Tentative Schedule:

	Date	Module	Topics
1	Aug. 25 (Mo)	Module 1. Spontaneous and Stimulated Transitions	Introduction, Spontaneous and Stimulated Transitions (Ch. 1)
2	Aug. 27 (We)		Spontaneous and Stimulated Transitions (Ch. 1) Homework 1: PH481 Ch.1 problems 1.4 &1.6. PH581 Ch.1 problems 1.4, 1.6 & 1.8 due Sep.3 before class
	Sep. 1 (Mo) No classes		**Labor Day Holiday**
3	Sep. 3 (We)	Module 2. Optical Frequency Amplifiers	Optical Frequency Amplifiers (Ch. 2.1-2.4) Problem solving for Ch.1
4	Sep. 8 (Mo)		Optical Frequency Amplifiers (Ch. 2.5-2.10)
5	Sep. 10 (We)		Optical Frequency Amplifiers (Ch. 2.5-2.10) Homework 2: PH481 Ch.2 problems 2.2 (a,b), 2.4 & 2.5 (a,b). PH581 Ch.2 problems 2.2 (a,b), 2.4 & 2.5 (a,b,c,d) due Sep.22 before class
6	Sep. 15 (Mo)	Module 3. Introduction to two practical Laser Systems	Problem solving for Ch.2 Introduction to two Practical Laser Systems (The Ruby Laser, The Helium Neon Laser) (Ch. 3)
7	Sep. 17 (We)		Review Chapters 1 & 2
8	Sep. 22 (Mo)		**Exam 1** Over Chapters 1-3; Grades for exam 1
9	Sep. 24 (We)	Module 4. Passive Optical Resonators	Exam 1 problem solving. Passive Optical Resonators – Lecture Notes
10	Sep. 29 (Mo)		Passive Optical Resonators – Lecture Notes.
11	Oct. 1 (We)		Passive Optical Resonators – Lecture Notes. Physical significance of χ' and χ'' (Ch.2.8-2.9). Homework 3: read Ch.2 & notes. Work out problems (see Canvas). Due Oct. 8
12	Oct. 6 (Mo)	Module 5. Optical Resonators Containing Amplifying Media	Optical Resonators Containing Amplifying Media (4.1-2).
13	Oct. 8 (We)		Optical Resonators Containing Amplifying Media (Ch.4.3-4.7) Homework 4: Ch. 4 problems 4.7 and 4.9. Due Oct 15.
14	Oct. 13 (Mo)	Module 6. Laser Radiation	Laser Radiation (Ch. 5.1-5.4)
15	Oct. 15 (We)	Module 7. Control of Laser Oscillations	Control of Laser Oscillators (6.1-6.3) Homework 5: Ch. 5 problems 5.1 and 5.5. Due Oct 29.
16	Oct. 20 (Mo)		Control of Laser Oscillators (6.4-6.5) and exam 2 review
17	Oct. 22 (We)	Module 8. Optically Pumped Solid State Lasers	Optically Pumped Solid State Lasers (7.1-7.11)
18	Oct. 27 (Mo)		Optically Pumped Solid State Lasers (7.1-7.11)
19	Oct. 29 (We)		**Exam 2 Over Chapters 4-6** Grades for exam 2 Exam 2 correct solution; Homework 6 Due Nov.5; see Canvas including article on Cr:CdSe
20	Nov. 3 (Mo)	Module 8. Optically Pumped Solid State Lasers	Optically Pumped Solid State Lasers (7.14-7.15)
21	Nov. 5 (We)		Optically Pumped Solid State Lasers (7.16-7.17) Homework 7 (see Canvas) Due Nov. 17
22	Nov. 10 (Mo)	Module 9. Spectroscopy of Common Lasers and Gas Lasers	Spectroscopy of Common Lasers and Gas Lasers (Ch. 8.1-8.10)
23	Nov. 12 (We)	Module10. Molecular Gas Lasers I	Molecular Gas lasers I (Ch. 9.1-9.5)
24	Nov. 17 (Mo)		Molecular Gas lasers I (Ch. 9.1-9.5) Homework 8 (see Canvas) Due Dec. 1
25	Nov. 19 (We)	Module 11. Molecular Gas Lasers II	Molecular Gas Lasers II (Ch. 10.1-10.8) and review for exam 3 (Ch. 10.1-0.8) Homework 9 (see Canvas) Due Dec. 1
	Nov. 24 (Mo) No classes		**Thanksgiving - no classes held**
	Nov.26 (We) No classes		**Thanksgiving - no classes held**
26	Dec. 1 (Mo)		**Exam 3** Over Chapters 7-10 Grades; Exam 3 Correct solution
27	Dec. 3 (We)		Review for Final
28	**Dec. 8 (We) in ESH 3160**		**FINAL EXAM Over Chapters 1-10 (4:15-6:45pm) in ESH 3160 Final Grades**

Laser Physics I

PH481/581-VT1 (Mirov)

Lecture 3. Chapter 1 problem solving

Fall 2025

C. Davis, “Lasers and Electro-optics”

Phase velocity, group velocity, refractive index, group refractive index - definitions

The **phase velocity** of light is the velocity with which phase fronts propagate in a medium. It is related to the **wavenumber *k*** and the (angular) **optical frequency *ω***:

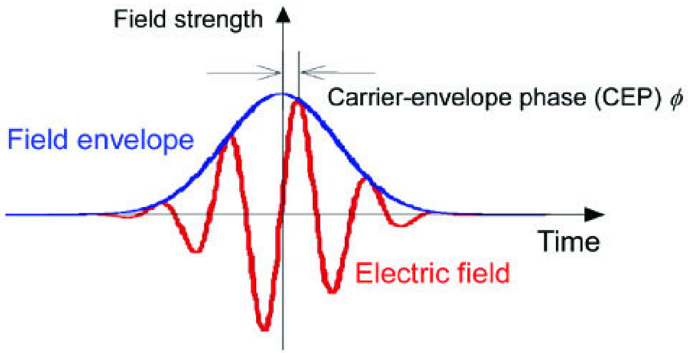
$$v_{ph} = \frac{\omega}{k}$$

In vacuum, the phase velocity is c = 299 792 458 m/s, independent of the optical frequency, and equals the **group velocity**. In a medium, the phase velocity is typically smaller by a factor n, called the **refractive index**, which is frequency-dependent In the visible spectral region, typical transparent crystals and optical glasses have refractive indices between 1.4 and 2.8. Semiconductors usually have higher values.

The **group velocity** of light in a medium is defined as the inverse of the derivative of the wavenumber with respect to angular optical frequency:

$$v_g = \left(\frac{\partial k}{\partial \omega}\right)^{-1} = c\left(\frac{\partial}{\partial \omega}(\omega n(\omega))\right)^{-1} = \frac{c}{n(\omega) + \omega \dfrac{\partial n}{\partial \omega}} = \frac{c}{n_g(\omega)}$$

where ***n(ω)*** is the **refractive index** and $\boldsymbol{n_g}$ is called the **group index**.



The group velocity is the velocity with which the **envelope of a pulse** propagates in a medium, assuming a not too short pulse with narrow bandwidth (so that higher-order chromatic dispersion is not relevant) and the absence of nonlinear effects (i.e., low enough optical intensities). Concerning the spatial shape, plane waves are assumed.

Due to **chromatic dispersion**, the group velocity in a medium is in general different from the phase velocity (typically smaller than the latter), and it is frequency-dependent; this effect is called **group velocity dispersion**. The difference between group velocity and phase velocity also changes the **carrier–envelope offset** of the pulse.

In analogy with the **refractive index**, the **group index** can be defined as the ratio of the group velocity in vacuum to the group velocity in the medium.

Problem (1.1)

In a dispersive medium the refractive index varies with wavelength. We can define a group refractive index by the relation

$$n_g = n - \lambda \frac{dn}{d\lambda}$$

(i) Prove that $n_g = n + \nu \frac{dn}{d\nu}$

$$\lambda = \frac{c}{\nu}$$

$$n_g = n - \frac{c}{\nu} \frac{dn}{d\frac{c}{\nu}} = n - \frac{\cancel{c}}{\nu} \frac{dn}{\cancel{c}\cdot(-\frac{1}{\nu^2}d\nu)} =$$

$$= n + \frac{\nu\, dn}{d\nu}$$

(ii) Prove that if a black-body cavity is filled with such a dispersive material then the radiation mode density, $\rho(\nu)$, satisfies

$$\rho(\nu) = \frac{8\pi\nu^2 n^2 n_g}{c_0^3}$$

See (1.57-1.59) ; $N_\nu = \frac{8\pi\nu^3}{3c^3} L^3$; $V = L^3$

$$\rho(\nu) = \frac{1}{V}\frac{dN_\nu(\nu)}{d\nu} = \frac{1}{L^3}\frac{d\left(\frac{8\pi\nu^3 n^3 L^3}{3c_0^3}\right)}{d\nu} = \frac{8\pi L^3}{L^3 \cdot 3c_0^3}\cdot\left[\frac{\nu^3 \cdot 3n^2 dn + n^3 3\nu^2 d\nu}{d\nu}\right] =$$

$$\left[\begin{array}{l} d(a \times b) = a\,db + b\,da \\ a = \nu^3 \\ b = n^3 \end{array}\right]$$

$$= \frac{8\pi \cancel{L^3}}{\cancel{L^3}\cdot\cancel{3}c_0^3}\cdot \cancel{3}\nu^2 n^2 \underbrace{\left[\frac{\nu\, dn}{d\nu} + n\right]}_{n_g} =$$

$$= \boxed{\frac{8\pi\nu^2 n^2 n_g}{c_0^3}}$$

Problem 1.1 continuation

(iii) Prove also that in such a situation

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3 n^2 n_g}{c_0^{\,3}}$$

$$\rho(\nu) = \frac{8\pi \nu^2 n^2 n_g}{c_0^{\,3}} \times h\nu \times \left(\frac{1}{e^{h\nu/kT}-1}\right)$$

$$0 = \frac{dN_2}{dt} = -N_2 B_{21}\,\rho(\nu) - A_{21}N_2 + N_1 B_{12}\,\rho(\nu)$$

$$N_2\left[B_{21}\cdot\frac{8\pi h\nu^3 n^2 n_g}{c_0^{\,3}(e^{h\nu/kT}-1)} + A_{21}\right] = N_1\left[B_{12}\,\frac{8\pi h\nu^3 n^2 n_g}{c_0^{\,3}(e^{h\nu/kT}-1)}\right]$$

$$\frac{N_2}{N_1} = e^{-\frac{h\nu}{kT}} \implies N_1 = N_2\cdot e^{\frac{h\nu}{kT}}$$

$$\cancel{N_2\cdot e^{\frac{h\nu}{kT}}} \qquad A_{21} = \frac{8\pi h\nu^3 n^2 n_g}{c_0^{\,3}(e^{h\nu/kT}-1)}\left[B_{12}\cdot e^{\left(\frac{h\nu}{kT}\right)} - B_{21}\right]$$

$$\frac{A_{21}}{B_{12}\,e^{h\nu/kT}-B_{21}} = \frac{8\pi h\nu^3 n^2 n_g}{c_0^{\,3}(e^{h\nu/kT}-1)}$$

$$\implies B_{12} = B_{21}$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3 n^2 n_g}{c_0^{\,3}\cancel{(e^{h\nu/kT}-1)}}\cdot\cancel{(e^{h\nu/kT}-1)} = \boxed{\frac{8\pi h\nu^3 n^2 n_g}{c_0^{\,3}}}$$

Problem 1.2.

Calculate the photon flux (photons $\cdot m^{-2} \cdot s^{-1}$) in a plane monochromatic wave of intensity 100 W/m^2 at a wavelength of

(a) 100 nm

(b) 100 μm

$$N_{photons} = \frac{I(\nu_{21})}{h\nu} \quad \left[\frac{photons}{m^2 \cdot s}\right]$$

$$\nu = \frac{c}{\lambda}$$

(a)

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^{8}\ m/s}{100 \times 10^{-9}\ m} = 3 \times 10^{15}\ Hz$$

$$h\nu = 6.62 \times 10^{-34}\ J \cdot s \times 3 \times 10^{15}\ Hz = 2 \times 10^{-18}\ J$$

$$N_{photons} = \frac{100\ \frac{J}{s \cdot m^2}}{2 \times 10^{-18}\ J} = \boxed{5 \times 10^{19}\ \frac{photons}{s \cdot m^2}}$$

(b)

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^{8}\ m/s}{100 \times 10^{-6}\ m} = 3 \times 10^{12}\ Hz$$

$$h\nu = 6.62 \times 10^{-34}\ J \cdot s \times 3 \times 10^{12}\ Hz = 2 \times 10^{-21}\ J$$

$$N_{photons} = \frac{100\ \frac{J}{s \cdot m^2}}{2 \times 10^{-21}\ J} = \boxed{5 \times 10^{22}\ \frac{photons}{m^2 \cdot s}}$$

Problem 1.3

What is the total # of modes per unit volume for visible light?

1) Visible light range is $\lambda = 400 \div 700$ nm

it corresponds to frequency range $\nu = \frac{c}{\lambda}\Big|_{700}^{400} =$

$$= \frac{(3\times10^{8} m/s)}{(400\times10^{-9} m)} \div \frac{(3\times10^{8} m/s)}{(700\times10^{-9} m)} = \underset{\nu_2 \quad \text{unit} \quad \nu_1}{\underline{7.5\times10^{14} \div 4.29\times10^{14}\ Hz}}$$

2) The # modes per unit volume per ν frequency interval $\rho(\nu) = \frac{8\pi\nu^2}{c^3}$

Total # of modes per unit volume in frequency interval from ν_1 to ν_2 is

$$\int_{\nu_1}^{\nu_2} \rho(\nu)\,d\nu = \int_{\nu_1}^{\nu_2} \frac{8\pi\nu^2}{c^3}\,d\nu =$$

$$= \frac{8\pi}{c^3}\int_{\nu_1}^{\nu_2} \nu^2\,d\nu = \frac{8\pi}{c^3}\frac{\nu^3}{3}\Bigg|_{4.29\times10^{14}}^{7.5\times10^{14}} =$$

$$= \frac{8\pi}{(3\times10^{8} m/s)^3}\cdot\left[(7.5\times10^{14})^3 - (4.29\times10^{14})^3\right] =$$

$$= \frac{8\pi\,(10^{14})^3}{(3\times10^{8} m/s)^3}\left[(7{,}5^3 - 4{,}29^3)\,Hz^3\right] =$$

$$= \frac{8\pi\cdot10^{42}}{27\times10^{24}}\cdot 343\,[m^{-3}] = 3.2\times10^{20}\ \frac{\text{modes}}{m^3}$$

Problem 1.9

At what temperature would be stimulated and spontaneous emission rates be equal for particles in a cavity and a transition at a wavelength of $1 \mu m$

1) $\cancel{N_2} B_{21} \, \rho(\nu) = A_{21} \cancel{N_2}$

2) $\rho(\nu) = \frac{8\pi h \nu^3}{c^3} \cdot \left(\frac{1}{e^{h\nu/KT} - 1} \right)$

3) $\nu = \frac{c}{\lambda} = \frac{3 \times 10^8 \, m/s}{1 \times 10^{-6} \, m} = 3 \times 10^{14} \, Hz$

4) $\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3}$

$$\cancel{B_{21}} \cdot \cancel{\frac{8\pi h\nu^3}{c^3}} \cdot \left(\frac{1}{e^{h\nu/KT} - 1} \right) = \cancel{B_{21}} \times \cancel{\frac{8\pi h\nu^3}{c^3}}.$$

$$\Rightarrow e^{h\nu/KT} - 1 = 1$$

$$e^{h\nu/KT} = 2$$

$$\frac{h\nu}{KT} = \ln 2$$

$$T = \frac{h\nu}{K \ln 2} = \frac{(6.62 \times 10^{-34} \, J \cdot s) \times 3 \times 10^{14} \, s^{-1}}{1.38 \times 10^{-23} \, J/K \cdot \ln 2} = \boxed{20762 \, ^\circ K}$$

Laser Physics I

PH481/581-VTA (Mirov)

Optical Frequency Amplifiers

Lectures 3-5 chapter 2

Fall 2025

C. Davis, “Lasers and Electro-optics”

Optical Frequency Amplifiers

The Intensity (I) of a light wave, propagating through a medium can be changed due to Stimulated emission and absorption processes.
I ↑ if a number of stimulated emissions is larger than — " ————— absorptions.
than we have built a light amplifier.

- Laser amplifier has useful gain over a particular frequency bandwidth.
- The operating frequency range will be determined by the lineshape of the transition.
- Line broadening affects in fundamental way not only the frequency bandwidth of the amplifier, but also its gain.

To turn a laser amplifier into an oscillator we need to supply an appropiate amount of positive feedback.
The level of oscillation will stabilize because the amplifier saturates.
Two categories of laser amplifiers that saturates in different ways.

- The homogeneously broadened amplifier consists of a number of amplifying particles that are equivalent.
- The inhomogeneously broadened amplifier consists of particles with a distribution of amplification characteristics.

Homogeneous Line Broadening

All energy states of atoms, molecules or ions are broadened over a finite range of energies.

At the fundamental level this broadening of the energy is caused by the uncertainty involved in the energy measurement process.

This gives rise to an intrinsic and unavoidable amount of line broadening called **natural broadening**.

Natural Broadening

The uncertainty in measured energy, ΔE, arises from the time, Δt, involved in making such measurement.

$$\boxed{\Delta E \cdot \Delta t \sim \hbar}$$

Heisenberg's Principle

$\hbar = \frac{h}{2\pi}$

$h = 6.62 \times 10^{-34}\ J \cdot s$

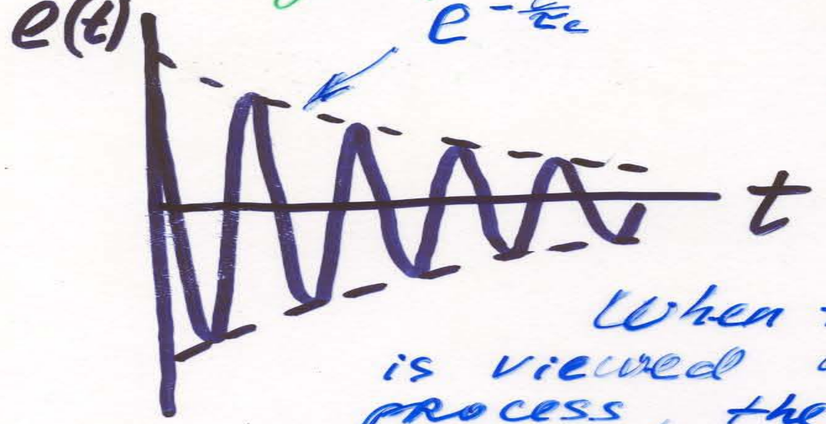
An excited particle can only be observed for a time $\Delta t \approx \tau$ $\Rightarrow$

$$\boxed{\Delta E \sim \frac{\hbar}{\tau} = A\hbar}$$

$$\boxed{\Delta \nu = \frac{\Delta E}{h} \sim \frac{\hbar}{h\tau} = \frac{\hbar \cdot A}{2\pi\hbar} = \frac{A}{2\pi}}$$

uncertainty in emitted frequency

Consider the exponential intensity decay of a group of excited atoms.



The decay of each individual excited atom is modelled as an exponentially decaying (damped) cosinusoidal oscillation.

When the decay of the excited atom is viewed as a photon emission process, the atom initially placed in the excited state at time $t'=0$, emits a photon at time t. The distribution of these times t among many such atoms varies as $e^{-\frac{t}{\tau_c}}$.

The knowledge of when the photon is likely to be emitted with respect to $t=0$, restricts our ability to be sure of its frequency.

The electric field of a decaying excited particle

$$e(t) = E_0 e^{-\frac{t}{\tau_c}} \cos \omega_0 t$$

The instantaneous intensity $i(t)$ emitted by an individual excited atom is

$$i(t) \sim |e(t)|^2 = E_0^2 e^{-\frac{2t}{\tau_c}} \cos^2 \omega_0 t$$

If we observe many such atoms the total observed intensity is

$$I(t) = \sum_{\text{particles}} i(t) = \sum_i E_0^2 e^{-\frac{2t}{\tau_c}} \cos^2(\omega_0 t + \varepsilon_i) =$$

$$= \sum_i \frac{E_0^2}{2} e^{-\frac{2t}{\tau_c}} \left[1 + \cos 2(\omega_0 t + \varepsilon_i)\right]$$

τ_c - time constant

$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\Rightarrow \boxed{\cos^2\alpha = \frac{\cos 2\alpha + 1}{2}}$

ε_i - is the phase of the wave emitted by atom i. individual atoms are emitting with random phases $\Rightarrow$ in the summation the cosine term gets smeared

$$\Rightarrow I(t) \sim e^{-\frac{2t}{\tau_c}} \text{ ; } \tau_c \text{ - time constant}$$

also we know $I(t) \sim e^{-\frac{t}{\tau}}$; τ - lifetime of the state.

$$\Rightarrow \tau_c = 2\tau$$

$$\Rightarrow \boxed{e(t) = E_0 e^{-\frac{t}{2\tau}} \cos \omega_0 t}$$

The electric field of ~~a~~ decaying excited particle

$$e(t) = E_0 \, e^{-\frac{t}{2\tau}} \cos \omega_0 t$$

To find the frequency distribution of this signal we take its Fourier transform

$$E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e(t)\, e^{-i\omega t}\, dt,$$

where

$$e(t) = \frac{E_0}{2}\left(e^{i(\omega_0 + \frac{i}{2\tau})t} + e^{-i(\omega_0 - \frac{i}{2\tau})t}\right) \text{ for } t > 0$$

$$e(t) = 0 \quad \text{for } t < 0$$

The start of the period of observation at $t=0$, taken at an instant when all the particles are pushed into the excited state, allows the lower limit of integration to be changed to 0.

$$E(\omega) = \frac{1}{2\pi} \int_{0}^{\infty} e(t)\, e^{-i\omega t}\, dt = \frac{E_0}{4\pi}\left[\frac{i}{(\omega_0 - \omega + \frac{i}{2\tau})} - \frac{i}{(\omega_0 + \omega - \frac{i}{2\tau})}\right]$$

$$e(t) = E_0 \, e^{-\frac{t}{2\tau}} \cos \omega_0 t$$

$$\boxed{\cos \omega_0 t = \frac{e^{+i\omega_0 t} + e^{-i\omega_0 t}}{2}}$$ Euler formula $\Rightarrow$

$$e(t) = E_0 \left[\frac{e^{(+i\omega_0 t - \frac{t}{2\tau})} + e^{(-i\omega_0 t - \frac{t}{2\tau})}}{2}\right] =$$

since $i^2 = -1$

$$= \frac{E_0}{2}\left[e^{it(\omega_0 + \frac{i}{2\tau})} + e^{-it(\omega_0 - \frac{i}{2\tau})}\right]$$

The intensity of emitted radiation is

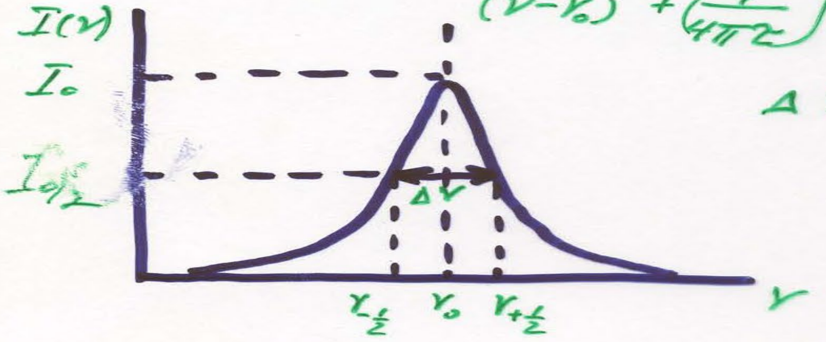
conjugate

$$I(\omega) \sim |E(\omega)|^2 = E(\omega)\cdot E^*(\omega) \sim \frac{1}{(\omega-\omega_0)^2+\left(\frac{1}{2\tau}\right)^2}$$

In terms of ordinary frequency

$$I(\nu) \sim \frac{1}{(\nu-\nu_0)^2+\left(\frac{1}{4\pi\tau}\right)^2}$$

Lorentzian lineshape function for natural broadening



$\Delta\nu$ – the full width at half maximum height (FWHM)

this occurs when

$$\left(\frac{1}{4\pi\tau}\right)^2 = \left(\nu_{\pm\frac{1}{2}}-\nu_0\right)^2$$

$$\Rightarrow \Delta\nu = \nu_{+\frac{1}{2}} - \nu_{-\frac{1}{2}} = \left(\nu_{+\frac{1}{2}}-\nu_0\right)+\left(\nu_0-\nu_{-\frac{1}{2}}\right) =$$

$$= 2\cdot\frac{1}{4\pi\tau} = \frac{1}{2\pi\tau} = \frac{A}{2\pi}$$

$$\Rightarrow I(\nu) \sim \frac{1}{(\nu-\nu_0)^2+\left(\frac{\Delta\nu}{2}\right)^2}$$

The lineshape function for natural broadening

$$\boxed{g(\nu)_N = \frac{\frac{2}{\pi\Delta\nu}}{1+[2(\nu-\nu_0)/\Delta\nu]^2}}$$

$$\boxed{= \frac{2\Delta\nu}{\pi[4(\nu-\nu_0)^2+\Delta\nu^2]}} = \frac{1}{2\pi}\frac{\Delta\nu}{(\nu-\nu_0)^2+\left(\frac{\Delta\nu^2}{2}\right)}$$

Natural broadening is the same for each particle => it is a homogeneous broadening mechanism.

Other mechanisms of homogeneous broadening

1. Collision of phonons with the particles of the lattice perturb the phase of any excited, emitting particles. – soft collision.

Constant vibrational motion of the crystalline lattice particles can carry energy in discrete amounts. The packets of acoustic energy are called phonons.

2. By pressure broadening: interaction of the emitting particle with its neighbors causes perturbation of its emitting frequency and broadening of the transition.

2a) Collisions with neutral particles.

2b) Collisions with charged particles

Stark broadening – external electric
caused by charged particle
field) perturbs the energy level of atom, ion, molecule

2c) Van der Waals and resonance interaction

Excited particle exchange energy with like neighbors.

Inhomogeneous Broadening

- When the environment of particles in an emitting sample are non-identical, inhomogeneous broadening can occur.
- The shifts and perturbations of emission frequencies differ from particle to particle.

In a real crystal the presence of imperfections and impurities in the crystal structure alters the physical environment of atoms from one lattice site to another. The random distribution of lattice point environments leads to a distribution of particles whose center frequencies are shifted in a random way throughout the crystal.

Doppler Broadening

- In a gas the random distribution of particle velocities leads to a distribution in the emission center frequencies of different emitting particles seen by stationary observer.
- if v_x – component of atom's velocity towards the observer than the observed frequency of the transition

$$\nu = \nu_0 + \frac{v_x}{c}\nu_0 \; ; \quad \nu_0 \text{ – stationary frequency}$$

- The Maxwell-Boltzman distribution of atomic velocities for particles of mass M at T.

$$f(v_x, v_y, v_z) = \left(\frac{M}{2\pi KT}\right)^{3/2} \exp\left[-\frac{M}{2KT}\left(v_x^2 + v_y^2 + v_z^2\right)\right]$$

- If N is the total # of atoms per unit volume than the # of atoms per unit volume that have velocities simultaneously in the range

$$v_x \to v_x + dv_x$$
$$v_y \to v_y + dv_y$$
$$v_z \to v_z + dv_z$$

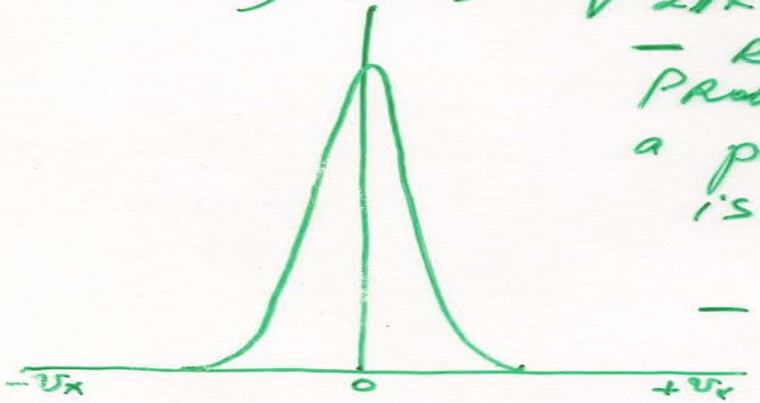
is $\boxed{N \cdot f(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z}$

- The $\left(\frac{M}{2\pi KT}\right)^{3/2}$ factor is a normalization constant that ensures

$$\iiint_{-\infty}^{\infty} f(v_x, v_y, v_z)\, dv_x\, dv_y\, dv_z = 1$$

- The normalized one-dimensional distribution of velocities for the particles in a gas

$$f(v_x) = \sqrt{\frac{M}{2\pi KT}} \cdot e^{-\frac{Mv_x^2}{2KT}}$$



— Represents the Probability that the velocity of a particle towards an observer is in a range $v_x \to v_x + dv_x$

— it is the same as probability that the frequency be in the range.

$$\nu_0 + \frac{v_x}{c}\nu_0 \to \nu_0 + \left(\frac{v_x + dv_x}{c}\right)\nu_0 =$$

$$= \nu_0 + \frac{v_x}{c}\nu_0 + \frac{dv_x}{c}\nu_0$$

- the probability that the frequency lies in the range $\nu \to \nu + d\nu$ is the same as the probability of finding the velocity in the range

$$\upsilon_x + d\upsilon_x = \frac{(\nu-\nu_0)c}{\nu_0} \to \frac{(\nu-\nu_0)c}{\nu_0} + \frac{c}{\nu_0}d\nu$$

$$\upsilon_x \to \upsilon_x + d\upsilon_x$$

$$\nu = \nu_0 + \frac{\upsilon_x}{c}\nu_0 \quad \text{- Doppler frequency}$$

$$\Rightarrow \upsilon_x = \frac{(\nu-\nu_0)c}{\nu_0}$$

$$\upsilon_x + d\upsilon_x = \frac{(\nu-\nu_0)c}{\nu_0} + \frac{c}{\nu_0}d\nu$$

$\Rightarrow$ the distribution of the emitted frequencies is

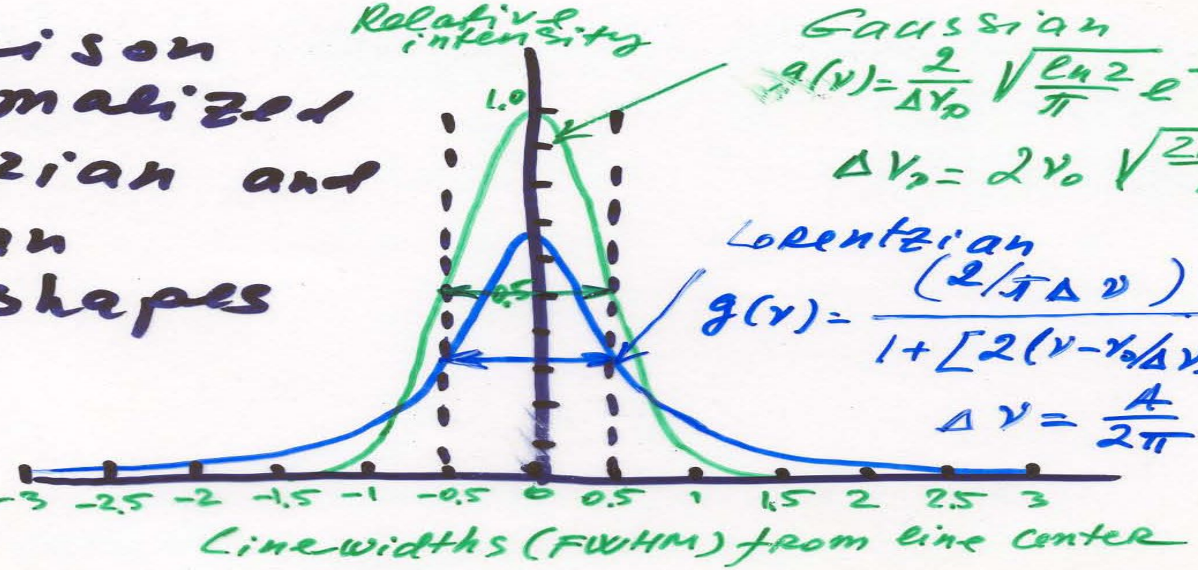
$$\boxed{g(\nu) = \frac{c}{\nu_0}\sqrt{\frac{M}{2\pi KT}}\exp\left[\left(-\frac{M}{2KT}\right)\left(\frac{c^2}{\nu_0^2}\right)(\nu-\nu_0)^2\right]}$$

normalized **Doppler broadened lineshape function**

FWHM $\boxed{\Delta\nu_D = 2\nu_0\sqrt{\frac{2KT\ln 2}{Mc^2}}}$

$$\boxed{g(\nu) = \frac{2}{\Delta\nu_D}\sqrt{\frac{\ln 2}{\pi}}\, e^{-[2(\nu-\nu_0)/\Delta\nu_D]^2 \ln 2}}$$

Comparison of normalized Lorentzian and Gaussian lineshapes



Gaussian

$$g(\nu)=\frac{2}{\Delta\nu_D}\sqrt{\frac{\ln 2}{\pi}}\, e^{-\left[2(\nu-\nu_0)/\Delta\nu_D\right]^2 \ln 2}$$

$$\Delta\nu_D = 2\nu_0\sqrt{\frac{2kT\ln 2}{Mc^2}}$$

Lorentzian

$$g(\nu)=\frac{(2/\pi\Delta\nu)}{1+\left[2(\nu-\nu_0)/\Delta\nu\right]^2}$$

$$\Delta\nu = \frac{A}{2\pi}$$

Example:

$\lambda_0 = 632.8\,nm$ transition of neon is the most important transition for laser oscillation in He-Ne laser. Atomic mass of neon is 20 $\frac{g}{mole}$

using

$$\Rightarrow M = \frac{\mu}{N_A} = \frac{20\,g/mole}{6.02\times10^{23}\,particles/mole} = 3.3\times10^{-23}\,g =$$

$$= \boxed{3.3\times10^{-26}\,kg}$$

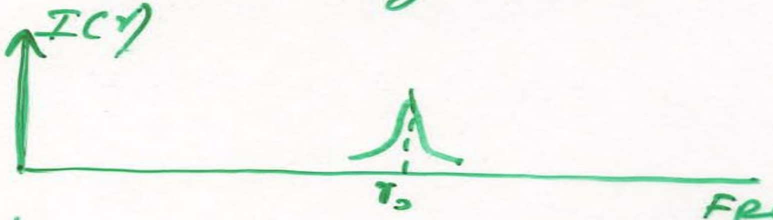
$$\nu_0 = \frac{c}{\lambda_0} = \frac{3\times10^{8}\,m/s}{632.8\times10^{-9}\,m} = 4.74\times10^{14}\,Hz$$

and $T = 400\,K$

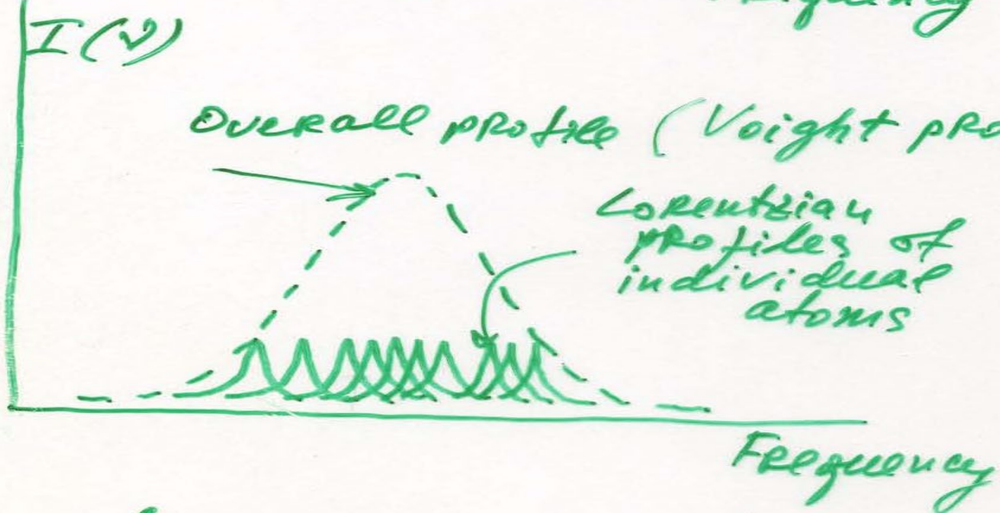
$$\Delta\nu_D = 2\nu_0\sqrt{\frac{2kT\ln 2}{Mc^2}} = 2\times(4.7\cdot10^{14}\,Hz)\sqrt{\frac{2\cdot(1.38\times10^{-23}\,J/K)(400\,K)\cdot\ln 2}{(3.3\times10^{-26}\,kg)\cdot(3\times10^{8}\,m/s)^2}} =$$

$$= 1.52\times10^{9}\,Hz = \underline{1.52\,GHz}$$

Homogeneous broadening always occurs at the same time as inhomogeneous broadening, to a greater or lesser degree.



Homogeneous broadening of a group of particles in a gas that have the same velocity v and center frequency ν_0



Overall lineshape results from the superposition of Lorentzian lineshapes spread across the Gaussian distribution of Doppler shifted center frequencies.

- if $\Delta\nu_L << \Delta\nu_D$ − overall lineshape is gaussian
  pure inhomogeneously broadened system
- if $\Delta\nu_D << \Delta\nu_L$ − homogeneously broadened system.

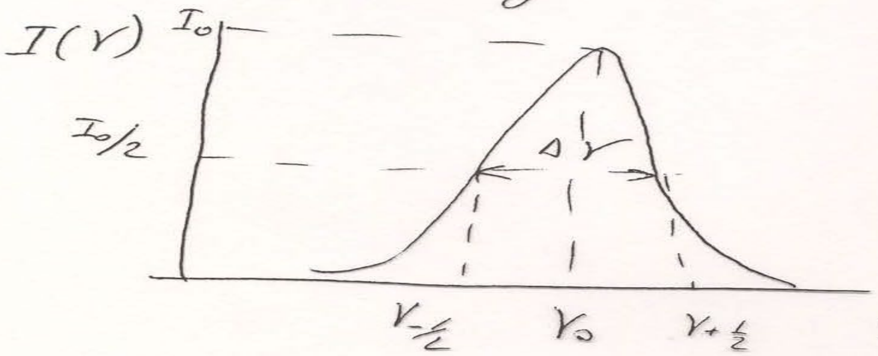
Homogeneous Line Broadening

1) $\Delta E \, \Delta t \sim \hbar$ Heisenberg's uncertainty principle

$\Delta t \simeq \tau \implies \Delta E \sim \frac{\hbar}{\tau} = A\hbar$

$\Delta \nu = \frac{\Delta E}{h} \implies \Delta \nu = \frac{A\hbar}{h} = \frac{A\,\cancel{h}}{2\pi\,\cancel{h}} = \frac{A}{2\pi}$

2) Lorentzian lineshape function for natural broadening

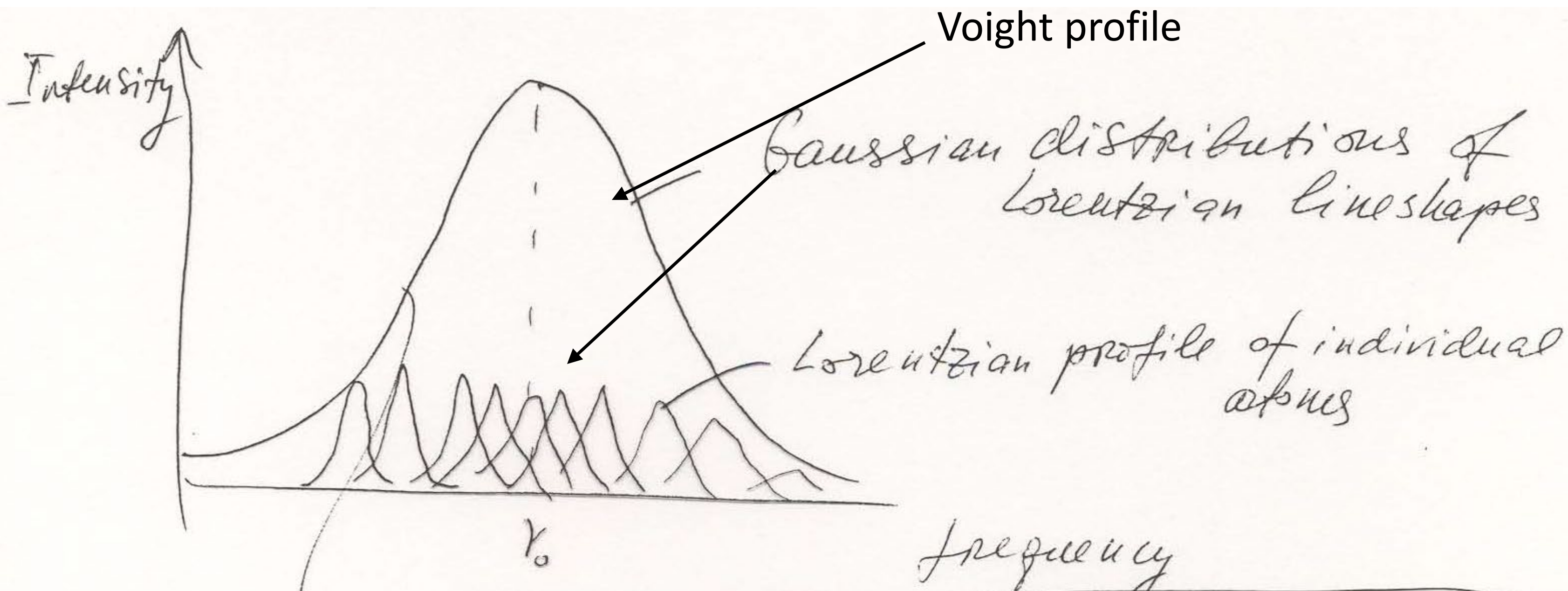


$$g(\nu)_N = \frac{\frac{2}{\pi \Delta \nu}}{1 + \left[2(\nu - \nu_0)/\Delta \nu\right]^2}$$

Mechanisms of homogeneous broadening

- natural broadening
- electron-phonon interactions
- collisional broadening
- Stark broadening
- resonance interactions with unexcited particles.

Inhomogeneous Broadening

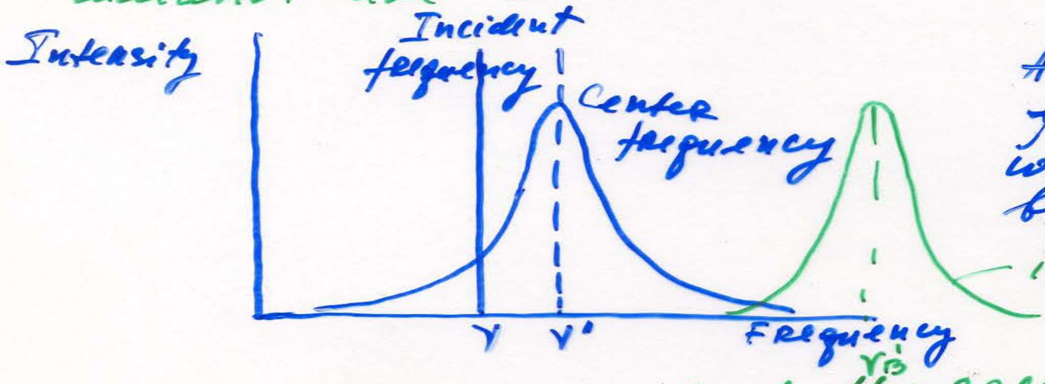


$$\boxed{g(\nu_0,\nu) = \frac{2}{\Delta\nu_D}\sqrt{\frac{\ln 2}{\pi}}\, e^{-\left[2(\nu-\nu_0)/\Delta\nu_D\right]^2 \ln 2}}$$

$$\boxed{\Delta\nu_D = 2\nu_0\sqrt{\frac{2kT\ln 2}{Mc^2}}}$$ for gas medium.

Optical Frequency Amplification with a Homogeneously Broadened Transition

Consider general case, when the monochromatic radiation field and the center frequency of the radiation are not the same.



A monochromatic field interacting with a homogeneously broadened system.

interaction with particle B having center freq. ν_B' is negligible.

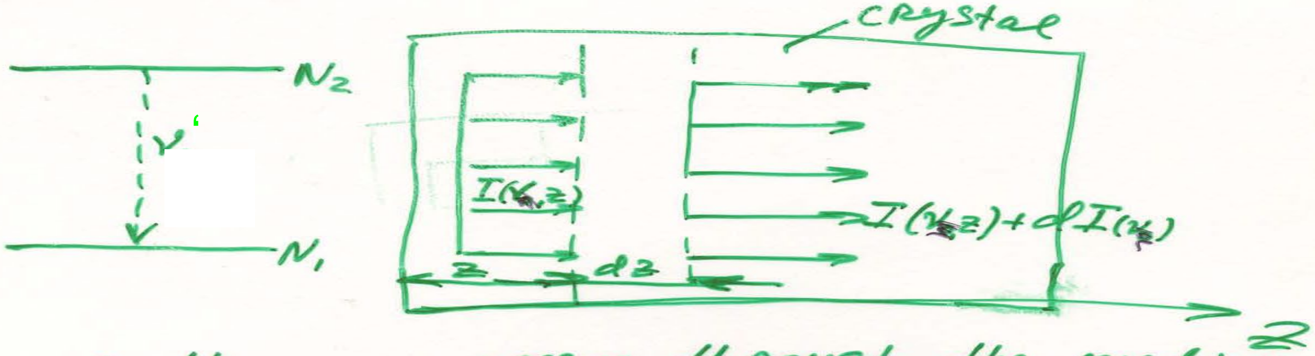
- the closer ν is to ν' the greater the # of transitions that can be stimulated
- the stimulated transitions occur at frequency of the stimulated radiation ν.
- the number of stimulated transitions

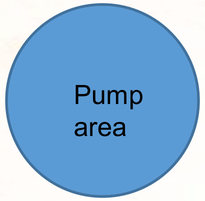
$$N_s = N_2 B_{21} \rho(\nu) g(\nu', \nu)$$

$$g(\nu', \nu) = \frac{(2/\pi \Delta\nu)}{1 + [2(\nu - \nu')/\Delta\nu]^2} = \frac{2}{\pi \Delta \nu} = \frac{2 \cdot 2\pi}{\pi A} = \frac{4}{A}$$

for $\nu = \nu'$

- $g(\nu', \nu)$ – homogeneous lineshape function of individ particles
- excited atom can interact with a monochromatic radiation that overlaps its homogeneous lineshape profile.





- As the wave passes through the medium it grows in intensity if the # of stimulated emissions exceeds the # of absorptions.
- the change in intensity of the wave in travelling a small distance dz through the medium:

$$dI_\nu = \frac{\#\text{ stim. emiss} - \#\text{ of abs}}{\text{volume}} \times h\nu \times dz =$$

$$= \left[N_2 B_{21} g(\nu',\nu)\frac{I_\nu}{c} - N_1 B_{12} g(\nu',\nu)\frac{I_\nu}{c}\right] \times h\nu \cdot dz$$

- using Einstein relations

$$dI_\nu = \frac{I_\nu}{c}\left(N_2 - \frac{g_2}{g_1}N_1\right)\frac{c^3 A_{21}}{8\pi h\nu^3}\cdot h\nu \cdot g(\nu',\nu)\,dz$$

$$\frac{dI_\nu}{dz} = \left(N_2 - \frac{g_2}{g_1}N_1\right)\frac{c^2 A_{21}}{8\pi\nu^2}\, g(\nu',\nu)\, I_\nu$$

solution $\boxed{I_\nu = I_\nu(0)\, e^{\gamma(\nu) z}}$ where $I_\nu(0)$ - int. at Z = 0

$$\boxed{\gamma = \left(N_2 - \frac{g_2}{g_1}N_1\right)\frac{c^2 A_{21}}{8\pi\nu^2}\, g(\nu',\nu)}$$ gain coefficient

Cross section of emission $\sigma = \frac{c^2 A_{21}}{8\pi\nu^2} g(\nu',\nu)$

$$\gamma = \left(N_2 - \frac{g_2}{g_1}N_1\right)\sigma$$

- if $N_2 > \frac{g_2}{g_1}N_1$ then $\gamma(\nu) > 0$ we have an optical frequency amplifier.

- if $N_2 < \frac{g_2}{g_1} N_1$ then $\gamma(\nu) < 0$ and we have net absorption of the incident radiation

- For a system in thermal equilibrium
$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT} \quad \text{for } T > 0 \quad e^{-\frac{h\nu}{kT}} < 1$$
we have no positive gain.

- For negative temperature we have population inversion $N_2 > \frac{g_2}{g_1} N_1$
we have positive gain.
  - It is not a true state of thermal equilibrium
  - it can be maintained by feeding energy into the system.

- in the discussion we have neglected the occurrence of spontaneous emission
  - total amount of spont. emissions into a small solid angle is very small.
$$N_2 A_{21} \, \delta\omega / 4\pi$$
  - there is no a constant phase relationship with the incident wave.

The Stimulated Emission rate in a Homogeneously Broadened Transition

The stimulated emission rate $W_{21}(\nu)$ is the # of stimulated emissions per particle per second per unit volume caused by a monochromatic input wave at frequency ν

$$A_{21} = B_{21}\frac{8\pi\nu^2}{c^3}h\nu;$$

$$W_{21}(\nu) = B_{21}\, g(\nu',\nu)\rho(\nu)$$

$$B_{21} = \frac{A_{21}c^3}{8\pi h\nu^3};\ \sigma = \frac{c^2 A_{21}}{8\pi\nu^2}g(\nu',\nu),\ \rho(\nu) = \frac{I(\nu)}{c}$$

$$W_{21}(\nu) = \frac{A_{21}c^2 I_\nu}{8\pi h\nu^3}g(\nu',\nu) = \frac{\sigma_e I_\nu}{h\nu}$$

$$\left[\frac{1}{m^3 \cdot s \cdot \text{particle}}\right]$$

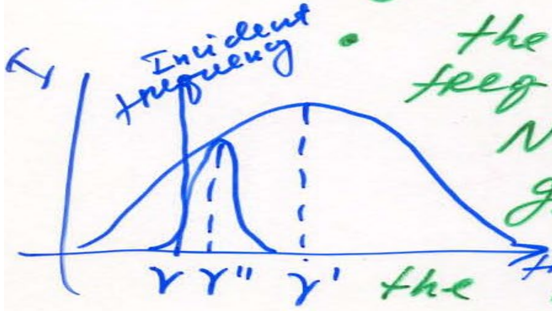
- frequency variation of $W_{21}(\nu)$ follows the lineshape function $g(\nu',\nu)$

The total # of stimulated emissions is

$$N_s = N_2 W_{21}(\nu)$$

Optical Frequency Amplification with Inhomogeneous Broadening

We can divide the atoms up into classes, each class consisting of atoms with a certain range of center emission frequencies and the same homogeneous lineshape.



- the class with center freq. ν'' in the freq. range $d\nu''$ has $N g_D(\nu'; \nu'')\, d\nu''$ atoms in it

  $g_D(\nu'; \nu'')$ normalized inhomogeneous distribution of center frequencies. — the inhomogeneous lineshape function centered at ν'.

- this class of atoms contributes to the change in intensity of a monochromatic wave at frequency ν as

$\Delta(dI_\nu)$ (from the group of particles in the band $d\nu''$) =

$$= \left[N_2 B_{21}\, g_D(\nu'; \nu'')\, d\nu''\, g_L(\nu''; \nu) \frac{I_\nu}{c} - N_1 B_{12}\, g_D(\nu'; \nu'' d\nu''\, g_L(\nu''; \nu) \frac{I_\nu}{c} \right] h\nu\, dz$$

where $g_L(\nu''; \nu)$ is the homogeneous lineshape function of an atom at center frequency ν''.

- The increase in intensity from all the classes of atoms is found by integrating over these classes over the range of frequencies ν''

$$dI_\nu = \frac{I_\nu}{c}(N_2 B_{21} - N_1 B_{12})\left[\int_{-\infty}^{+\infty} g_D(\nu_1',\nu'')\, g_L(\nu_1'',\nu)\, d\nu''\right] h\nu\, dz$$

Solution $\quad I_\nu = I_\nu(0)\, e^{\gamma(\nu) z}$

- $$\gamma(\nu) = \left(N_2 - \frac{g_2}{g_1}N_1\right)\frac{c^2 A_{21}}{8\pi\nu^2}\, g(\nu_1',\nu)$$

  where $g(\nu_1',\nu)$ is the overall lineshape function defined as

  $$g(\nu_1',\nu) = \int_{-\infty}^{+\infty} g_D(\nu_1',\nu'')\, g_L(\nu_1'',\nu)\, d\nu''$$

  convolution of the homogeneous and inhomogeneous lineshape functions.

- if we measure frequency relative to the center frequency of the overall lineshape $\nu' = 0$

$$g(0,\nu) = \int_{-\infty}^{+\infty} g_D(0,\nu'')\, g_L(\nu'',\nu)\, d\nu'' =$$

$$= \int_{-\infty}^{\infty} g_D(0,\nu'')\, g_L(0,\nu-\nu'')\, d\nu''$$

$$g(\nu) = \int_{-\infty}^{\infty} g_D(\nu'') g_L(\nu - \nu'') d\nu''$$

it is a standard convolution integral of two functions $g_D(\nu)$ and $g_L(\nu)$

- if $g_D(\nu'; \nu'')$ is Gaussian lineshape $g_L(\nu''; \nu)$ is Lorentzian then

$$g(\nu', \nu) = \frac{2}{\Delta\nu_D}\sqrt{\frac{\ln 2}{\pi}}\,\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{e^{-t^2}}{y^2 + (x-t)^2}dt \quad \text{Normalized Voight Profile}$$

$$y = \frac{\Delta\nu_L}{\Delta\nu_D}\sqrt{\ln 2},\ x = \frac{2(\nu - \nu')\sqrt{\ln 2}}{\Delta\nu_D},\ \text{and } t = \frac{2\delta\sqrt{\ln 2}}{\Delta\nu_D},$$

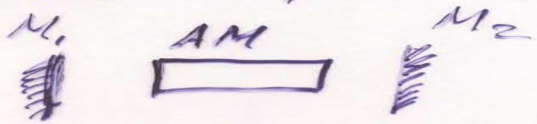
where δ is shift in central frequency due to molecules collision

The integral in Voight profile cannot be evaluated analytically but must be evaluated numerically.

- $\nu - \nu'$: distance from line center
- $\Delta\nu_L$: Lorentz half-width
- $\Delta\nu_D$: Doppler half-width

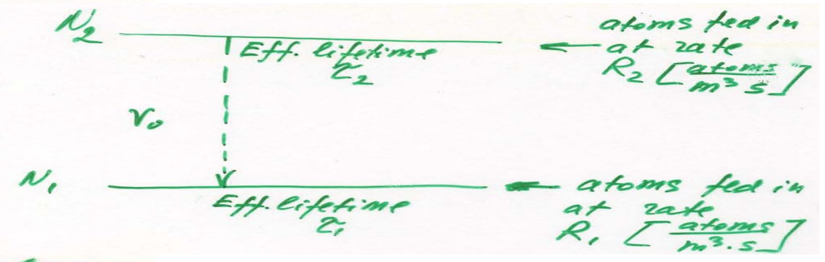
Optical Frequency Oscillation - Saturation

- if we can force a medium into a state of population inversion for a pair of its energy levels → the transitions between these levels forms an optical frequency amplifier
- to turn amplifier into an oscillator → apply positive feedback by inserting the medium between a pair of appropriate mirrors

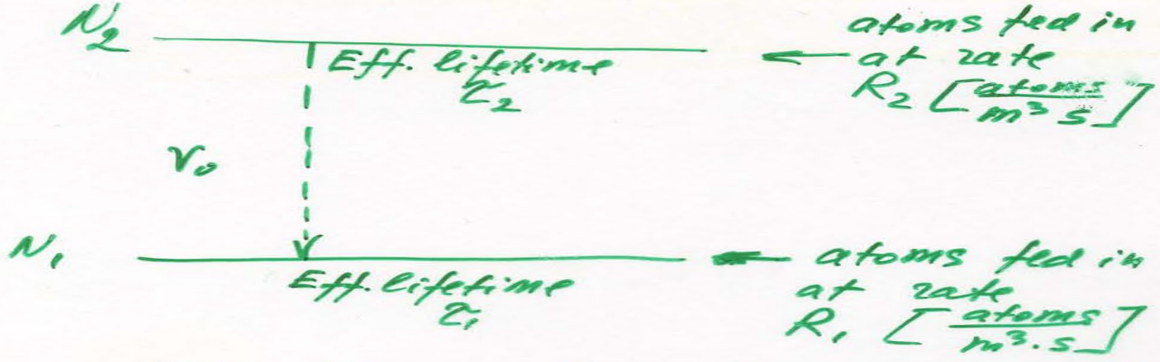


- oscillation when gain > losses
- the level at which the oscillation stabilizes is set by the way in which the amplifier saturates.

Homogeneous System



- consider an amplifying transition at center frequency ν_0 between two energy levels of an atom.
- maintain this pair of levels in population inversion by feeding in energy.
- in equilibrium if $g(\nu)=0$ (absence of an ext. field) the rates R_2 and R_1 at which atoms are fed into this levels must be balanced by spontaneous emission & nonradiative loss processes.



Effective lifetimes include the effect of non-radiative deactivation

If X_{2j} is the rate per particle per unit volume by which collisions depopulate level 2 and cause a particle to end up in a lower state j

$$\frac{1}{\tau_2} = \sum_j \left(A_{2j} + X_{2j} \right)$$

In equilibrium

$$\frac{dN_2^\circ}{dt} = R_2 - \frac{N_2^\circ}{\tau_2} = 0 \qquad \text{for level 2}$$

where $\frac{N_2^\circ}{\tau_2}$ - total loss rate per unit volume from spontaneous emission + other deactivation processes.

$$\rightarrow \quad N_2^\circ = R_2 \tau_2$$

$N_2^\circ \rightarrow$ indicates that the population is calculated in the absence of a radiat. field.

$$\frac{dN_1^\circ}{dt} = R_1 + N_2^\circ A_{21} - \frac{N_1^\circ}{\tau_1} \qquad \text{for level 1}$$

$$\Rightarrow \quad N_1^\circ = (R_1 + N_2^\circ A_{21})\tau_1 = (R_1 + R_2 \tau_2 A_{21})\tau_1$$

$$\left(N_2^\circ - \frac{g_2}{g_1} N_1^\circ\right) = \Delta N^\circ = R_2 \tau_2 - \frac{g_2}{g_1} \tau_1 (R_1 + R_2 \tau_2 A_{21})$$

the population of inversion.

when $g_1 = g_2$

$$\Delta N^\circ = R_2 \tau_2 - \tau_1 (R_1 + R_2 \tau_2 A_{21})$$

! Now feed in an external electric field radiation,

$$\rho(\nu) = \frac{I(\nu)}{c}$$

The rate at which this signal causes stimul. emis.

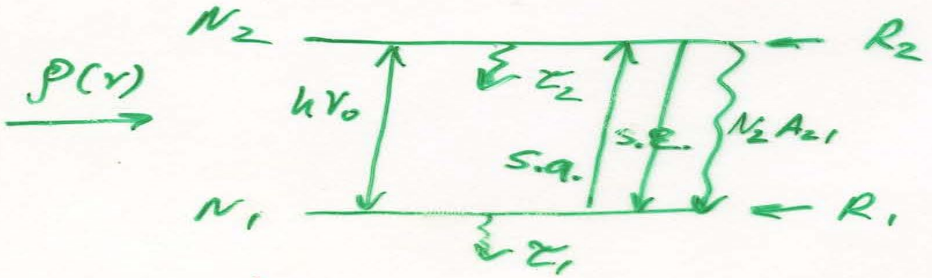
$$W_{21}(\nu) = \int_{-\infty}^{\infty} B_{21}\, g(\nu_0, \nu)\, \rho(\nu)\, d\nu$$

$g(\nu_0, \nu)$ — homogeneous lineshape function.

for a white radiation $W_{21}(\nu) = B_{21}\, \rho(\nu) \int_{-\infty}^{\infty} g(\nu_0, \nu)\, d\nu = B_{21}\, \rho(\nu)$

for a monochrom. plane wave $W_{21}(\nu) = \int_{-\infty}^{\infty} B_{21}\, g(\nu_0, \nu)\, \rho(\nu)\, d\nu =$

$$= \frac{I_\nu}{c} B_{21}\, g(\nu_0, \nu) \int_{-\infty}^{\infty} \delta(\nu - \nu'')\, d\nu'' = B_{21}\, g(\nu_0, \nu) \frac{I_\nu}{c}$$



$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - N_2 B_{21}\, g(\nu_0, \nu)\, \rho(\nu) + N_1 B_{12}\, g(\nu_0, \nu)\, \rho(\nu) = 0$$

$$\frac{dN_1}{dt} = R_1 + N_2 A_{21} - \frac{N_1}{\tau_1} + N_2 B_{21}\, g(\nu_0, \nu)\, \rho(\nu) - N_1 B_{12}\, g(\nu_0, \nu)\, \rho(\nu) = 0$$

if $B_{21}\, g(\nu_0, \nu)\, \rho(\nu) = W_{12}(\nu)$ and $g_1 = g_2$

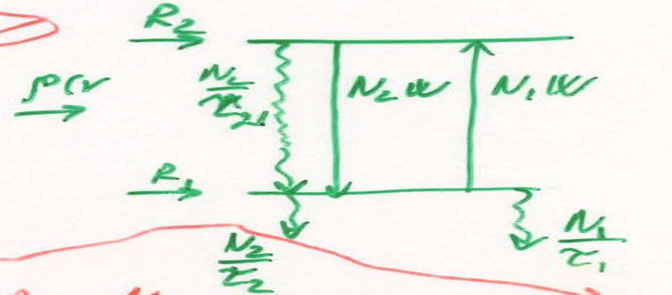
$W_{12}(\nu) = W_{21}(\nu) = W$ $\Rightarrow$

$$R_2 - \frac{N_2}{\tau_2} - N_2 W + N_1 W = 0 \qquad N_2 = \frac{N_1 W + R_2}{\frac{1}{\tau_2} + W}$$

$$R_1 + N_2 A_{21} - \frac{N_1}{\tau_1} + N_2 W - N_1 W = 0 \qquad N_2 = \frac{\frac{N_1}{\tau_1} + N_1 W - R_1}{A_{21} + W} \Rightarrow$$

$$\frac{N_1 W + R_2}{\frac{1}{\tau_2} + W} = \frac{\frac{N_1}{\tau_1} + N_1 W - R_1}{A_{21} + W} \Rightarrow$$

$$\Rightarrow N_1 = \frac{R_1/\tau_2 + R_2 A_{21} + W(R_1 + R_2)}{\frac{1}{\tau_1 \tau_2} + W\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} - A_{21}\right)}$$



$$N_2 - N_1 = \frac{N_1 W + R_2}{\frac{1}{\tau_2} + W} - N_1 = \frac{\cancel{N_1 W} + R_2 - \frac{N_1}{\tau_2} - \cancel{N_1 W}}{\frac{1}{\tau_2} + W} = \frac{R_2 - \frac{N_1}{\tau_2}}{\frac{1}{\tau_2} + W}$$

$$N_2 - N_1 = \frac{R_2 \tau_2 - R_1 \tau_1 - R_2 \tau_1 \tau_2 A_{21}}{1 + W\tau_2(1 + \tau_1/\tau_2 - A_{21}\tau_1)} = \frac{\Delta N^\circ}{1 + W\tau_2\left(1 + \frac{\tau_1}{\tau_2} - A_{21}\tau_1\right)} =$$

$$= \frac{\Delta N^\circ}{1 + \phi W/A_{21}}$$ where $\phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right]$

we know that $W = \frac{c^2 A_{21}}{8\pi h \nu^3} I_\nu \, g(\nu_0, \nu)$

if we define $I_s(\nu) = \frac{8\pi h \nu^3}{c^2 \phi \, g(\nu_0, \nu)}$

$$\boxed{N_2 - N_1 = \frac{\Delta N^\circ}{1 + I_\nu / I_s(\nu)}}$$

gain of a laser amplifier

$$\gamma(\nu) = (N_2 - N_1)\frac{c^2 A_{21}}{8\pi \nu^2} g(\nu_0, \nu)$$

The gain as a function of intensity in a homogeneously broadened system

$$\boxed{\gamma(\nu) = \frac{\Delta N^0}{\left[1 + I_\nu / I_s(\nu)\right]} \frac{c^2 A_{21}}{8\pi \nu^2} g(\nu_0, \nu)} \qquad I_s = \frac{8\pi h\nu^3}{c^2 \phi g(\nu_0, \nu)}$$

- gain saturates as the strength of the amplified signal increases.
- good amplifier should have a large value of I_s
- $I_s \uparrow \quad \phi \downarrow \qquad \phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right]$

$$A_{21} \approx \frac{1}{\tau_2} \Rightarrow \phi \approx 1$$

$$\gamma_0 = \left(N_2 - \frac{g_2}{g_1} N_1\right)\underbrace{\left(\frac{c^2 A_{21}}{8\pi \nu^2} g(\nu_0, \nu)\right)}_{\sigma_0} = \left(N_2 - \frac{g_2}{g_1} N_1\right)\sigma_0$$

$$I_s = \frac{8\pi h\nu^3}{c^2 \phi g(\nu_0, \nu)} = \frac{\boxed{8\pi \nu^2} \times h\nu}{\boxed{c^2 A_{21}\, g(\nu_0, \nu)} \times \tau_2} = \frac{h\nu}{\sigma(\nu)\tau_2}$$

$$\phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right] = A_{21}\tau_2$$

$$\boxed{I_s = \frac{h\nu}{\sigma(\nu)\tau_2}}$$

Power output from a laser amplifier

- if saturation is neglected for a laser amplifier of length l and gain coefficient $\gamma(\nu)$ the output intensity for a monochromatic input intensity $I_0 \left(\frac{W}{m^2}\right)$ at frequency ν is

$$I = I_0 e^{\gamma(\nu) l}$$

- if there is saturation $dI = I \gamma(\nu) dz \Rightarrow$

$$\gamma(\nu) = \frac{1}{I}\frac{dI}{dz}$$

- for a homogeneously broadened amplifier solution: $\gamma(\nu) = \dfrac{\gamma_0(\nu)}{1 + I/I_s(\nu)} = \dfrac{1}{I}\dfrac{dI}{dz}$

can be rewritten

$$\frac{dI}{I} + \frac{dI}{I_s(\nu)} = \gamma_0(\nu) dz$$

γ_0 - small signal gain.

$$\boxed{I = I_0 e^{\gamma_0(\nu) l - \frac{(I - I_0)}{I_s(\nu)}}}$$

A homogeneously broadened optical amplifier with a small-signal gain 13 dB (i.e., $G_o(dB)=10\log[I_{out}/I_{in}]$) is irradiated with a wave with intensity of $5W/cm^2$. The output intensity is $30\ W/cm^2$.

(a) What is the saturation intensity?

(b) If the saturation intensity were $20\ W/cm^2$, what is the maximum power (per unit area) extractable from the amplifier?

a) $I = I_0 \cdot e^{G = \gamma_0 l - \frac{I - I_0}{I_s}}$ ① . Find relationship between G and G_o

for small-signal gain $I = I_0 e^{G}$ ②

$\Rightarrow G = \ln \frac{I}{I_0}$ ③; Since $G_o = 10 \log \frac{I}{I_0}$ ④

$\frac{I}{I_0} = 10^{\frac{G_o (dB)}{10}}$ ⑤ $\Rightarrow$ Substituting ⑤ in ③

$G = \ln \frac{I}{I_0} = \ln 10^{\frac{G_o}{10}} = \frac{G_o}{10} \ln 10 = \frac{G_o}{10} \cdot 2.3 = 3$ ⑥

$\Rightarrow G = 3$ ⑥

Substituting ⑥ in ①

$I = I_0 e^{3 - \frac{I - I_0}{I_s}}$

$30 = 5 \cdot e^{3 - \frac{30-5}{I_s}} \Rightarrow 6 = e^{3 - \frac{25}{I_s}}$; $\ln 6 = 3 - \frac{25}{I_s}$

$1.79 = 3 - \frac{25}{I_s}$; $\frac{25}{I_s} = 1.21 \Rightarrow I_s = 21 \frac{W}{cm^2} = 2 \times 10^1 \frac{W}{cm^2}$

b) $I = I_0 e^{3 - \frac{I - I_0}{I_s}} = 5 e^{3 - \frac{I-5}{20}}$

Guess	Calc.
I = 20	47.4
21	45.1
19	49
20.1	47
26	35
29	30.2
29.5	29.5

$\Rightarrow I = 29.5 \frac{W}{cm^2} = 3 \times 10^1 \frac{W}{cm^2}$

Inhomogeneous system

- In a gas a plane monochromatic wave at freq. ν interacts with a medium whose individual particles have Lorentzian homogeneous lineshapes with FWHM $\Delta\nu_N$
- Center frequencies are distributed over an inhomogeneous (Doppler) broadened profile of width (FWHM) $\Delta\nu_D$.

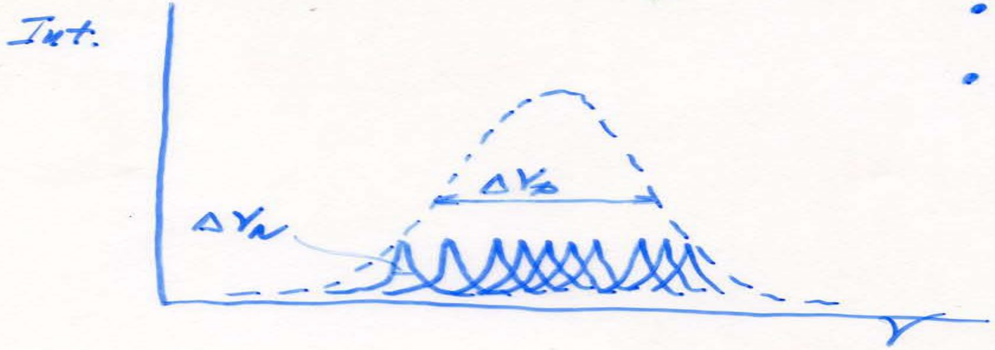
$$g_L(\nu',\nu) = \frac{\frac{2}{\pi\Delta\nu_N}}{1+[2(\nu-\nu')/\Delta\nu_N]^2} \quad \text{– Lorentzian}$$

ν' – center freq. of the particle.

$$g_D(\nu_0,\nu') = \frac{2}{\Delta\nu_D}\sqrt{\frac{\ln 2}{\pi}}\, e^{-[(2(\nu'-\nu_0)/\Delta\nu_D)^2 \ln 2]}$$

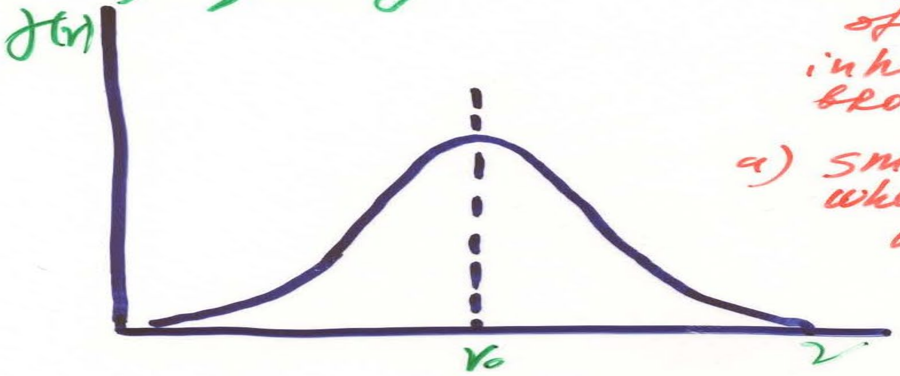
ν_0 – central frequency of a particle at rest.

The overall lineshape from all the particles is a sum of Lorentzian profiles spread across the particle velocity distribution.



- if $\Delta\nu_N >> \Delta\nu_D$ – homogeneous
- if $\Delta\nu_D >> \Delta\nu_N$ – inhomogeneous broadening.

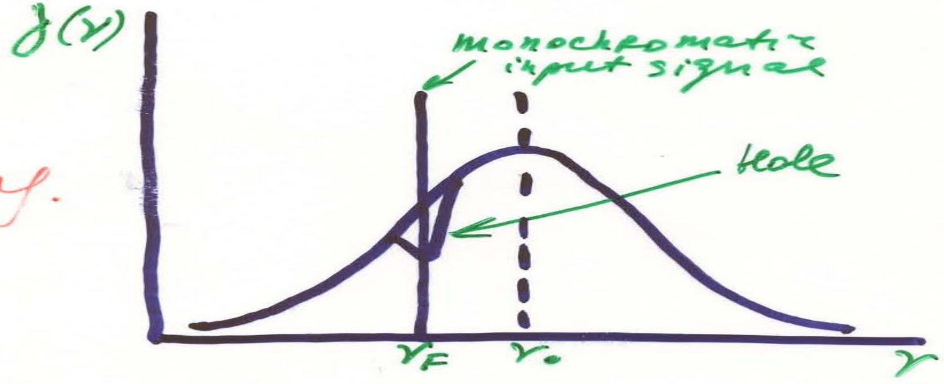
• The gain in the system comes largely from those particles whose frequencies are within (roughly) a homogeneous linewidth of the input radiation frequency.



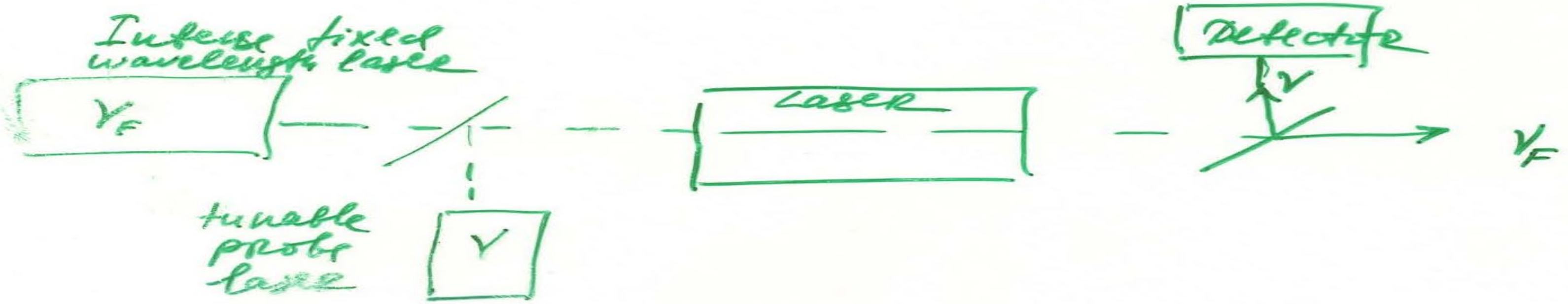
Gain as a function of frequency in an inhomogeneously broadened amplifier

a) small-signal situation when no saturation has occurred

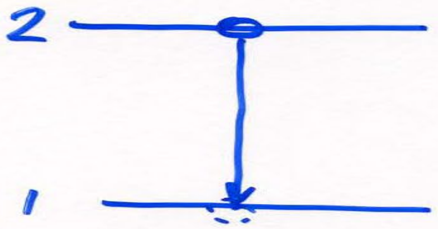
Hole burning.



Production of a hole in the gain curve by a strong monochromatic input at frequency ν_F.



The electron oscillator model of a radiative transition



- when a particle decays from an excited state 2 into a lower state 1, the resultant electr. field can be modeled as a damped oscillation.

- Analogy between the decay of an excited particle and the damped oscillation of an electric circuit.

RLC circuit

resonant frequency

$$\nu_0 = \frac{1}{2\pi}\sqrt{LC}$$

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V(t)$$

$$V = V_0 e^{i\omega t}$$

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt} \rightarrow I = I_0 e^{i\omega t}$$

$$|I_0| = \frac{|V_0|}{\sqrt{(1/\omega C - L\omega)^2 + R^2}} = \quad \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

$$= \frac{|V_0|/R}{\sqrt{1 + [2(\omega - \omega_0)/\omega_0^2 RC]^2}}$$

The power $W(\omega)$ dissipated in the circuit is $R|I_0|^2$

$$W = \frac{|V_0|^2/R}{1 + [2(\omega-\omega_0)/\omega_0^2 RC]^2}$$

which is a Lorentzian-shaped resonance curve with FWHM

$$\Delta \omega = \omega_0^2 RC$$

The quality factor Q of the circuit is defined as

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{1}{\omega_0 RC}$$

- the Power spectrum of the decaying electric current is Lorentzian as it is for spont. transition.
- the FWHM of the circuit resonance

$$\boxed{\Delta\nu = \frac{\nu_0}{Q}}$$

Q is the quality factor of the circuit

$\Delta\nu$ is analogous to the homogeneously broadened linewidth.

How particle responds to EM radiation?
(Classical Approach)

- each of the n electrons attached to the particle is treated as a damped harmonic oscillator

- the nucleus moves in the direction of the field
- electr. cloud moves in the opposite direction
- ω of electric field $\uparrow$ up to optical ω we can neglect motion of the nucleus due to its great inertia
- if $\vec{X_i}$ – is the vector displacem. of the i-th electron of the atom from equilibrium than atom has acquired a dipole moment

$$\overline{\mu} = -\sum_{i=1}^{n} e\,\overline{X_i}$$

- the magnitude of displacement depends on $\overline{E_i}$ electr. field at the electron.

$K_i \overline{X_i} = -e\,\overline{E_i}$; K_i is a force constant.

- A time varying field $\bar{E}$ leads to a time-varying dipole moment. μ

  μ is large if there is a resonance between $\bar{E}$ and an electron.

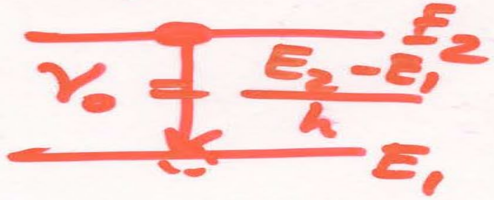
  it happens when the frequency of the field is near the natural oscillation frequency of a particular electron.

  $$\omega_i = \sqrt{\frac{k_i}{m}}$$

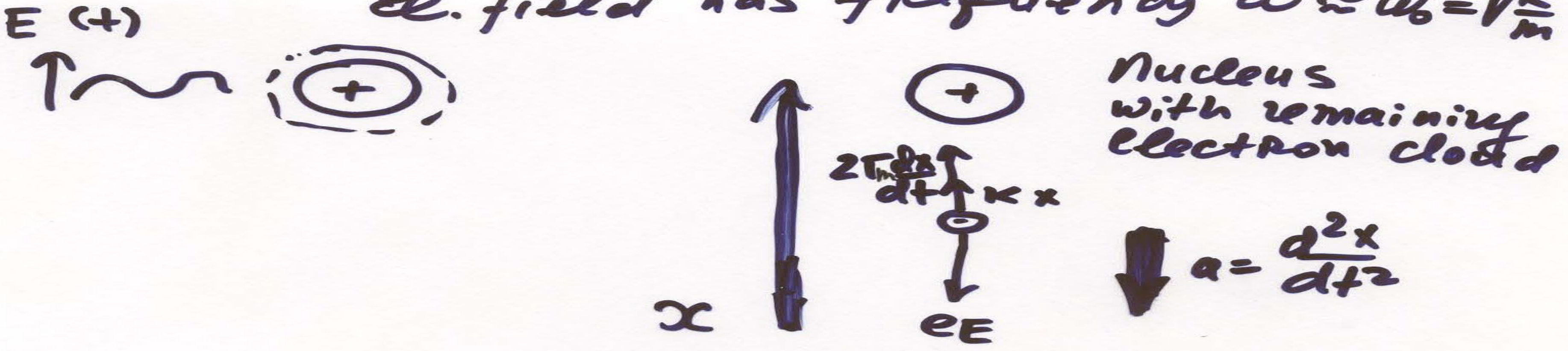
- if ω of Electr. field is close to ω_i than <u>one electron</u> makes a dominant contribution to the dipole moment $\mu = -\sum_{i=1}^{n} e x_i$

  ! We can treat the atom (n electron) as a single electron oscillator

- resonant frequency of the electron correspond to the frequency of transition 2 → 1



El. field has frequency $\omega \approx \omega_0 = \sqrt{\frac{K}{m}}$



Kx - restoring force

$2\Gamma m\frac{dx}{dt}$ - damping force (viscous drag) due to interaction of an electron with the other electrons of the particle

eE electric force (Colomb)

$$-eE + Kx + 2\Gamma m\frac{dx}{dt} = -ma$$

$$m\frac{d^2x}{dt^2} + 2\Gamma m\frac{dx}{dt} + Kx = eE$$

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \frac{K}{m}x = \frac{e}{m}E(t)$$

$$\frac{d^2x}{dt^2} + 2\Gamma\frac{dx}{dt} + \frac{K}{m}x = -\frac{e}{m}E(t)$$

$e = 1.6\times10^{-19}\,C$

take
$$E(t) = R\left[E e^{i\omega t}\right]$$
$$x(t) = R\left[X(\omega) e^{i\omega t}\right]$$
$$\omega_0 = \sqrt{\frac{k}{m}}$$

$\Rightarrow$ diff. eq. becomes

$$(\omega_0^2 - \omega^2) X + 2i\omega\Gamma X = -\frac{e}{m} E$$

$$X = \frac{-\left(\frac{e}{m}\right)E}{\omega_0^2 - \omega^2 + 2i\omega\Gamma}$$

– amplitude of the displacement of the electron from its equilibR. position as $f(\omega)$ of the applied field.

- near resonance $\omega \simeq \omega_0$

$$X(\omega \simeq \omega_0) = \frac{-\left(\frac{e}{m}\right)E}{2\omega_0(\omega_0 - \omega) + 2i\omega_0\Gamma}$$

- dipole moment of a single electron

$$\mu(t) = -e[x(t)]$$

- net polarization (dipole moment per unit volume)

$$P(t) = -Ne\,x(t)$$

$$P(t) = -Ne\,x(t) = R\left[P(\omega)e^{i\omega t}\right]$$

$P(\omega)$ - complex amplitude of the polarization

$$P(\omega) = -Ne\,X(\omega) = \frac{\left(\frac{Ne^2}{m}\right)E_0}{2\omega_0(\omega_0-\omega)+2i\omega_0\Gamma} =$$

$$= \frac{-i\left[Ne^2/(2m\omega_0\Gamma)\right]}{1+i(\omega-\omega_0)/\Gamma}E_0$$

- Electronic susceptibility $\chi(\omega)$ is defined by the equation

$$P(\omega) = \varepsilon_0\,\chi(\omega)E_0$$

ε_0 - permittivity of free space

$\chi(\omega)$ is complex

$$\chi(\omega) = \chi'(\omega) - i\,\chi''(\omega)$$

net polarization

- $\Rightarrow P(t) = R\left[\varepsilon_0\,\chi(\omega)E_0e^{i\omega t}\right] =$

$$= \varepsilon_0 E_0\,\chi'(\omega)\cos\omega t + \varepsilon_0 E_0\,\chi''\sin\omega t$$

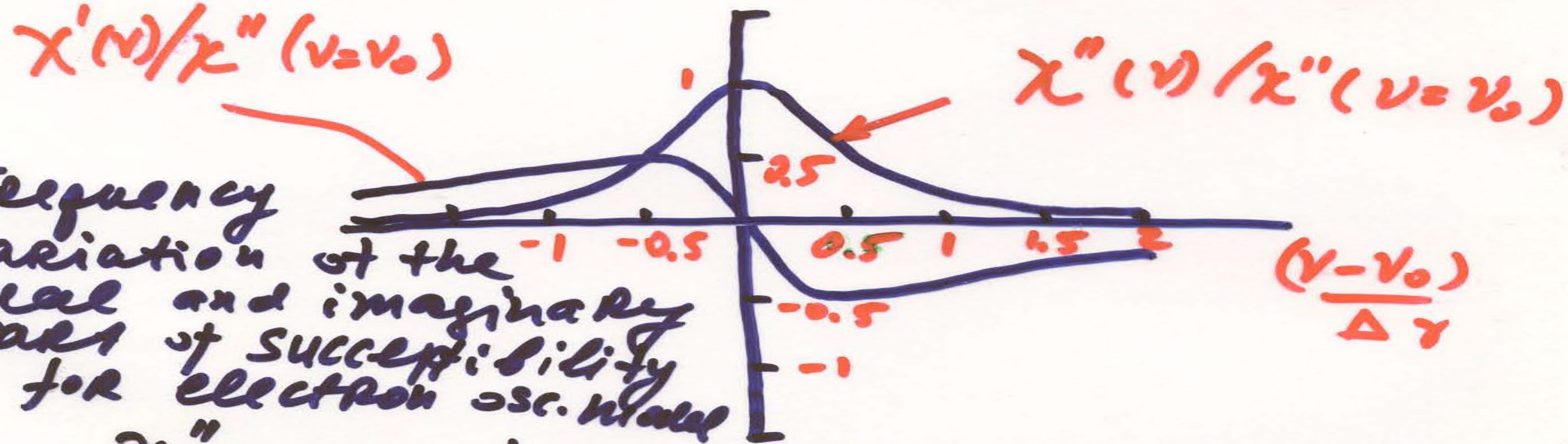
$\chi'(\omega)$ – real part of the susceptibility is related to the in-phase polariz.

$\chi''(\omega)$ – complex part, is related to the out of phase component

(E (t) and x (t) can be not in phase

$$\chi''(\nu) = \left(\frac{Ne^2}{16\pi^2 m\nu_0 \varepsilon_0}\right) \frac{\Delta\nu}{\left(\Delta\nu/2\right)^2 + (\nu-\nu_0)^2}$$

$$\chi'(\nu) = \left(\frac{Ne^2}{8\pi^2 m\nu_0 \varepsilon_0}\right) \frac{\nu_0-\nu}{\left(\frac{\Delta\nu}{2}\right)^2 + (\nu-\nu_0)^2}$$



Frequency variation of the real and imaginary part of susceptibility for electron osc. model

χ'' and χ' normalized to the peak value χ_0'' of χ''. χ'' has the Lorentzian shape

What are the Physical Significances of χ' and χ''?

- the relationship between the applied electric field $\vec{E}$ and the electron displacement vector $\vec{D}$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad = \varepsilon_0 (1+\chi)\vec{E}$$

- by introducing the diel. constant $\varepsilon_r = 1+\chi$ can be written as

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

- $n = \sqrt{\varepsilon_r}$

- when external $\vec{E}$ interacts with a group of particles there are 2 contributions to the induced polarization
  - a macroscopic contribution $\vec{P_m}$ from the collective properties of the particles.
  - $\vec{P_t}$ associated with transitions in the medium

$$\vec{P} = \vec{P_m} + \vec{P_t}$$

- only one transition will be near resonance with the frequency of an applied field.
  - $\vec{P_t}$ is dominated when we have resonance
  - $\vec{P_t} \to 0$ when far from resonance.

- far from resonance.

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P_m} = \varepsilon_0 \vec{E} + \chi_m \varepsilon_0 \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$$

χ_m - macroscopic susceptibility (nonresonant)

- if we are close to resonance.

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}_m + \vec{P}_t = \varepsilon_o \varepsilon_r \vec{E} + \vec{P}_t$$

- $\vec{P}_t$ is related to the complex susceptibility that results from the transition according to $\vec{P}_t = \varepsilon_o \chi(\omega) \vec{E}$

- $\Rightarrow$ $D = \varepsilon_o \left[\varepsilon_r + \chi(\omega)\right] \vec{E} = \varepsilon_o \varepsilon_r^{*} \vec{E}$

- when an e-m wave propagates through a medium with a complex susceptability, both the amplitude and phase velocity of the wave will be affected.

- Example. - plane wave propagating in the z direction with a field variation $\sim e^{i(\omega t - kz)}$

  K is propagation constant

  $$k = \omega \sqrt{\mu \varepsilon} = \omega \sqrt{\mu_r \mu_o \varepsilon_r \varepsilon_o}$$

  μ_r for optical materials ≈ 1.

- for a complex diellectric constant k can be rewritten as

$$K' = \omega \sqrt{\mu_o \varepsilon_r \varepsilon_o} \sqrt{1 + \frac{\chi(\omega)}{\varepsilon_r}} = K \sqrt{1 + \frac{\chi(\omega)}{\varepsilon_r}}$$

where K' is now the new propagation constant.

- $k' = \omega\sqrt{\mu_0 \varepsilon_r \varepsilon_0}\sqrt{1 + \frac{\chi(\omega)}{\varepsilon_r}} = k\sqrt{1 + \frac{\chi(\omega)}{\varepsilon_r}}$

  k' is the new propagation constant, which differs from the nonresonant propagation constant k because of the complex susceptibility resulting from a transition.

- if $|\chi(\omega)| << \varepsilon_r$

  $$k' = k\left[1 + \frac{\chi(\omega)}{2\varepsilon_r}\right] = k\left[1 + \frac{\chi'(\omega)}{2\varepsilon_r} - i\frac{\chi''(\omega)}{2\varepsilon_r}\right]$$

- the wave now propagates through the medium as $e^{-ik'z}$
- the electric field varies as

$$E = E_0 \exp\left(i\left\{\omega t - k\left[1 + \frac{\chi'(\omega)}{2\varepsilon_r} - \frac{i\chi''(\omega)}{2\varepsilon_r}\right]z\right\}\right) =$$

$$= E_0 \exp\left(i\left\{\omega t - k\left[1 + \frac{\chi'(\omega)}{2\varepsilon_r}\right]z\right\}\right)\exp\left[-\frac{k\chi''(\omega)}{2\varepsilon_r}z\right]$$

- Clearly this is a wave whose phase velocity is $c' = \frac{\omega}{k[1 + \chi'(\omega)/2\varepsilon_r]} = \frac{\omega}{k + \Delta k}$

  and whose field amplitude changes exponentially with distance.