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<td>Spontaneous and Stimulated Transitions (Ch. 1) – Lecture Notes</td>
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<td>Sep. 4 (Mo) No classes</td>
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<td>Sep. 6 (We) 394; 5:00-6:15</td>
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<td>Exam 1 problem solving. Passive Optical Resonators (Lecture notes)</td>
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<td>Oct. 2 (Mo) 394; 5:00-6:15</td>
<td>Passive Optical Resonators (Lecture notes). Physical significance of $\chi'$ and $\chi''$ (Ch.2.8-2.9). <strong>Homework 3</strong>: read Ch.2 &amp; notes. Work out problems. Due Oct. 9</td>
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<td>Oct. 4 (We) 394; 5:00-6:15</td>
<td>Optical Resonators Containing Amplifying Media (4.1-2).</td>
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<td>Oct. 9 (Mo) 394; 5:00-6:15</td>
<td>Optical Resonators Containing Amplifying Media (Ch.4.3-4.7) <strong>Homework 4</strong>: Ch. 4 problems 4.7 and 4.9. Due Oct 16.</td>
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<td>Oct. 11 (We) 394; 5:00-6:15</td>
<td>Laser Radiation (Ch. 5.1-5.4)</td>
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<td>Oct. 16 (Mo) 394; 5:00-6:15</td>
<td>Control of Laser Oscillators (6.1-6.3) <strong>Homework 5</strong>: Ch. 5 problems 5.1 and 5.5. Due Oct 23.</td>
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<td>Optically Pumped Solid State Lasers (7.16-7.17) <strong>Homework 7 Due Nov. 13</strong></td>
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<td>Optically Pumped Solid State Lasers (7.14-7.15) Suplemental material for Homework 6–diode pumped LiF:F2 laser</td>
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<td>Spectroscopy of Common Lasers and Gas Lasers (Ch. 8.1-8.10 and class material)</td>
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<td>Gas lasers (Ch. 8.4-8.10); Molecular Gas lasers I (Ch. 9.1-9.5)</td>
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<td>Nov. 20 (Mo) No classes</td>
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<td>Nov. 22 (We) No classes</td>
<td>Thanksgiving - no classes held</td>
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<td>Nov. 27 (Mo) 394; 5:00-6:15</td>
<td>Molecular Gas lasers I (Ch. 9.1-9.5) <strong>Homework 8 Due Dec, 4</strong></td>
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<td>Dec. 4 (Mo) 394; 5:00-6:15</td>
<td><strong>Exam 3 Over Chapters 7-10</strong> Grades; Exam 3 Correct solution</td>
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<td>Dec. 6 (We) 394; 5:00-6:15</td>
<td>Review for Final</td>
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<td>Dec. 11 (Mon) in CH 394</td>
<td><strong>FINAL EXAM Over Chapters 1-10 (4:15-6:45pm) in CH 394 Final Grades</strong></td>
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Laser Physics I

PH581-VTA (Mirov)

Optical Resonators Containing Amplifying Media
Lectures 11-12 chapter 4

Fall 2017
C. Davis, “Lasers and Electro-optics”
Optical resonators containing amplifying media

- Combination of our knowledge of optical frequency amplification and feedback characteristics of Fabry-Perot systems.
- Laser oscillation will occur at specific frequencies if the gain of the medium is large enough to overcome the loss of energy through the mirrors and by other mechanisms within the laser medium.
- Once laser oscillation is established, it stabilizes at a level that depends on the saturation intensity of amplifying medium and the reflectance of the laser mirrors.

Factors (saturation intensity & reflectance of the laser mirrors) affect the output power that can be obtained from a laser and how this can be optimized.

**Fabry-Perot Resonator Containing an Amplifying Medium**

- Consider a F-P resonator with plane mirrors that is filled with an amplifying medium.
- Consider the complex amplitudes of the waves bouncing backwards and forwards normally between the resonator mirrors.
- These waves result from an incident beam \( E_0 \) with electric vector at the first mirror.
The amplitudes of the electric field vectors of the successively transmitted amplified and reflected waves.
The output beam through the right mirror arises from the transmission of waves travelling to the right. Its total electric field amplitude is:

\[ E_t = E_0 \frac{t^2 e^{-i k G}}{1 - r \frac{1}{2} e^{-2i k L}} = E_0 \frac{t^2 e^{-i k L}}{1 - r \frac{1}{2} e^{-2i k L}} \]

where \( f(x) = \frac{N_e - (\varepsilon^2 x^2)}{\varepsilon^2} \frac{C_e}{x^2} \) \( g(x) \)

The ratio of output to input intensities is:

\[ \frac{I_t}{I_0} = \frac{E_x \cdot E_x^*}{E_0 \cdot E_0^*} = \frac{E_0 t^2 e^{-i k (x + 2x)} e^{-2i k x}}{E_0^2 [1 - r \frac{1}{2} e^{-2i k L}] e^{-2i k x}} \]

\[ = \begin{cases} e^{2i k x} \text{ for } x, k \text{ - complex} \\ e^{-i k (x + 2x)} x e^{i k x} = 1 \end{cases} \]

\[ = \frac{t^2 e^{-i k x} e^{i k x} e = 1}{1 + r \frac{1}{2} e^{-2i k L} e^{-2i k x}} \]
\[ \frac{I_t}{I_0} = \frac{\text{constant}}{1 + C - d e^{2i(k+\Delta k)l} - 2iC e^{(d-2l)l} \cos 2(k+\Delta k)l} \]

As \( d - l \) increases from 0, the numerator approaches \( \infty \) and the whole expression blows up when

\[ \frac{1}{1, \frac{i}{2} e^{-2i(k+\Delta k)l} e^{(d-2l)l}} = 1 \]

we have an infinite amplitude transmitted wave for a finite amplitude incident wave.

or a finite amplitude transmitted wave for zero incident wave — **oscillation**.

- Physically, this condition must be satisfied for a wave to make a complete round trip inside the resonator and return to its starting point with the same amplitude and phase.

- It is an amplitude condition for oscillation that gives an expression for the threshold gain constant \( \gamma \) from \( \nu \)

\[ \frac{1}{\frac{i}{2} e^{\text{constant} - 2.7l}} = 1 \]

- To satisfy \( \frac{1}{\frac{i}{2} e^{-2i(k+\Delta k)l} e^{(d-2l)l}} = 1 \)
  \( e^{-2i(k+\Delta k)l} e^{(d-2l)l} \) must be real, so we have
  
  \[ e^{i\theta} = \cos \theta + i \sin \theta \]
  
  \[ e^{-i\theta} = \cos \theta - i \sin \theta \]

\[ 2(k+\Delta k)l = 2\pi M; \quad M = 1, 2, 3, \ldots \]

- Threshold gain coefficient

\[ \gamma = \Delta - \frac{2}{l} \ln r \]

- Population inversion needed for oscillation

\[ \left( N_2 - \frac{2}{\gamma} N_1 \right) = \frac{1}{g(k\nu)A_2} \Delta \left( d - \frac{2}{l} \ln r \right) \]
For a homogeneously broadened transition

\[ \frac{N_2 - \frac{g_2}{g_1} N_1}{\frac{\Delta v}{A_2, A^2}} \]

For an inhomogeneously broadened transition

\[ \frac{N_2 - \frac{g_2}{g_1} N_1}{\frac{1}{A^2}} \]

Lower inversions are needed to achieve laser oscillation at longer wavelengths.

Let us find population inversion at threshold using time constant of the cavity.

Consider a passive F-P resonator having small distributed losses \( \Delta \).

A wave starting with intensity \( I \) inside the resonator will after one complete round trip have intensity \( I R_1 R_2 e^{-2zL} \)

If \( R_1 = \frac{r_1^2}{1 + r_1^2} \); \( R_2 = \frac{r_2^2}{1 + r_2^2} \)

\[ \frac{dI}{dt} = I R_1 R_2 e^{-2zL} - I = I (R_1 R_2 e^{-2zL} - 1) \]

This loss occurs in a time \( \Delta t = \frac{2L}{c} \)

\[ \Rightarrow \frac{dI}{dt} = c I \left[ R_1 R_2 e^{-2zL} - 1 \right] \]

Solution \( I = I_0 \exp \left( \frac{-1 + R_1 R_2 e^{-2zL}}{2L} \right) \)

\( I_0 \) - intensity at \( t=0 \)

The time constant of the cavity for intensity loss

\[ \tau_0 = \frac{2L}{c (1 - R_1 R_2 e^{-2zL})} \]
If \( R_1 R_2 e^{-2dL} \approx 1 \) then:

\[
(1-R_1 R_2 e^{-2dL}) \approx -\ln (R_1 R_2 e^{-2dL}) = -\ln (R_1, R_2) + 2dL
\]

Use the relation \( 1-x \approx -\ln x \) when \( x \ll 1 \).

\[
\zeta_0 = \frac{2L}{c (1-R_1 R_2 e^{-2dL})} = \frac{2L}{c (2dL - \ln R_1, R_2)} = \frac{1}{\left[ dL \left( 1 - \frac{L}{dL} \right) \right]}
\]

The Threshold population inversion:

\[
N_4 = \frac{8\pi L}{\hbar \omega L^2 g(v)} \zeta_0.
\]

Threshold Population Inversion Numerical Example

For the 632.8 nm transition of He-Ne laser:

\[
\lambda = 632.8 \text{ nm}, \quad \tau = 10^{-7} \text{ s}, \quad L = 12 \text{ cm}
\]

\[
\frac{1}{g(v)} \approx \Delta v \times 10^3 \text{ Hz} \quad (g(v,v) = \frac{\hbar}{\Delta v} \sqrt{\frac{C_0}{\eta \gamma} - \frac{0.94}{\Delta v^2}})
\]

Assume \( \lambda = 0 \) and \( R_1 = R_2 = 0.98 \).

Since \( R_1 R_2 \approx 1 \) we use approximation

\[
-\ln x = 1-x, \quad x \ll 1 \quad \text{to write}
\]

\[
\zeta_0 = \frac{2L}{c (2dL - \ln R_1, R_2)} = \frac{2L}{c (-\ln R_1, R_2)} = \frac{2L}{c (1-R_1 R_2)} = \frac{2L}{c (5 \times 10^{-4})(1-0.96)} = 2.0 \times 10^{-5}
\]

\[
N_4 = \frac{8\pi \times 10^9}{\hbar \omega L^2 (632.8 \times 10^{-3})^2 (3 \times 10^9) \times 2 \times 10^{-8}} = 1 \times 10^{15} \text{ m}^{-3} \approx \frac{9}{10 \text{ cm}^{-3}}
\]
The oscillation frequency

To determine the frequency at which laser oscillation can occur, return to the phase condition of oscillation:

\[(k + \Delta k) \ell = m \pi\]

\[(k + \Delta k) \ell = (k + \frac{k\lambda'(\nu)}{2\hbar^2}) \ell = k \ell \left[1 + \frac{\lambda'(\nu)}{2\hbar^2}\right] = m \pi\]

We remember that \[\lambda'(\nu) = \frac{2(\nu_0 - \nu)\lambda''(\nu)}{\Delta \nu}\]

\[\lambda''(\nu) = -\frac{k\lambda''(\nu)}{\hbar^2}\]

\[-\frac{k}{\hbar^2} \lambda''(\nu) = \lambda''(\nu) \frac{-\Delta \nu}{\Delta \nu} = \lambda''(\nu) \frac{-\lambda'(\nu)}{\lambda''(\nu)} = \lambda'(\nu) \frac{-\lambda''(\nu)}{\lambda''(\nu)} = \lambda'(\nu)\]

\[= \left[1 + \frac{\lambda'(\nu)}{2\hbar^2}\right] \lambda'(\nu) = m \pi\]

\[k \ell \left[1 - \frac{(\nu_0 - \nu)}{\Delta \nu}\right] \lambda'(\nu) = \left[1 - \frac{(\nu_0 - \nu)}{\Delta \nu}\right] \lambda'(\nu) = m \pi\]

\[r \left[1 - \frac{(\nu_0 - \nu)}{\Delta \nu}\right] \lambda'(\nu) = \frac{mc}{2e} = \nu_m\]

where \(\nu_m\) is the mth resonance of the passive laser resonator in normal incidence.

\[\nu = \frac{(\nu_0 - \nu)}{\Delta \nu}\]

\[\nu = \nu_m - \frac{(\nu - \nu_0)}{2 \pi \Delta \nu}\]
Since \( y \) close to \( v_m \)

\[
(y - v_o) \sim (v_m - v_o) \quad \text{and} \quad J(y) \approx J(v_m)
\]

\[
\Rightarrow \quad y = v_m - (v_m - v_o) \frac{J(v_m)}{2\pi \alpha v}
\]

- At threshold:
  \[ J_t(v_m) = L - \frac{\alpha_k}{\alpha_s} \alpha_s z_1^2 \]
  if \( L = 0 \) and \( z_1 = z_2 = \sqrt{R} \)

\[
J_t(v_m) = \frac{1 - R}{\alpha_s} \Rightarrow \quad y = v_m - (v_m - v_o) \frac{1 - R}{2\pi \alpha v}
\]

- FWHM of intensity maxing of F-P in transmission:

\[
\Delta y_{1/2} = \frac{\Delta V_{se}}{F} = \frac{C}{2 \alpha} \cdot \frac{1 - R}{\sqrt{R}} \quad \text{for} \quad R \approx 1
\]

- \( y = v_m - (v_m - v_o) \frac{\Delta V_{se}}{\Delta y} \)

if \( v_m \neq v_o \) - oscillation takes place near \( v_m \)
but is shifted slightly towards \( v_o \).

It is called Mode - pulling.

Relative position of line center, Fabry-Perot resonances and scale oscillation frequencies that satisfy the phase condition

Pulled oscillation frequencies

Cavity resonances:

\[
\frac{\text{Frequency}}{\text{Line Center}}\quad \frac{\text{Frequency}}{\text{Frequency}}
\]
Multimode laser oscillation

- When gain reaches a threshold value $J_0 (\nu) = \frac{1}{2} \frac{\nu}{\nu_0}$, oscillation will occur in a laser system.
- For gain coefficients greater than this, oscillation can occur at, or near (because of mode-pulling effect), one or more of the passive resonance frequencies of the ring laser cavity.
- The resulting oscillations of the system are called longitudinal modes.
- As the resulting oscillations at a particular one of these mode frequencies builds up, the growing intracavity energy density depletes the inverted population and gain saturation sets in. The reduction of gain continues until $J (\nu) = J_0 (\nu) = \frac{1}{2} \frac{\nu}{\nu_0}$
- Further reduction of $J (\nu)$ below $J_0 (\nu)$ does not occur, otherwise the oscillation would cease.
- Gain stability at the loss level $\frac{1}{2} \frac{\nu}{\nu_0}$

- Usually $\lambda$, $\nu$, and $\nu_0$ are constant over the frequency range covered by typical transitions ($10^6$ Hz). $\lambda = \frac{\nu}{\nu_0}$ as a function of $\nu$ is a straight line $11$ to the frequency axis.
In a homogeneously broadened laser, because the reduction in gain caused by a monochromatic field is uniform across the whole gain profile, the clamping of the gain at $J_g(\nu)$ leads to final oscillation at only one of the cavity resonance frequencies, the one where the original unsaturated gain was the highest.

- It can be shown by plotting $J_g(\nu)$ at various stages of oscillation buildup.

- Saturation of gain of a homogeneously broadened transition by a monochromatic signal, whose intensity increases from 1 to 2 to 3.

The gain profile is suppressed uniformly even though the saturating signal is not at the line center.
As oscillation begins, several such monochromatic fields start to build up at where gain exceeds loss.

Fig. 5.5. Oscillation building up in a homogeneously broadened laser. Gain saturation has already suppressed oscillation at two of the cavity modes that were above the loss line in Fig. (5.4).

In a homogeneously broadened laser, oscillation occurs at one longitudinal mode frequency.

Fig. 5.6. Oscillation stabilized in a homogeneously broadened laser. The gain has been uniformly saturated until only one mode remains at the loss line.
In an inhomogeneously broadened laser, the onset of gain saturation due to a monochromatic signal only reduces the gain locally over a region which is of the order of homogeneous width. Only particles whose center emission frequencies lie within a homogeneous width of the monochrom. field do not interact strongly with it.

Fig. 5.7. Localized gain saturation in an inhomogeneously broadened amplifier produced by a monochromatic signal whose intensity increases from 1→2→3.

![Diagram](image)

A localized hole in the gain profile occurs.

If only one cavity resonance has a small signal gain above the loss line then only this longitudinal mode oscillates.

![Diagram](image)

The situation is not quite as simple.
Oscillation at single longitudinal mode frequency implies waves travelling in both directions inside the laser cavity.

\[ i(\omega t - kz) \]

1) Wave travelling to the right \( E_0 e^{i(\omega t - kx)} \)
2) \( E_0 e^{i(\omega t + kx)} \)

\[ \omega = 2\pi v < 2\pi v_o \quad \text{for convenience.} \]

Wave 1) interacts with particles with center frequency near \( v \).

These particles are moving away from the observer looking into laser from right to left.

\[ v = v_0 - \frac{1}{c} v_0 \]

Wave 2) which is travelling in the opposite direction (to the left) and is monitored at \( v = v_0 \) by a second observer. Cannot second observer interact with the same velocity group of particles as wave 1).

The particles which interacted with wave 1) were moving away from the first observer and were Doppler shifted to lower frequencies so at to satisfy

\[ v = v_0 - \frac{1}{c} v_0 \]

The second observer sees these particles approaching and their center frequency as

\[ v = v_0 + \frac{1}{c} v_0 \]

so they cannot interact with wave 2)

Wave 2) interacts with particles moving away from the second observer. Their velocities - solution

\[ v = v_0 - \frac{1}{c} v_0 \]

These particles would be monitored by 1st observer at

\[ v = v_0 + \frac{1}{c} v_0 \]
The oscillating waves interact with two groups of particles. This leads to saturation of the gain by a single laser mode in an inhomogeneously broadened laser, both at the frequency of the mode \( \nu_0 \) and at a frequency \( \nu_0 + (\nu_0 - \nu) \), which is equally spaced on the opposite side of line center.

Fig. 5.9. Production of two holes in the velocity distribution of a collection of amplifying particles by a single cavity mode.

A single oscillating frequency interacts with two groups of atoms. Hole produced by interaction with left→right travelling wave. Hole produced by interaction with right→left travelling wave.

Negative velocity (right→left) 0 Negative velocity (left→right)

Single oscillating mode Image

\( \nu(v) \)

Image hole Cavity resonances

Loss \( \nu_0 \) Loss

The power output of the laser comes from these groups of particles that have gone into stimulated emission and left the two holes.

The combined area of these two holes gives a measure of the laser power.
If the frequency of the oscillating mode is moved in toward the line center, the main hole and image hole begin to overlap.

This corresponds to the fact that, as the travelling waves begin to interact with the small group of particles.

As the oscillating modes moves in toward the line center, the holes overlap further, the combined area decreases, the laser output power falls, reaching a minimum.

Because hole-burning in gain saturation in inhomogeneously broadened near the frequency of a cavity mode.

Simultaneous oscillation at several closely spaced frequencies (5-10 apart) can be observed with a high resolution spectrometer.

The multiple modes are almost exactly 50% in frequency apart, but are not equally spaced because of mode-pulling.
4.1

A four-level laser is pumped into its pump band at a rate $10^{24} \text{m}^{-2} \text{s}^{-1}$, the transfer efficiency to the upper level is 0.5. The lifetime of the upper laser level is $7 \times 10^{-5}$ s. For the laser transition $\lambda_0 = 1 \mu m$, the upper laser level population is $n = 1.6$. The amplifying medium is 20 mm long. Assume $n = 1.6$. The amplifying medium is 20 mm long. Assume $n = 1.6$. The amplifying medium is 20 mm long.

a) What is the gain at the center?

b) What is the gain of $P_2$ needed to get oscillation if mirror 1 has $R_1 = 1$? Assume $\lambda = 0$.

\[
\begin{align*}
N_2 &= 0.5 \times 10^{-24} \text{ m}^{-2} \text{s}^{-1} \\
\alpha &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{dN_2}{dt} &= R_2 - \frac{N_2}{t_2} = 0 \quad \Rightarrow \quad N_2 = R_2 t_2 \\
\frac{dN_1}{dt} &= N_2 A_2 = 0 \\
\frac{dN_1}{dt} &= 0
\end{align*}
\]

\[
\begin{align*}
\int (v) &= (N_2 - \frac{v^2}{2}) \cdot \frac{A_2}{8 \pi v} \\
\int (v, \frac{v}{v_0}) &= \frac{A_2}{8 \pi} \cdot \frac{v^2}{v_0^2} \\
\int &\approx 1.2 \times 10^{-5} \text{ m}^{-1}
\end{align*}
\]

\[
\begin{align*}
\int (v) &= \frac{1}{v} - \frac{1}{v_0} \sqrt{\frac{v}{v_0}} \approx \frac{1}{v} \\
\int (v) &= \frac{1}{v_0} \\
L_2 &= \frac{1}{v_0} = \frac{1}{v} \\
L_2 &= \frac{1}{v_0} \approx 0.9999
\end{align*}
\]
\[ r(v) = \Delta N \frac{c^2 A_1}{8\pi v^2} g(v, v) = \frac{\Delta N c^2 A_1}{8\pi v^2} \frac{2}{\Delta v_0} \sqrt{\frac{v_0^2 - v^2}{v_0^2}} e^{-\frac{(v - v_0)^2}{\Delta v_0^2}} \]

\[ r(v) = \Delta N \frac{c^2 A_1}{8\pi v_0^2} g(v_0, v) = \frac{\Delta N c^2 A_1}{8\pi v_0^2} \frac{2}{\Delta v_0} \sqrt{\frac{v_0^2 - v^2}{v_0^2}} \]

\[ \Rightarrow r'(v) = r'(v_0) e^{-\frac{(v - v_0)^2}{\Delta v_0^2}} \]
Problem
How many longitudinal modes will oscillate in an inhomogeneously broadened gas laser with \( \Gamma = 1 \text{ m}^{-1} \):
\[ f(Y_0) = 1 \text{ m}^{-1}, \quad \Gamma_1 = \Gamma_2 = 99\%. \]
Distributed loss = 0.001 m^{-1};
\( \lambda_0 = 500 \text{ nm} ; \quad \Delta Y_0 = 3 \text{ GHz} \).

1)
\[ J_h(Y) = 2 - \frac{Y}{Y_0} + \frac{Y_0^2}{Y} = 0.001 \text{ m}^{-1} \]
\[ = \frac{1}{\ln(599^2)} = 1.05 \times 10^{-2} \text{ m}^{-1} \]

2)
Cavity resonances \( \nu = \frac{mc^2}{2\hbar} \).
Does \( \nu \) coincide with line center for some \( m \)?
\[ m = \nu_m \cdot 2\hbar c = \nu_m \cdot \frac{2.11 \times 10^8}{\nu_c} = \frac{3 \times 10^8}{500 \times 10^3} = 4 \times 10^6 \text{ each} \]

\[ \Rightarrow \text{ laser radiation will occur at the cavity resonances separated by } \frac{c}{2\hbar} = 150 \text{ MHz}. \]

3)
For an inhomogeneously broadened laser with \( f(Y) \):
\[ f(Y) = f(Y_0) e^{-\left[ \frac{2(Y - Y_0)}{\Gamma_0^2} \right]^2} \]
we are interested in determining the number of oscillating modes with a gain greater than the loss.
Frequency corresponds to the situation where loss = gain.
\[ 1.05 \times 10^{-2} = 1 \cdot e^{-E_2 \frac{\Delta Y^2}{2 \nu_c^2} \gamma_n} \]
\[ (Y_m - Y_0)^2 = -\ln \left( \frac{1.05 \times 10^{-2}}{\nu_c^2} \right) \Delta Y_0^2 = 1.46 \times 10^8 \text{ Hz} \]
\[ V_{th} = Y_0 \pm V_{th} \times 10^8 = Y_0 \pm 3.82 \times 10^8 \text{ Hz} \]
The number of oscillating modes:
\[ \frac{1}{\Delta Y_{FSR}} \approx 7 \pm 1 = 6 \]
Mode-Beating

- We shine the light from a two-mode laser on a square-law detector (respects to the intensity, not the electric field of an incident light signal).

- The incident electric field is:
  \[ E_i = R \left( E_1 e^{i(\omega_1 t + \phi_1)} + E_2 e^{i(\omega_2 t + \phi_2)} \right) \]

- The complex amplitudes of the two modes and their frequency spacing determine:
  \[ E_1, E_2, \omega_1 - \omega_2 \]

- The output current from the detector is:
  \[ i \propto E_1 E_2 \cos \left[ (\omega_1 + \omega_2) t + \phi_2 - \phi_1 \right] + \cos \left( \omega_1 t + \phi_1 \right) \cos \left( \omega_2 t + \phi_2 \right) + \frac{1}{2} \left( E_1^2 \cos^2 \left[ (\omega_1 + \omega_2) t + \phi_2 - \phi_1 \right] + \left( E_2^2 \cos^2 \left[ (\omega_1 + \omega_2) t + \phi_2 - \phi_1 \right] \right) \]

- Since \( E_1^2 \cos^2 \left[ \omega_1 t + \phi_1 \right] = \frac{1}{2} E_1^2 \cos \left[ (\omega_1 + \omega_2) t + \phi_1 + \phi_2 - \phi_1 \right] \)
  
- The output frequency spectrum of the detector contains frequencies \( 2\omega_1, 2(\omega_1 + \omega_2), 2\omega_2 \) and very high frequencies do not appear in the output of the detector.

- Only the difference frequency \( \omega_1 - \omega_2 \) is observed.
If the output from the square-law detector is analyzed with a radio-frequency spectrum analyzer (freq. range where different frequency differences between longitudinal laser modes are observed) different displays are obtained according to how many longitudinal modes are oscillating.

Fig. 5.14. Schematic mode-beating spectra observed with a square-law detector and a multimode laser.
Problem 4.6

A laser is exactly 1 m long and has a wavelength $\lambda_0 = 632.8 \text{ nm}$. The mirrors of the laser have $R = 99\%$. The index of refraction of the amplifying medium is exactly 1.0001, $\Delta \nu = 100,000 \text{ MHz}$. The laser operates on only the two modes nearest to the line center. The laser output illuminates a photodiode whose output is mixed with a 150 MHz local oscillator. What is the frequency of the lowest beat signal observed? Take $c_0 = 2.997 \times 10^8 \text{ m/s}$.

**1) Calculate the frequency splitting**

$$\nu = \nu_m - (\nu_m - \nu_0) \frac{\Delta \nu_0}{\Delta \nu}$$

**2) Find the frequency splitting**

$$\Delta \nu_{m,m+1} = \nu_{m+1} - (\nu_{m+1} - \nu_0) \frac{\Delta \nu_0}{\Delta \nu} = \nu_m + (\nu_m - \nu_0) \frac{\Delta \nu_0}{\Delta \nu} = (\nu_{m+1} - \nu_m) + (\nu_m - \nu_{m+1}) \frac{\Delta \nu_0}{\Delta \nu} = (\nu_{m+1} - \nu_m) \left(1 - \frac{\Delta \nu_0}{\Delta \nu}\right) = \frac{(m+1)c_0}{2\pi n e} \left(1 - \frac{c_0 (1-R)}{2 \pi e / \Delta \nu} \right) = \frac{c_0}{2 \pi n e} \left[1 - \frac{2.997 \times 10^8 (1-0.99)}{2 \pi \times 2.10001 \times 100 \times 10^6}ight] = 149.1 \text{ MHz}$$

**3) If we mix this frequency with a 150 MHz oscillator, the observed beat frequency would be**

$$150 - 149.1 = 0.9 \text{ MHz}$$
*If a predominantly inhomogeneously broadened laser also has a significant amount of homogeneous broadening, the holes burn in the gain curve start to overlap. (when $\Delta \nu \approx \frac{\nu}{2\epsilon}$)*

Fig. 5.15: Schematic illustration of mode competition in an inhomogeneously broadened laser in which there is significant homogeneous broadening.

---

*If $\Delta \nu$ is large – neighboring oscillating modes compete and may lead to oscillation on a strong mode suppressing its weaker neighbors.*
The power output of a laser

When a laser oscillates, the intracavity field grows in the loss amplitude until saturation reduces the gain to the loss for each oscillating mode. For an asymmetrical resonator, whose mirror reflectances are not equal, the distribution of standing wave energy within the resonator is not symmetric.

If \( L_2 > L_1 \), the distribution of intracavity travelling wave intensity is:

\[
\frac{I_2}{I_2} = \frac{L_2}{L_2} \quad \text{and} \quad \frac{I_3}{I_4} = \frac{L_3}{L_4}
\]

The left travelling wave \( I_- \) grows in intensity from \( I_3 \) to \( I_4 \) on a single pass.

The right travelling wave \( I_+ \) grows from \( I_2 \) to \( I_4 \).

The total output intensity:

\[
I_{\text{out}} = I_2 + I_3 + I_4
\]

Consider purely homogeneously broadened system with saturated gain:

\[
\mathcal{F}(z) = \frac{J_0}{1 + (I_+ + I_-) / I_3}
\]

\( I_- \) and \( I_+ \) grows according to:

\[
\frac{d[I_+ I_-]}{dz} = I_+ \frac{dI_+}{dz} + I_- \frac{dI_-}{dz} = -\gamma_2 I_+ I_- + \gamma_1 I_+ I_- = 0
\]
For the right travelling wave
\[ \frac{dI_2}{dt} = j(z) = j_0 \frac{1 + \frac{c}{v_2}}{1 + \frac{c}{v_2}} \]

Integration gives
\[ I_2 = I_0 (\frac{I_0}{I_1}) + \frac{(I_2 - I_1)}{I_3} - \frac{C}{I_3} (\frac{1}{I_2} - \frac{1}{I_1}) \]

For the left travelling wave
\[ I_0 = m \left( \frac{I_0}{I_2} \right) + \frac{I_0 - I_3}{I_5} - \frac{C}{I_5} (\frac{1}{I_4} - \frac{1}{I_3}) \]

using
\[ \frac{I_0}{I_2} = r_2; \quad \frac{I_3}{I_4} = r_1 \]
\[ I_4 I_1 = I_2 I_3 = c \]
\[ \frac{I_0}{I_2} = \sqrt{\frac{r_1}{r_2}} \]

\[ I_2 = \frac{I_3 VR_1 (I_0 + C_0 VR, R_2)}{(1/R_1 + VR_2)(1 - VR_2)} \]

\[ I_4 = I_2 \frac{\sqrt{R_2}}{R_1}; \quad T_1 = 1 - R_1 - A_1; \quad T_2 = 1 - R_2 - A_2 \]

\[ I_{out} = \frac{R_2}{I_2} I_2 + T_1 I_1 \]

if \( A_1 = A_2 = A \)

\[ I_{out} = \frac{I_3}{1 - VR_2} (1 - A - VR_2) (I_0 + C_0 VR, R_2) \]

For \( T_1 = 0; \quad R_1 = 1 \)
\[ I_{out} = \frac{T_2 I_2}{I_2 I_2} \frac{I_0 + C_0}{A_2 + T_2} \]

For a symmetrical solution
\[ r_1 = R_2 \]
\[ R = 1 - A - T \]
\[ I_{out} = \frac{I_3}{2} \frac{(1 - A - R)}{(1 - R) (I_0 + C_0 R)} \]
to maximize the output intensity from the symmetric resonator we must find the value of $R$ such that

$$\frac{\partial I_{out}}{\partial R} = 0$$

$$\frac{T_{opt}}{A} = \left( \frac{1 - A - T_{opt}}{A + T_{opt}} \right) \left[ \delta_{opt} + \ln (1 - A - T_{opt}) \right]$$

Fig. 5.17. Calculated optimum coupling for a symmetrical resonator for various values of the loss parameter $A$ and the unsaturated gain.

For small losses $A + T_{opt} \ll 1$

$$\frac{T_{opt}}{A} = \sqrt{\frac{\delta_{opt}}{A}} - 1$$
A laser ($\lambda=2.09$ µm, $\sigma=1.15\times10^{-20}$ cm$^2$, $\tau=8$ ms) measured to have an intensity of 100 W/cm$^2$ emerging from one end of the laser, which has two identical mirrors each with transmission of 15%. The gain of the laser is also measured to be 0.5.

a) What is the loss parameter “A” in the cavity?

b) What is the optimum output mirror transmission?

\[
\begin{align*}
\text{a) } I_{out} &= \frac{I_s}{2} (1 - A - R) \left( \gamma_o L + \ln R \right) \\
I_s &= \frac{h\nu}{\sigma \tau} = \frac{6.62 \times 10^{-34} \cdot 1.435 \times 10^{14}}{1.15 \times 10^{-20} \cdot 8 \times 10^{-3}} = 1.03 \text{ kW/cm}^2 \\
\nu &= \frac{c}{\lambda} = \frac{3 \times 10^8}{2.09 \times 10^{-6}} = 1.435 \times 10^{14} \text{ Hz} \\
R &= 0.85; \quad I_{out} = 100 \text{ W/cm}^2; \quad \gamma_o L = 0.5 \\
\text{Everything is given. Let us find } A \\
100 &= 1030 \frac{(1 - A - 0.85)}{(1 - 0.85)} (0.5 + \ln 0.85) \\
50 &= 515 \frac{(0.15 - A)}{0.15} \cdot 0.337; \quad A = 0.107
\end{align*}
\]

\[
\begin{align*}
\text{b) Use } &\frac{T_{opt}}{A} = \sqrt{\frac{\gamma_o L}{A}} - 1 \text{ to find } T_{opt} \\
T_{opt} &= A \sqrt{\frac{\gamma_o L}{A}} - A = 0.107 \sqrt{\frac{0.5}{0.107}} - 0.107 = 0.124 = 12.4\%
\end{align*}
\]