medium absorbs energy from the wave.

Although a laser is a device that requires quantum mechanics for a complete description of its behavior, much can be learned from a classical analysis. In this context we describe a medium that has gain or absorption as one whose dielectric properties have been modified in a special way. When an electromagnetic wave propagates in the $z$ direction through a
medium with complex dielectric constant $\epsilon_r = \epsilon' - j\epsilon''$ the variation of the fields of the wave is, taking the electric field as an example,

$$E = E_0 e^{j(\omega t - k'z)},$$  \hspace{1cm} (1.12)

where the new propagation constant is

$$k' = \omega \sqrt{\mu_0 \mu_r \epsilon_r \epsilon_0}. \hspace{1cm} (1.13)$$

Both $\epsilon'$ and $\epsilon''$ vary with frequency. If we make the usual assumption that optical media are generally not strongly magnetic then we can take $\mu_r = 1$ and write

$$k' = \frac{\omega}{c_0} \sqrt{\epsilon' - j\epsilon''}. \hspace{1cm} (1.14)$$

For a medium that does not have large absorption or gain, we can make the assumption that $\epsilon'' \ll \epsilon'$, and Eq. (1.14) becomes

$$k' = \frac{\omega}{c} \left( 1 - j\frac{\epsilon''}{2\epsilon'} \right), \hspace{1cm} (1.15)$$

where $c = c_0/\sqrt{\epsilon'}$ is the modified velocity of light in the medium. The variation of the amplitude of the electric field of the wave as it propagates is now

$$E = E_0 e^{-\frac{k'z}{2}} e^{j(\omega t - k''z)}, \hspace{1cm} (1.16)$$

where $k = \omega/c$ and $\alpha = k\epsilon''/\epsilon'$. If we write $n = \sqrt{\epsilon'}$ and note from our previous discussion that $\epsilon'' = \chi''$, then

$$\alpha = \frac{k\chi''}{n^2}. \hspace{1cm} (1.17)$$

If the complex susceptibility were to become negative then we could write

$$\gamma = -\frac{k\chi''}{n^2}, \hspace{1cm} (1.18)$$

and the wave would increase in field amplitude as it propagated according to

$$E = E_0 e^{\frac{\chi''z}{2}} e^{j(\omega t - k''z)}, \hspace{1cm} (1.19)$$

meaning that the medium has become an amplifier.
In conventional electronics (f < 10^6 Hz) an oscillator is constructed by applying an appropriate amount of positive feedback to an amplifier.

\[ V_i + B V_o \]

\[ A = \frac{V_o}{V_i} \quad \text{overall gain of the system} \]

\[ V_o = A_0 (V_i + B V_o) \Rightarrow V_o = \frac{A_0 V_i}{1 - BA_0} \]

\[ A = \frac{V_o}{V_i} = \frac{A_0 V_i}{(1 - BA_0) V_i} = \frac{A_0}{1 - BA_0} = A \]

If \( BA_0 = 1 \) \( \Rightarrow A \rightarrow \infty \)

\( BA_0 \) - function of frequency

Output does not grow infinitely large because as signal grows \( A_0 \) falls - saturation
1.8 The energy levels of atoms, molecules, and condensed matter

All particles in nature have distinct states\footnote{10} that they can occupy. These states in general have different energies, although it is possible for particles in different states to have the same energy. The term “energy level” is used to describe a particle with a specific, distinct energy, without implying any particular information about its (quantum) state. The lowest energy state, in which a particle is stable, is called the \textit{ground state}. All higher energy states are called \textit{excited} states. Excited states are intrinsically unstable, and a particle occupying one will eventually lose energy and fall to lower energy states. When a particle falls from a higher energy state to a lower, energy is conserved. The energy $\Delta E$ lost by the particle can be emitted as a photon with energy $h\nu = \Delta E$: this is radiative energy loss. The particle can also lose energy \textit{non-radiatively}, in which case the energy is dissipated into heating. Atomic systems have only electronic states, which in the simple Bohr model of the atom correspond to different configurations of electron orbits. The types of energy state that exist in a molecular system are more varied, and include electronic, vibrational, and rotational states. In a molecule, changes in the internuclear separation of the constituent atoms give rise to \textit{vibrational} energy states, which have quantized energies. The various characteristic vibrational motions of a molecule are called its \textit{normal modes}, which for a molecule with $N$ atoms number $3N - 6$, unless the molecule is linear, in which case they number $3N - 5$. The quantized energies of a normal mode can be written as (see Chapter 10 for more details)

$$E_{\text{vib}} \simeq \left(n + \frac{1}{2}\right) h\nu_{\text{vib}},$$

(1.23)

and form a ladder of (almost) equally spaced energy levels. Molecules also have quantized rotational energy levels, whose energies can be written as

$$E_{\text{rot}} \simeq BJ(J + 1),$$

(1.24)

where $B$ is a rotational energy constant, and $J$ is called the rotational quantum number. The overall energy state of a molecule thus has electronic, vibrational, and rotational components. A molecule in a particular combination of electronic and vibrational states is described as being in a \textit{vibronic} state. A state with a specific combination of vibrational and rotational energies would be described as being in a \textit{vibrot} state. As a rough rule of thumb, transitions between different vibronic states where the electronic state changes lie in the visible spectrum with energy spacings\footnote{11} $\sim 20,000 \text{ cm}^{-1}$, which corresponds to an energy spacing of $3 \times 10^{10} h/\lambda$ J. Transitions between vibrot states where the electronic energy does not change, but the vibrational state changes, are typically of energy $\sim 1,000 \text{ cm}^{-1}$. Transitions between different rotational states where the electronic and vibrational states do
not change are typically of energy $\sim 100$ cm$^{-1}$. In practical terms vibrational transitions are typically in the 3–20-$\mu$m range, and rotational transitions are typically in the 50–1,000-$\mu$m range. In the gas phase the energy levels of atoms or molecules are quite sharp and distinct, as shown schematically in Fig. 1.4, although we shall see later that even these precise energies are "broadened." This broadening occurs for several reasons, but perhaps most importantly because of the interactions between neighboring particles.
Bands in crystalline solids

Sketch of the change in the energy values originally assigned to the ground and first excited states of an atom as more and more atoms are combined to form a solid. Note that the band gap energy can be identified with the original atomic level spacing, but is generally different in size.

In a crystalline solid the allowed electron energy levels occur in bands of closely spaced levels. Between these allowed energy bands are forbidden gaps.
Bands in crystalline solids

In condensed matter, whether this be in the solid or in the liquid state, there are very many particles close to any individual particle of interest, and inter-particle interactions are strong. Consequently, the allowed energies of particles in the medium occupy broad, continuous ranges of energy called energy “bands.” The lowest-lying energy band, which is analogous to the ground state of an isolated particle, is called the valence band. The next highest band of allowed energies is called the conduction band. An energy band can be thought of as the result of very many sharp isolated energy states having their energies “smeared” out so that they overlap. We will reserve further discussion of the energy bands in condensed matter until a little later, and for the moment will consider the energy levels of particles as relatively sharp, and not strongly influenced by inter-particle interactions.

In a good conductor of electricity (a), the highest occupied band is only partially filled with electrons, whereas in a good insulator (b) it is filled. In (b) the energy gap $E_g$ between the valence band and the conduction band is large. In the case (c) of a semiconductor, however, this gap is small, and electrons in the valence band can easily be promoted to the conduction band.
Basic laser structure

A laser (maser) is an optical (microwave) frequency oscillator constructed from an optical (microwave) frequency amplifier with positive feedback.

1) : Optical amplifier
2) : Laser cavity (resonator)

1) Light wave which become amplified on traversing the amplifier are returned through the amplifier by the reflectors and grow in intensity. Since A0 saturates, I growth doesn’t continue indefinitely.
2) The arrangement of mirrors (or other components) provide the positive feedback.

How to construct an amplifier at optical frequencies.

Lasers 10.98 μm (2.73 \times 10^{15} \text{Hz}) - para-hydrogen gas
2.659 μm (1.13 \times 10^{15} \text{Hz}) - methyl bromide

\[ \lambda \rightarrow A = \text{Laser} \]
\[ \lambda \rightarrow A = \text{maser} \]
Amplifier of optical frequencies

To construct an amplifier at optical frequencies, we use the energy delivered as the particles that constitute the amplifying medium make jumps between their different energy levels.

- Gaseous
- Liquid
- Crystalline
- Glassy
- Insulating materials
- Semiconductor

Types of laser (amplifying medium)

The electrons that are bound within the particles of the amplifying medium (atoms, molecules, or ions) can occupy only certain discrete energy levels.

Electrons can make jumps between these levels in three ways:

- Energy: $E_i$
- Excited state: $E_j$
- Ground state: $E_0$
Spontaneous Emission

When an electron falls from a higher energy level \( E_i \) to a lower one \( E_j \), the emitted photon has:

1. frequency \( \nu_{ij} = \frac{E_i - E_j}{h} \)
2. direction - random
3. polarization - arbitrary
4. momentum - \( \frac{\hbar}{\lambda} \) (where \( \lambda = \frac{E}{c} = \frac{hv}{c} = \frac{h}{\lambda} \))

The emitting particle recoils in the opposite direction.

The probability of a spontaneous jump is given quantitatively by the Einstein coefficient:

\[ A_{ij} = \text{probability per second of a spont. jump from level } i \text{ to level } j. \]

If population density of level \( i \) is \( N_i \) particles per unit volume \( [cm^{-3}] \) then \( N_i A_{ij} \) particles per second make jumps to level \( j \).

The total rate \( \frac{dN_j}{dt} = \sum_i N_i A_{ij} \) since pop. of i \( \leq \)

\[ A_i = \sum_j A_{ij} \]

- total probability that \( i \) will make a spont. jump to any lower level.
The total rate at which the population of level $i$ changes by spontaneous emission is:

$$\frac{dN_i}{dt} = -N_i A_i$$

Solution:

$$N_i = N_i^0 e^{-A_i t}$$

If at time $t = 0$, $N_i = N_i^0$,

$$N_i = N_i^0 e^{-A_i t}$$

With time, as electrons leave by spontaneous emission:

$$T_i = \frac{1}{A_i} = \frac{1}{\sum A_{ij}}$$

natural lifetime of level $i$.

The time at which the population falls to $\frac{1}{e}$ of its initial value.

Allowed Transitions - jumps which are likely to occur.

Forbidden Transitions - jumps are unlikely to occur.

Metastable Transitions - jumps are so unlikely that levels whose electrons can only fall to lower levels by such jumps are very long lived. Levels can only decay slowly by forbidden transitions.
The lineshape function, $g(\nu)$

Real energy levels are not infinitely sharp. They are broadened.

A particle in a given energy level can actually have any energy within a finite range.

The lineshape function, $g(\nu)$ describes the frequency spectrum of the spontaneous radiation.

$\int g(\nu) \, d\nu = 1$

$g(\nu)\, d\nu$ - probability that a photon will be emitted spontaneously in the frequency range $\nu + d\nu$

$g(\nu)$ is a true probability function for the spectrum of emitted radiation - it is sparsely peaked near some frequency $\nu_0$.

$\nu_1, \nu_2, \nu$ - to highlight it we write it as $g(\nu_0, \nu)$. 

Lineshape function. Example

If the lineshape function could be approximated by the graph shown below, what is the magnitude of $g(v_0)$?

$$
\int_{-\infty}^{+\infty} g(v_0, v) \, dv = 1
$$

\[
\int_{-v_0}^{+v_0} g(v_0, v) \, dv = \frac{1}{2} g(v_0) (v_0 - v_0 + 2) + \frac{1}{2} g(v_0) (v_0 + 1 - v_0) = \frac{1}{2} g(v_0) \cdot 2 + \frac{1}{2} g(v_0) \cdot 1 = \frac{3}{2} g(v_0) (GHz) = 1
\]

$$
\left. \right|_{v_0 - 2 (GHz)}^{v_0} = \frac{2}{3} (GHz)^{-1} = \frac{2}{3 \cdot 10^9} = \frac{2}{3} \times 10^{-9} \text{ S}
$$
Radiant intensity, $I_e(v)$ - describes the amount of radiation emitted spontaneously by a collection of particles.

Unit: \[ \text{Watt/Steradian} \]

Steradian - the unit of solid angle, $\Sigma$.

The surface of a sphere encompasses a solid angle of $4\pi$ Steradians.

The total power emitted in a given frequency interval $dv$ is:

$$W(v)dv = \int_{v} I_e(v)dvds \Sigma$$

$s$ - is a closed surface surrounding the emitted particles.

The total power emitted is:

$$W_0 = \int_{-\infty}^{\infty} W(v)dv$$

$$W(v) = W_0 \cdot g(v)$$

For a collection of $N_i$ identical particles, the total spontaneously emitted power per frequency interval is:

$$W(v) = N_i \cdot A_i \cdot h \cdot v \cdot g(v)$$

Intensity \[ \text{[W/m}^2] \] for plane electromagnetic wave is the average amount of energy per second transported across unit area in the direction of travel of the wave.

$$I(v) = I_0 \cdot g(v)$$
Stimulated Emission

Electron can be stimulated to make a jump from a higher level to a lower level by the action of an external radiation field. 

\[ S(v) = \frac{y}{m^2 \cdot Hz} \]

If \( v = v_2 \), the rate at which stimulated emission occurs is

\[ \frac{1}{N_2 B_{21}(v_2 \nu)} \left( \frac{\nu}{Hz \cdot m^3} \right) \]

where \( B_{21}(v) \) is a function specific to the electron jump.

\[ B_{21} = B_{21} \Theta(v) \]

- Frequency dependence of \( B_{21} \) is the same as the lineshape function.

\[ B_{21} \text{- Einstein coeff. for stimulated emission.} \]

\[ dN_2 \frac{dt}{dt} = -N_2 \int B_{21}(v) S(v) dv = -N_2 B_{21} \int S(v) \Theta(v) dv \]

This is the total rate of change of population density by stimulated emission.

To evaluate the integral, we consider how dens. of energy relates to intensity and how they vary with frequency.
The relation between energy density and intensity

\[ \rho(v) = \frac{I(v)}{c} \]

- \( \rho(v) \) - the energy density of a radiation field [\( \frac{J}{m^3 \cdot Hz} \)]
- \( I(v) \) - the spectral distribution of intensity [\( \frac{W}{m^2 \cdot Hz} \)]
- \( c \) - velocity of light in medium \( c = \frac{c_0}{n} \)

The intensity is the average amount of energy transported per second across unit area in the direction of propagation of the wave. All the energy stored in the volume of length \( c \) passes across the plane A in one second.

\[ \rho(v) \times A \times c = I(v) \times A \]
$\rho(v)$ as a function of frequency

1) $\rho(v)$ is independent of frequency
   
   A white energy density spectrum

2) $\rho(v)$
   
   Narrow energy density profile at frequency $v_1$
   Ideal monochromatic energy density spectrum

3) General radiation field
   
   $\int \rho(v) \, dv$ - the total energy stored per unit volume between frequencies $v_1$ and $v_2$
For a monochromatic radiation field at frequency $v_0$,

$$p(v) = p_{21} \times \delta(v - v_{21})$$

$\delta$-function has the property
1) $\delta(v - v_{21}) = 0$ for $v \neq v_{21}$
2) $\int_{-\infty}^{\infty} \delta(v - v_{21}) \, dv = 1$

For a monochromatic radiation field the total stored energy per unit volume is

$$\int_{-\infty}^{\infty} p(v) \, dv = \int_{-\infty}^{\infty} S_{21} \delta(v - v_{21}) \, dv = S_{21}$$

The total rate of change of population density by stimulated emission is:

$$\frac{dN_2}{dt} = -N_2 \int_{-\infty}^{\infty} B_{21}(v) p(v) \, dv = -N_2 B_{21} \int_{-\infty}^{\infty} g(v, v) \delta(v - v_{21}) \, dv$$

$$= -N_2 B_{21} \int_{-\infty}^{\infty} g(v, v) P_{21} \delta(v - v_{21}) \, dv = -N_2 B_{21} g(v_0, v_{21})$$

The rate of stimulated emissions produced by the input monochromatic radiation is directly proportional to the value of the lineshape function at the input frequency.

$$\Delta N_2 = \max \left( \frac{dN_2}{dt} \right)$$

If the input radiation is at the line center frequency.
If the stimulating radiation field has a spectrum that is broad → assume that $P(v) = \text{const}$ over the range of frequencies where $g(\nu, v)$ is significant.

$$\frac{dN_2}{dt} = -N_2 B_{12} J(\nu_0, \nu) \frac{1}{\hbar \nu_0}$$

where $P(v) = P(\nu_0)$ is the energy density in the frequency range where transitions take place.

**Stimulated Absorption**

As well as making jumps in a downward direction, $e$ may make transitions in an upward direction between energy levels of a particle by absorbing energy from an electromagnetic field.

$$N_1 P(y) B_{12} J(\nu_0, \nu) \frac{1}{\hbar \nu_0}$$

$$\frac{dN_1}{dt} = -N_1 B_{12} \int J(\nu_0, \nu) P(v) dv$$

The rate at which electrons leave the lower level, the total rate of change of population of level 1 by stimulated absorption.
Be is a constant specific for upward jump - Einstein coefficient of stimulated absorption.

There is no analog in the absorption process to spontaneous emission. A particle cannot spontaneously gain energy without an external energy supply.

**Stimulated Emission and absorption as photon-particle collision process**

- Stimulated emission
  - the incident photon produces an identical photon by colliding with an electron in the excited level
  - both photons have the same direction
  - the same polarization
  - stimulated photon has exactly the same frequency as the stimulating photon.

- Stimulated absorption
  - the incident photon disappears
Intensity of a Beam of Electromagnetic Radiation in Terms of Photon Flux

If the intensity of a beam of light is \( I(y) \left( \frac{W}{m^2 \cdot Hz} \right) \) then the number of photons in the beam crossing unit area per unit time is

\[
N_{\text{photons}} = \frac{I(y)}{h \nu} \left[ \frac{\text{photons}}{m^2 \cdot s} \right]
\]

If the beam is monochromatic and has total intensity \( I(y_\nu) \) then

\[
N_{\text{photons}} = \frac{I(y_\nu)}{h \nu} \left[ \frac{\text{photons}}{s \cdot m^2} \right]
\]
Black-Body Radiation

A close approximation to a black-body absorber and emitter is an enclosed cavity containing a small hole. Radiation that enters the hole has a little chance of escaping. If the inside of the cavity is in thermal equilibrium, it must lose as much energy as it absorbs, and the emission from the hole is characteristic of the equilibrium temperature inside the cavity. This type of radiation is called "thermodynamic cavity" radiation.

Attempts to explain the form of the radiation were based on treating (e.m.) radiation as a collection of oscillators with each on its own. The problem was to determine how many oscillations at a given frequency could fit side by side.
- Thermodynamically, the shape of the cavity for which the calculation is performed is arbitrary.

- For convenience, we choose a cubical cavity with sides of length $L$.

- A plane electromagnetic wave will "fit" inside the cavity if it satisfies appropriate periodic boundary conditions:

$$e^{iK_x x} = e^{iK_x (x+L)}$$ (components of the wave)

for the wave that has a spatial variation written in complex notation as $e^{iKz}$.

- Boundary conditions are satisfied if

$$K_x = \frac{2\pi l}{L}; \quad K_y = \frac{2\pi m}{L}; \quad K_z = \frac{2\pi n}{L}$$

where $K_x$, $K_y$, $K_z$ - components of $K$,

$L$, $m$, $n$ - integers.

$$K = \frac{\omega}{c} \hat{K}; \quad |K| = \frac{2\pi}{L}$$

$\omega$ - angular frequency.

$c$ - light velocity.

$\hat{K}$ - a unit vector in the direction of wave vector.
In a three-dimensional space whose axes are $K_x$, $K_y$, and $K_z$ ($K$-Space) the possible $K$ values that are periodic inside the cube form a lattice.

Each cell corresponds to one possible mode characterized by its own values of $K_x$, $K_y$, $K_z$ which are periodic inside the cube.

The spacing of adjacent modes in the $K_x$ direction is $\frac{2\pi}{L}$. The permitted values of $K_x$ are $0, \frac{2\pi}{L}, \frac{4\pi}{L}, \frac{6\pi}{L},$ etc., corresponding to oscillation wavelengths $\lambda = 0, L, \frac{2L}{3}, \frac{L}{3},$ etc.

The total number of modes of oscillation with $|K| \leq K$ is

$$N_K = \frac{\text{total volume of } K\text{-space with } |K| \leq K}{\text{volume of unit cell}}$$

$$= 2 \times \frac{\frac{4}{3} \pi k^3}{(2\pi/L)^3}$$
The factor of 2 enters because we must take account of the two distinct polarizations of the radiation field.

\[ N_\kappa = \frac{k^3}{\pi^2} \]

Since \( k = \frac{2\pi \nu}{c} \), the number of modes with frequency \( \leq \nu \) is

\[ N_\nu = \frac{2\pi \nu}{3c^3} \]

The mode-density (per unit volume, per unit frequency interval) is

\[ P(\nu) = \frac{1}{V} \frac{dN_\nu(\nu)}{d\nu} = \frac{8\pi \nu^2}{c^3} \]

Rayleigh-Jeans predicted that the intensity distribution of the black body radiation would be

\[ I(\nu) \sim \frac{8\pi \nu^2}{c^3} \]

At high frequencies, dramatic conflict.

Ultraviolet catastrophe.
Planck resolved this difficulty with his quantum hypothesis. Each oscillation mode could only take on certain quantized energies:

\[ E_n = (n+\frac{1}{2}) h\nu; \quad n = 0, 1, 2, 3. \]

\( \frac{1}{2} h\nu \) - zero point energy.

The probability of finding energy \( E_n \) in a particular mode of oscillation is given by classical Maxwell-Boltzmann statistics:

\[
\frac{P(n)}{P(0)} = e^{-\frac{E_n}{kT}} = e^{-\frac{n}{kT} \frac{1}{2} h\nu}
\]

\( P(0) \) - probability of finding the lowest energy in the mode.

The average energy of the mode is:

\[
E_\tau = \sum_{n=0}^{\infty} P(n) E_n = \sum_{n=0}^{\infty} \frac{e^{-\frac{n}{kT} \frac{1}{2} h\nu}}{e^{-\frac{1}{kT} \frac{1}{2} h\nu}} (n+\frac{1}{2}) h\nu
\]

If a particular oscillation is excited it must be in one of the quantized states:

\[
\sum_{n=0}^{\infty} P(n) = 1
\]

so

\[
\sum_{n=0}^{\infty} P(n) e^{-\frac{n}{kT} \frac{1}{2} h\nu} = 1
\]
Thus \( P(0) = \frac{1}{\sum_{n=0}^{\infty} e^{-\frac{n}{2}} \frac{n}{2}} \)

and \( E' = \frac{\sum_{n=0}^{\infty} (n+\frac{1}{2}) \frac{n}{2} e^{-\frac{n}{2}}}{\sum_{n=0}^{\infty} e^{-\frac{n}{2}}} = \ldots = \frac{1}{2} \hbar \nu + \frac{\hbar \nu}{(e^{\hbar \nu/kT} - 1)} \) 

So

\[
E' = \frac{1}{2} \hbar \nu + \frac{\hbar \nu}{(e^{\hbar \nu/kT} - 1)}
\]

Average energy per mode

The stored energy in the black-body radiation field treated as a collection of quantized oscillators is

\[
P(\nu) = P(\nu) E' = \frac{8 \pi \hbar^3}{c^3} \left( \frac{1}{2} + \frac{1}{e^{\hbar \nu/kT} - 1} \right)
\]

the \( \frac{1}{2} \) factor comes from zero point energy that cannot be released, so the available stored energy in the field is

\[
P(\nu) = \frac{8 \pi \nu^2}{c^3} \times \hbar \nu \times (\frac{1}{e^{\hbar \nu/kT} - 1})
\]

\( \frac{8 \pi \nu^2}{c^3} \) \# of modes per volume per frequency interval.

\( \hbar \nu \) - photon energy

\( (e^{\hbar \nu/kT} - 1) \) - average \# of photons in each mode. or occupation \# of the modes of the field.
Relation Between the Einstein A and B Coefficients

We can derive a useful relationship between Einstein A and B coefficients by considering a collection of atoms in thermal equilibrium inside a cavity at temperature T. The energy density of the radiation within the cavity is given by

$$\rho(\nu) = \frac{8\pi \hbar \nu^3}{c^2} \left( \frac{1}{e^{\frac{\hbar \nu}{kT}} - 1} \right)$$

Since the radiation in thermal equilibrium in the cavity will be black-body radiation.

Consider the dynamic equilibrium between any two energy levels of atoms. In thermal equilibrium

$$N_1 = N_2 \text{ const}$$

$$\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0.$$

$$0 = \frac{dN_2}{dt} = -N_2 B_2 \rho(\nu) - A_2 N_2 + N_1 B_1 \rho(\nu)$$

$$\Rightarrow N_2 \left[ B_2 - \frac{8\pi \hbar \nu^3}{c^3 (e^{\frac{\hbar \nu}{kT}} - 1)} + A_2 \right] = N_1 \left[ B_1 - \frac{8\pi \hbar \nu^3}{c^3 (e^{\frac{\hbar \nu}{kT}} - 1)} \right]$$

For particles that obey statistics in thermal equilibrium

$$\frac{N_2}{N_1} = e^{-\frac{\hbar \nu}{kT}}$$

Maxwell-Boltzmann
Einstein relations

\[ \frac{8 \pi h c^3}{C^3 (e^{h \nu \kappa} - 1)} = \frac{A_{21}}{B_{21} e^{h \nu \kappa} - B_{21}} \]

It can be satisfied if:

- \( B_{21} = B_{21} \)
- \( \frac{A_{21}}{B_{21}} = \frac{8 \pi h c^3}{C^3} \)

Einstein relations

The stimulated emission rate is

\[ W_{21} = B_{21}, \quad P(\nu) = \frac{C^3 A_{21}}{8 \pi h c^3} P(\nu) \]

Proportional to energy density

The spontaneous emission rate \( A_{21} \) is independent of external radiation.
The effect of level degeneracy

In real systems containing atoms, molecules, or ions, it happens that different configurations of the system can have exactly the same energy.

If a given energy level corresponds to a number of different arrangements specified by an integer \( g \), we call \( g \) the degeneracy of the level. All the separate states of the system with the same energy we call sub-levels.

For each of the sub-levels of levels 1 and 2

\[
\frac{n_2}{n_1} = e^{-\frac{\Delta E}{kT}}
\]

\[
N_1 = g, \quad n_1 = N_2 = g \frac{n_2}{n_1}
\]

Einstein relations

\[
\frac{g_1 B_{12}}{B_{21}} = \frac{g_2 B_{21}}{B_{21}} \quad \frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3}
\]
Ratio of Spontaneous and Stimulated Transitions

Let us examine the relative rates at which spontaneous and stimulated processes occur in a system in equilibrium at temperature \( T \).

\[
R = \frac{A_{21}}{B_{21} \sigma(\nu)}
\]

\( \sigma(\nu) \) - black-body radiation, since such radiation is always present to interact with an excited atom that is contained within an enclosure at temperature \( T \).

\[
A = \frac{8 \pi \nu^3}{C^2} \quad R = \frac{A}{B \sigma(\nu)} = \left( e^{\nu/kT} - 1 \right)
\]

For \( T = 300 \, \text{K} \) and microwave region (for example \( v = 10^{10} \, \text{Hz})

\[
\frac{\nu}{kT} = \frac{6.626 \times 10^{-34}}{4.38 \times 10^{-23} \times 300} = 16 \times 10^{-3}
\]

\[
R = e^{0.0016} - 1 \approx 0.0016
\]

Stimulated emission dominates over spontaneous -

\( S(\nu) \) of created \( S(\nu) \) black-body
In the visible region

\[ \nu = 10^{15}; \quad \frac{h \nu}{kT} \approx 160 \]

\[ A \Rightarrow B P(\nu) \]

In the visible and near IR region spontaneous emission dominates over stimulated unless we can arrange for there to be several photons in a mode.

The average \# of photons in a mode in the case of black-body radiation is

\[ \bar{n}(\nu) = \frac{1}{e^{h \nu/kT} - 1} \]

which is very small in the visible and IR.