PH 221-1D Spring 2013

ROTATION

Lectures 24-26

Chapter 10
(Halliday/Resnick/Walker, Fundamentals of Physics 9th edition)
Chapter 10
Rotation

In this chapter we will study the rotational motion of rigid bodies about a fixed axis. To describe this type of motion we will introduce the following new concepts:

- Angular displacement
- Average and instantaneous angular velocity (symbol: $\omega$)
- Average and instantaneous angular acceleration (symbol: $\alpha$)
- Rotational inertia also known as moment of inertia (symbol: $I$)
- Torque (symbol: $\tau$)

We will also calculate the kinetic energy associated with rotation, write Newton’s second law for rotational motion, and introduce the work-kinetic energy for rotational motion
The Rotational Variables

In this chapter we will study the rotational motion of rigid bodies about fixed axes. A **rigid** body is defined as one that can rotate with all its parts locked together and without any change of its shape. A **fixed** axis means that the object rotates about an axis that does not move. We can describe the motion of a rigid body rotating about a fixed axis by specifying just one parameter. Consider the rigid body of the figure.

We take the the z-axis to be the fixed axis of rotation. We define a reference line which is fixed in the rigid body and is perpendicular to the rotational axis. A top view is shown in the lower picture. The angular position of the reference line at any time \( t \) is defined by the angle \( \theta(t) \) that the reference lines makes with the position at \( t = 0 \). The angle \( \theta(t) \) also defines the position of all the points on the rigid body because all the points are locked as they rotate. The angle \( \theta \) is related to the arc length \( s \) traveled by a point at a distance \( r \) from the axis via the equation: 

\[
\theta = \frac{s}{r}
\]

**Note:** The angle \( \theta \) is measured in radians.
Angular Displacement

In the picture we show the reference line at a time $t_1$ and at a later time $t_2$. Between $t_1$ and $t_2$ the body undergoes an angular displacement $\Delta \theta = \theta_2 - \theta_1$. All the points of the rigid body have the same angular displacement because they rotate locked together.

We define as average angular velocity for the time interval $(t_1, t_2)$ the ratio:

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta \theta}{\Delta t}$$

The SI unit for angular velocity is radians/second.

We define as the instantaneous angular velocity the limit of $\frac{\Delta \theta}{\Delta t}$ as $\Delta t \to 0$:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}$$

This is the definition of the first derivative with $t$.

Algebraic sign of angular frequency: If a rigid body rotates counterclockwise (CCW) $\omega$ has a positive sign. If on the other hand the rotation is clockwise (CW) $\omega$ has a negative sign.
Angular Acceleration

If the angular velocity of a rotating rigid object changes with time we can describe the time rate of change of $\omega$ by defining the angular acceleration

In the figure we show the reference line at a time $t_1$ and at a later time $t_2$. The angular velocity of the rotating body is equal to $\omega_1$ at $t_1$ and $\omega_2$ at $t_2$. We define as average angular acceleration for the time interval $(t_1, t_2)$ the ratio:

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta \omega}{\Delta t}$$

The SI unit for angular velocity is radians/second$^2$

We define as the instantaneous angular acceleration the limit of $\frac{\Delta \omega}{\Delta t}$ as $\Delta t \to 0$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t}$$

This is the definition of the first derivative with $t$
Angular Velocity Vector

For rotations of rigid bodies about a fixed axis we can describe accurately the angular velocity by assigning an algebraic sign. Positive for counterclockwise rotation and negative for clockwise rotation.

We can actually use the vector notation to describe rotational motion which is more complicated. The angular velocity vector is defined as follows:

- The direction of $\vec{\omega}$ is along the rotation axis.
- The sense of $\vec{\omega}$ is defined by the right hand rule (RHL).

Right hand rule: Curl the right hand so that the fingers point in the direction of the rotation. The thumb of the right hand gives the sense of $\vec{\omega}$.
Rotation with Constant Angular Acceleration

When the angular acceleration $\alpha$ is constant we can derive simple expressions that give us the angular velocity $\omega$ and the angular position $\theta$ as function of time. We could derive these equations in the same way we did in chapter 2. Instead we will simply write the solutions by exploiting the analogy between translational and rotational motion using the following correspondance between the two motions:

<table>
<thead>
<tr>
<th>Translational Motion</th>
<th>Rotational Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ $\leftrightarrow$ $\theta$</td>
<td></td>
</tr>
<tr>
<td>$v$ $\leftrightarrow$ $\omega$</td>
<td></td>
</tr>
<tr>
<td>$a$ $\leftrightarrow$ $\alpha$</td>
<td></td>
</tr>
</tbody>
</table>

$v = v_0 + at$ $\leftrightarrow$ $\omega = \omega_0 + \alpha t$ \hspace{1cm} (eqs.1)

$x = x_o + v_o t + \frac{at^2}{2}$ $\leftrightarrow$ $\theta = \theta + \omega_o t + \frac{\alpha t^2}{2}$ \hspace{1cm} (eqs.2)

$v^2 - v_o^2 = 2a(x - x_o)$ $\leftrightarrow$ $\omega^2 - \omega_o^2 = 2\alpha(\theta - \theta_o)$ \hspace{1cm} (eqs.3)
Consider a point P on a rigid body rotating about a fixed axis. At $t = 0$ the reference line which connects the origin O with point P is on the x-axis (point A). During the time interval $t$ point P moves along arc $AP$ and covers a distance $s$. At the same time the reference line OP rotates by an angle $\theta$.

Relation between angular velocity and speed

The arc length $s$ and the angle $\theta$ are connected by the equation:

$$s = r\theta$$

where $r$ is the distance OP. The speed of point P

$$v = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r \frac{d\theta}{dt}$$

$$v = r\omega$$

The period $T$ of revolution is given by:

$$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{v} = \frac{2\pi r}{r\omega} = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\omega} \quad T = \frac{1}{f} \quad \omega = 2\pi f$$
The Acceleration

The acceleration of point P is a vector that has two components. A "radial" component along the radius and pointing towards point O. We have encountered this component in chapter 4 where we called it "centripetal" acceleration. Its magnitude is:

$$a_r = \frac{v^2}{r} = \omega^2 r$$

The second component is along the tangent to the circular path of P and is thus known as the "tangential" component. Its magnitude is:

$$a_t = \frac{dv}{dt} = \frac{d(\omega r)}{dt} = r \frac{d\omega}{dt} = r \alpha$$

The magnitude of the acceleration vector is:

$$a = \sqrt{a_t^2 + a_r^2}$$
Example. Motion with constant angular acceleration.

When you turn off the motor, a phonograph turntable initially rotating at $33.3 \text{ rad/min}$ makes $25$ revolutions before it stops. Calculate the angular acceleration of this turntable, assume it is constant.

**Reasoning/Strategy**
1. Make a drawing to represent the situation being studied. Showing the direction of rotation.
2. Decide which direction of rotation $\phi$ is to be called $+$, and which $\omega$. Let us choose counterclockwise $+$. Do not change your decision during the course of a calculation.
3. Write down the values (with appropriate signs) that are given for any of the free kinematic variables. ($\phi$, $\omega$, $\omega_0$, and $t$). Identify the variables that you are asked to determine.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\omega_0$</th>
<th>$\omega$</th>
<th>$\omega_0$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-25 \text{ rad} = \frac{-25 \text{ rev}}{60 \text{ sec}}$</td>
<td>$-33.3 \text{ rad/min}$</td>
<td>$0$</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

4. Verify that the given information contains at least three of five kinematic variables. Choose one of the equations:
   a) $\omega = \omega_0 + \alpha t$
   b) $\phi = \frac{1}{2} (\omega_0 + \omega) t$
   c) $\phi = \omega_0 t + \frac{1}{2} \alpha t^2$
   d) $\omega^2 = \omega_0^2 + 2 \alpha \phi$
   e) $\omega_0^2 = \omega^2 + 2 \alpha \phi$  

Choose e.g. d)

$$\omega = \frac{\omega_0^2 - \omega^2}{2 \phi} = \frac{-(-3.49 \text{ rad/s})^2}{2 \left( -50 \text{ rad} \right)} = \frac{\sqrt{3.9 \times 10^{-2} \text{ rad}^2}}{s^2}$$

The positive sign in the answer indicates that the direction of the angular acceleration is opposite to the direction of angular velocity.
Rotation with constant angular acceleration.

Problem: An automatic dryer spins wet clothes at an angular speed of \(5.2\, \text{rad/s}^2\). Starting from rest, the dryer reaches its operating speed with an average angular acceleration of \(4.0\, \text{rad/s}^2\). How long does it take the dryer to come up to speed?

1) **drawing**
2) direction of rotation
3) 

<table>
<thead>
<tr>
<th>(\phi)</th>
<th>(\omega_0)</th>
<th>(\omega)</th>
<th>(\alpha)</th>
<th>(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(5.2,\text{rad/s})</td>
<td>(4.0,\text{rad/s}^2)</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

4) verify that at least 3 of 5 kinematic variables are given.

5) Choose equation: 
   a) \(\omega = \omega_0 + \alpha t\)
   b) \(\phi = \frac{1}{2} (\omega_0 + \omega) t\)
   c) \(\phi = \omega_0 t + \frac{1}{2} \alpha t^2\)
   d) \(\omega^2 = \omega_0^2 + 2\alpha \phi\)

\[
t = \frac{\omega - \omega_0}{\alpha} = \frac{5.2\,\text{rad/s} - 0\,\text{rad/s}}{\frac{4.0\,\text{rad/s}^2\cdot\text{s}}{5}} = 1.35\]


The equations of rotational kinematics.

Problem

A 9.0-m long string is wrapped around a pulley that is free to rotate about a fixed axle. The pulley has a radius of 6.0 cm and initially at rest. The thickness of the string is negligible. Someone pulls on the free end of the string, thereby unwinding it and giving the pulley an angular acceleration of \( +17 \text{ rad/s}^2 \).

1) Through what angle has the pulley rotated when the string is completely unwound?
2) What is the final angular velocity of the pulley?

Given:
- \( \theta = 6.0 \text{ cm} \)
- \( \alpha = 17 \text{ rad/s}^2 \)
- \( \omega_0 = 0 \)
- \( S = 9.0 \text{ m} \)

3) Drawing
1) \( \theta = ?/\omega \)?
2) Direction of rotation

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \omega_0 )</th>
<th>( \omega )</th>
<th>( L )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>90(^\circ)</td>
<td>6.0 cm</td>
<td>0</td>
<td>( 17 \text{ rad/s} )</td>
<td>( 150 \text{ rad} )</td>
</tr>
</tbody>
</table>

4) Verify that at least 3 of 5 variables are given

5) Choose

\[ \omega^2 = \omega_0^2 + 2\alpha \theta \]

\[ \omega = \pm \sqrt{\omega_0^2 + 2\alpha \theta} = \pm \sqrt{(0 \text{ rad/s})^2 + 2(17 \text{ rad/s}^2)(150 \text{ rad})} = \pm 71 \text{ rad/s} \]

The wheel starts from rest and has a positive angular acceleration \( \Rightarrow \) \( \omega \) is positive \( \omega = \pm 71 \text{ rad/s} \)
The equations of kinematics.

Problem: A motorcyclist is traveling along a race and accelerates for 4.50 s to pass another cyclist. The angular acceleration of each wheel is +6.70 rad/s². Just after passing, the angular velocity of each wheel is +74.5 rad/s, where the plus signs indicate counterclockwise directions. What is the angular displacement of each wheel during this time?

\[ \theta = w_0 t + \frac{1}{2} a t^2 \]

\[ w = w_0 + \frac{d}{dt} \]

\[ \theta = \frac{1}{2} (w_0 + w)t \]

\[ \theta = w_0 t + \frac{1}{2} a t^2 \]

\[ \theta = w_0 t - \frac{1}{2} a t^2 \]

\[ \theta = (+74.5 \text{ rad/s})(4.50 \text{ s}) - \frac{1}{2} (+6.70 \text{ rad/s}^2)(4.50 \text{ s})^2 \]

\[ = 2672 \text{ rad} \]
Angular variables and tangential variables.

An auto race is held on a problem circular track. A car completes one lap in a time of 18.9 s, with an average tangential speed of 42.6 m/s. Find (a) the average angular speed and (b) the radius of the track.

Given
\[
\begin{align*}
& V_{r,av} = 42.6 \text{ m/s} \\
& t = 18.9 \text{ s.}
\end{align*}
\]

Find
\[
\begin{align*}
& \omega_{av} = ? \\
& \theta = ?
\end{align*}
\]

(a) In one lap, the car undergoes an angular displacement of \(2\pi\) radians.
\[
\theta = \frac{\Delta \phi}{\Delta t} = \frac{2\pi \text{ rad}}{18.9 \text{ s}} = 10.332 \text{ rad/s}
\]

(b) \[ V_{r,av} = \theta \omega_{av} \]
\[
\omega_{av} = \frac{V_{r,av}}{\theta} = \frac{42.6 \text{ m/s}}{10.332 \text{ rad/s}} = 128 \text{ m/s}
\]

Notice that the unit "rad" being dimensionless, does not appear in the final answer.
Angular variables and tangential variables.

Problem

A thin rod (length = 1.50 m) is oriented vertically, with bottom end attached to the floor by means of a frictionless hinge. The mass of the rod may be ignored, compared to the mass of the object fixed to the top of the rod. The rod, starting from rest, tips over and rotates downward.

1) What is the angular speed of the rod just before it strikes the floor? (Consider the principle of conservation of mechanical energy)

2) What is the magnitude of the angular acceleration of the rod just before it strikes the floor?

\[ E_i = \frac{1}{2} I \omega^2 \]

\[ mgL = \frac{1}{2} mv^2 \]

\[ v = \sqrt{2gL} \]

2) As the object rotates downward it travels in a circle of radius \( L \). Its speed just before it strikes the floor is its tangential speed.

\[ \omega L = v \]

\[ \omega = \frac{v}{L} = \frac{\sqrt{2gL}}{L} = \sqrt{\frac{2g}{L}} = \sqrt{\frac{2(9.80 \text{ m/s}^2)}{1.50 \text{ m}}} = 3.61 \text{ rad/s} \]

6) \[ a_T = \frac{L}{L} \cdot a_T \text{ is acceleration due to gravity.} \]

\[ L = \frac{a_T}{r} = \frac{9.80 \text{ m/s}^2}{1.50 \text{ m}} = 6.53 \text{ rad/s}^2 \]
Centripetal acceleration and tangential acceleration.

Problem

A floppy disk for a personal computer rotates with a constant angular speed of 31.4 rad/s about an axis perpendicular to the disk at its center.

1) Find the tangential speed of a point that is 0.0410 m from the center of the disk.

\[ v_T = r \omega = (0.0410 \text{ m})(31.4 \text{ rad/s}) = 1.29 \text{ m/s} \]

2) What is the magnitude of the centripetal acceleration at this point?

\[ a_c = r \omega^2 = (0.0410 \text{ m})(31.4 \text{ rad/s})^2 = 40.4 \text{ m/s}^2 \]
Kinetic Energy of Rotation

Consider the rotating rigid body shown in the figure. We divide the body into parts of masses \( m_1, m_2, m_3, \ldots, m_i, \ldots \). The part (or "element") at P has an index \( i \) and mass \( m_i \).

The kinetic energy of rotation is the sum of the kinetic energies of the parts \( K = \sum \frac{1}{2} m_i v_i^2 \). The speed of the \( i \)-th element \( v_i = \omega r_i \rightarrow K = \sum \frac{1}{2} m_i (\omega r_i)^2 \).

\[ K = \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2 \]

The term \( I = \sum m_i r_i^2 \) is known as rotational inertia or moment of inertia about the axis of rotation. The axis of rotation must be specified because the value of \( I \) for a rigid body depends on its mass, its shape as well as on the position of the rotation axis. The rotational inertia of an object describes how the mass is distributed about the rotation axis.

\[ I = \sum m_i r_i^2 \quad I = \int r^2 dm \quad K = \frac{1}{2} I \omega^2 \]
In the table below we list the rotational inertias for some rigid bodies

\[ I = \int r^2 \, dm \]
An automobile of mass 1400kg has wheels 0.75m in diameter weighing 27kg each. Taking into account the rotational kinetic energy of the wheels about their axes, what is the total kinetic energy of the automobile traveling at 80km/h? What percent of kinetic energy belongs to the rotational motion of the wheels about their axes? Pretend that each wheel has a mass distribution equivalent to that of a uniform disk.

Given:
- \( M = 1400 \text{ kg} \)
- \( m = 27 \text{ kg} \)
- \( R = 0.375 \text{ m} \)
- \( V = 80 \text{ km/h} \)

Translational kinetic energy,
\[
K_t = \frac{1}{2} M V^2 = \frac{1}{2} 1400 \times \left(\frac{80}{3.6}\right)^2 = 3.46 \times 10^5 \text{ J}
\]

Angular speed of the wheel
\[
\omega = \frac{V}{R} = \frac{80}{3.6 \times 0.375} = 59.3 \text{ rad/s}
\]

Rotational kinetic energy
\[
K_R = 4 \left(\frac{1}{2} I \omega^2\right) = 2 I \omega^2 = I \omega^2 = \frac{1}{2} m R^2
\]
\[
= 2 \left[\frac{1}{2} (27)(0.375)^2\right] (59.3)^2 = 1.33 \times 10^4 \text{ J}
\]

Total
\[
K_{tor} = K_t + K_R = (3.59 \times 10^5 \text{ J})
\]

\[
\frac{K_R}{K_{tor}} = \frac{1.33 \times 10^4}{3.59 \times 10^5} \times 100\% = 3.7\%
\]
Calculating the Rotational Inertia

The rotational inertia \( I = \sum_i m_i r_i^2 \)  This expression is useful for a rigid body that has a discreet distribution of mass.  For a continuous distribution of mass the sum becomes an integral \( I = \int r^2 dm \)

Parallel-Axis Theorem

We saw earlier that \( I \) depends on the position of the rotation axis. For a new axis we must recalculate the integral for \( I \). A simpler method takes advantage of the parallel-axis theorem.

Consider the rigid body of mass \( M \) shown in the figure. We assume that we know the rotational inertia \( I_{\text{com}} \) about a rotation axis that passes through the center of mass \( O \) and is perpendicular to the page.

The rotational inertia \( I \) about an axis parallel to the axis through \( O \) that passes through point \( P \), a distance \( h \) from \( O \) is given by the equation:

\[ I = I_{\text{com}} + Mh^2 \]
Proof of the Parallel-Axis Theorem  We take the origin $O$ to coincide with the center of mass of the rigid body shown in the figure. We assume that we know the rotational inertia $I_{com}$ for an axis that is perpendicular to the page and passes through $O$. 

We wish to calculate the rotational inertia $I$ about a new axis perpendicular to the page and passes through point $P$ with coordinates $(a, b)$. Consider an element of mass $dm$ at point $A$ with coordinates $(x, y)$. The distance $r$ between points $A$ and $P$ is: 

$$r = \sqrt{(x-a)^2 + (y-b)^2}$$

Rotational Inertia about $P$: 

$$I = \int r^2 dm = \int \left[ (x-a)^2 + (y-b)^2 \right] dm$$

$$I = \int \left( x^2 + y^2 \right) dm - 2a \int x dm - 2b \int y dm + \int \left( a^2 + b^2 \right) dm$$

The second and third integrals are zero. The first integral is $I_{com}$. The term $(a^2 + b^2) = h^2$

Thus the fourth integral is equal to $h^2 \int dm = Mh^2 \rightarrow I = I_{com} + Mh^2$
According to spectroscopic measurements, the moment of inertia of an oxygen molecule about an axis through the center of mass and perpendicular to the line joining the atoms is $1.95 \times 10^{-46} \text{kg} \cdot \text{m}^2$. The mass of an oxygen atom is $2.66 \times 10^{-26} \text{g}$. What is the distance between atoms?

\[
I = \sum m_i R_i^2
\]

where $R_i$ is the distance from the CM to the oxygen atoms; $R_1 = R_2 = R$.

\[
m_1 = m_2 = m
\]

So,

\[
I = 2mR^2
\]

\[
R = \sqrt{\frac{I}{2m}} = \sqrt{\frac{1.95 \times 10^{-46} \text{kg} \cdot \text{m}^2}{2 \times 2.66 \times 10^{-26} \text{kg}}} =
\]

\[
= 6.05 \times 10^{-10} \text{m}
\]

\[
2R \text{ (distance between oxygen atoms)} = 2 \times 6.05 \times 10^{-10} \text{m} = 1.21 \times 10^{-9} \text{m}
\]
Torque
In fig.a we show a body which can rotate about an axis through point O under the action of a force $\vec{F}$ applied at point P a distance $r$ from O. In fig.b we resolve $\vec{F}$ into two components, radial and tangential. The radial component $F_r$ cannot cause any rotation because it acts along a line that passes through O. The tangential component $F_t = F \sin \phi$ on the other hand causes the rotation of the object about O. The ability of $\vec{F}$ to rotate the body depends on the magnitude $F_t$ and also on the distance $r$ between points P and O. Thus we define as torque $\tau = rF_t = rF \sin \phi = r_{\perp}F$

The distance $r_{\perp}$ is known as the moment arm and it is the perpendicular distance between point O and the vector $\vec{F}$. The algebraic sign of the torque is assigned as follows:
If a force $\vec{F}$ tends to rotate an object in the counterclockwise direction the sign is positive. If a force $\vec{F}$ tends to rotate an object in the clockwise direction the sign is negative.
Newton's Second Law for Rotation

For translational motion Newton's second law connects the force acting on a particle with the resulting acceleration. There is a similar relationship between the torque of a force applied on a rigid object and the resulting angular acceleration.

This equation is known as Newton's second law for rotation. We will explore this law by studying a simple body which consists of a point mass $m$ at the end of a massless rod of length $r$. A force $\vec{F}$ is applied on the particle and rotates the system about an axis at the origin. As we did earlier, we resolve $\vec{F}$ into a tangential and a radial component. The tangential component is responsible for the rotation. We first apply Newton's second law for $F_t$. $F_t = ma_t$ (eqs.1)

The torque $\tau$ acting on the particle is: $\tau = F_t r$ (eqs.2) We eliminate $F_t$ between equations 1 and 2: $\tau = ma_t r = m(\alpha r)r = (mr^2)\alpha = I\alpha$

$$\boxed{\tau = I\alpha}$$ (compare with: $F = ma$)
Newton's Second Law for Rotation

We have derived Newton's second law for rotation for a special case. A rigid body which consists of a point mass $m$ at the end of a massless rod of length $r$. We will now derive the same equation for a general case.

Consider the rod-like object shown in the figure which can rotate about an axis through point O under the action of a net torque $\tau_{net}$. We divide the body into parts or "elements" and label them. The elements have masses $m_1, m_2, m_3, \ldots, m_n$ and they are located at distances $r_1, r_2, r_3, \ldots, r_n$ from O. We apply Newton's second law for rotation to each element: $\tau_1 = I_1 \alpha$ (eqs.1), $\tau_2 = I_2 \alpha$ (eqs.2), $\tau_3 = I_3 \alpha$ (eqs.3), etc. If we add all these equations we get:

$$\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n = (I_1 + I_2 + I_3 + \ldots + I_n) \alpha.$$  Here $I_i = m_i r_i^2$ is the rotational inertia of the $i$-th element. The sum $\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n$ is the net torque $\tau_{net}$ applied. The sum $I_1 + I_2 + I_3 + \ldots + I_n$ is the rotational inertia $I$ of the body.

Thus we end up with the equation: $\tau_{net} = I \alpha$
A block of mass $m$ hangs from a string wrapped around a frictionless pulley of mass $M$ and radius $R$. If the block descends from rest under the influence of gravity, what is the angular acceleration of the pulley?
In chapter 7 we saw that if a force does work \( W \) on an object, this results in a change of its kinetic energy \( \Delta K = W \). In a similar way, when a torque does work \( W \) on a rotating rigid body, it changes its rotational kinetic energy by the same amount.

Consider the simple rigid body shown in the figure which consists of a mass \( m \) at the end of a massless rod of length \( r \). The force \( \vec{F} \) does work \( dW = F_r r d\theta = \tau d\theta \). The radial component \( F_r \) does zero work because it is at right angles to the motion.

The work \( W = \int F_r r d\theta = \int \tau d\theta \). By virtue of the work-kinetic energy theorem we have a change in kinetic energy \( \Delta K = W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \frac{1}{2} mr^2 \omega_f^2 - \frac{1}{2} mr^2 \omega_i^2 \rightarrow W = \Delta K \)

\[
\Delta K = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 \\
W = \int_{\theta_i}^{\theta_f} \tau d\theta
\]
Conservation of Energy in Rotational Motion

A meter stick is initially standing vertically on the floor. If it falls over, with what angular velocity will it hit the floor? Assume that the end in contact with the floor doesn’t slip.

1. Given: $l = 1\text{m}$
2. Initial position: $\theta_i = 0, \omega_i = 0$
3. Find $\omega_f$?
Power

Power has been defined as the rate at which work is done by a force and in the case of rotational motion by a torque. We saw that a torque $\tau$ produces work $dW = \tau d\theta$ as it rotates an object by an angle $d\theta$.

$$P = \frac{dW}{dt} = \frac{d}{dt} \left( \tau d\theta \right) = \tau \frac{d\theta}{dt} = \tau \omega \quad \text{(Compare with } P = Fv)$$

Below we summarize the results of the work-rotational kinetic energy theorem:

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

For constant torque

$$W = \tau \left( \theta_f - \theta_i \right)$$

Work-Rotational Kinetic Energy Theorem

$$W = \Delta K = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

$$P = \tau \omega$$
Analogies between translational and rotational Motion

<table>
<thead>
<tr>
<th>Translational Motion</th>
<th>Rotational Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>( v )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>( a )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( v = v_0 + at )</td>
<td>( \omega = \omega_0 + \alpha t )</td>
</tr>
<tr>
<td>( x = x_o + v_o t + \frac{at^2}{2} )</td>
<td>( \theta = \theta_o + \omega_o t + \frac{\alpha t^2}{2} )</td>
</tr>
<tr>
<td>( v^2 - v_o^2 = 2a(x - x_o) )</td>
<td>( \omega^2 - \omega_o^2 = 2\alpha(\theta - \theta_o) )</td>
</tr>
<tr>
<td>( K = \frac{mv^2}{2} )</td>
<td>( K = \frac{I\omega^2}{2} )</td>
</tr>
<tr>
<td>( m )</td>
<td>( I )</td>
</tr>
<tr>
<td>( F = ma )</td>
<td>( \tau = I\alpha )</td>
</tr>
<tr>
<td>( F )</td>
<td>( \tau )</td>
</tr>
<tr>
<td>( P = Fv )</td>
<td>( P = \tau\omega )</td>
</tr>
</tbody>
</table>