

PH 221-1D Spring 2013

EQUILIBRIUM and ELASTICITY

Lectures 30-32

Chapter 12

(Halliday/Resnick/Walker, Fundamentals of Physics 9th edition)

Chapter 12

Equilibrium and Elasticity

In this chapter we will define equilibrium and find the conditions needed so that an object is at equilibrium.

We will then apply these conditions to a variety of practical engineering problems of static equilibrium.

We will also examine how a “rigid” body can be deformed by an external force. In this section we will introduce the following concepts:

Stress and strain

Young’s modulus (in connection with tension and compression)

Shear modulus (in connection with shearing)

Bulk modulus (in connection to hydraulic stress)

Equilibrium

We say that an object is in equilibrium when the following two conditions are satisfied:

1. The linear momentum \vec{P} of the center of mass is constant
2. The angular momentum \vec{L} about the center of mass or any other point is a constant

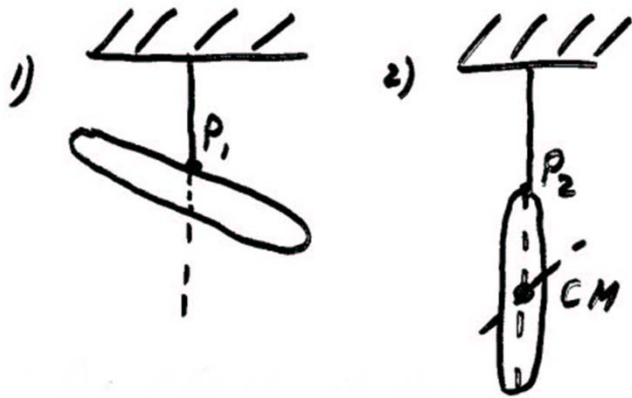
Our concern in this chapter is with situations in which $\vec{P} = 0$ and $\vec{L} = 0$

That is we are interested in objects that are not moving in any way (this includes translational as well as rotational motion) in the reference frame from which we observe them. Such objects are said to be in **static equilibrium**

In chapter 8 we differentiated between stable and unstable static equilibrium

If a body that is in static equilibrium is displaced slightly from this position the forces on it may return it to its old position. In this case we say that the equilibrium is **stable**. If the body does not return to its old position then the equilibrium is **unstable**.

A simple method for the experimental determination of the center of mass of a body of a complicated shape

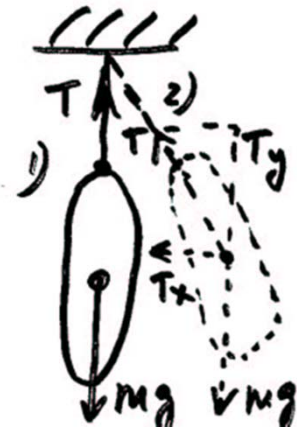


A rigid body supported by an upward force acting at a point on the vertical line through its center of mass is in equilibrium

The center of mass is at the intersection of the new and old prolongations of the string.

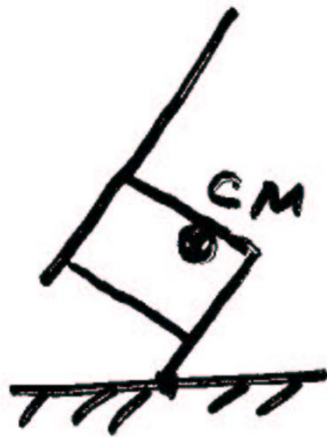
Stable, unstable and neutral equilibrium

A body suspended from a point above its center of mass is in stable equilibrium



2) The force of gravity and supporting force produce a torque that tends to return the body to the equilibrium position.

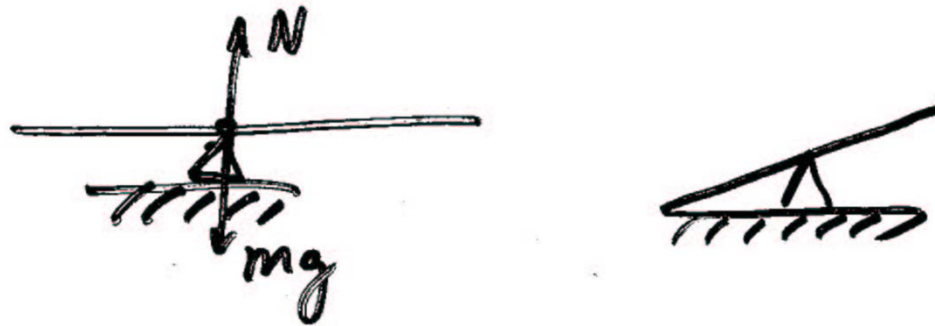
If the body is supported by a force applied at a point below the center of mass it is in unstable equilibrium

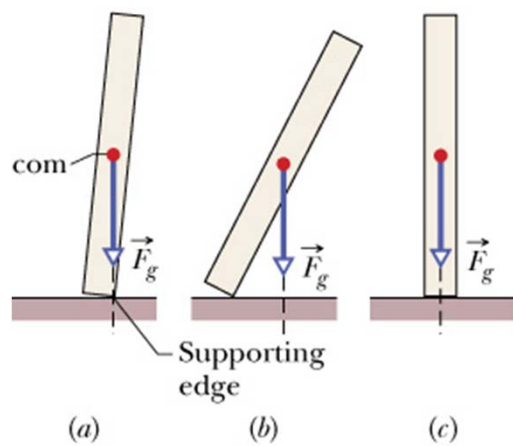


A chair supported at a point directly below the center of mass. If we turn the chair slightly the chair tends to topple over.

A body supported at its center of mass is in neutral equilibrium

If we turn such a body, it remains in equilibrium in its new position and exhibits no tendency to return to its original position or to turn farther away.





An example of **unstable equilibrium** is shown in the figures. In fig.a we balance a domino with the domino's center of mass vertically above the supporting edge. The torque of the gravitational force \vec{F}_g about the supporting edge is zero because the line of action of \vec{F}_g passes through the edge.

Thus the domino is in equilibrium. Even a slight force on the domino ends the equilibrium. As the line of action of \vec{F}_g moves to one side of the supporting edge (see fig.b) the torque due to \vec{F}_g is non-zero and the domino rotates in the clockwise direction away from its equilibrium position of fig.a. The domino in fig.a is in a position of unstable equilibrium. The domino in fig.c is not quite as unstable. To topple the domino the applied force would have to rotate it through and beyond the position of fig.a. A flick of the finger against the domino can topple it.

The Conditions of equilibrium

In chapter 9 we calculated the rate of change for the linear momentum of the center

of mass of an object. $\frac{d\vec{P}}{dt} = \vec{F}_{net}$ If an object is in translational equilibrium then

$$\vec{P} = \text{constant} \quad \text{and thus} \quad \frac{d\vec{P}}{dt} = 0 \rightarrow \vec{F}_{net} = 0$$

In chapter 11 we analyzed rotational motion and saw that Newton's second law takes the

form: $\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$ For an object in rotational equilibrium we have: $\vec{L} = \text{constant}$

$$\rightarrow \frac{d\vec{L}}{dt} = 0 \rightarrow \vec{\tau}_{net} = 0$$

The two requirements for a body to be in equilibrium are:

1. The vector sum of all the external forces on the body must be zero

$$\vec{F}_{net} = 0$$

2. The vector sum of all the external torques that act on the body measured about any point must be zero

$$\vec{\tau}_{net} = 0$$

In component form the conditions of equilibrium are:

$$\text{Balance of forces: } F_{net,x} = 0 \quad F_{net,y} = 0 \quad F_{net,z} = 0$$

$$\text{Balance of torques: } \tau_{net,x} = 0 \quad \tau_{net,y} = 0 \quad \tau_{net,z} = 0$$

We shall simplify matters by considering only problems in which all the forces that act on the body lie in the xy-plane. This means that the only torques generated by these forces tend to cause rotation about an axis parallel to the z-axis. With this assumption the conditions for equilibrium become:

$$\text{Balance of forces: } F_{net,x} = 0 \quad F_{net,y} = 0$$

$$\text{Balance of torques: } \tau_{net,z} = 0$$

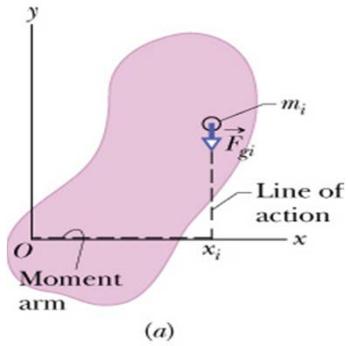
Here $\tau_{net,z}$ is the net torque produced by all external forces either about the z-axis or about any axis parallel to it.

Finally for static equilibrium the linear momentum \vec{P} of the center of mass must be zero: $\vec{P} = 0$

$$\begin{array}{l} F_{net,x} = 0 \quad F_{net,y} = 0 \\ \tau_{net,z} = 0 \end{array}$$

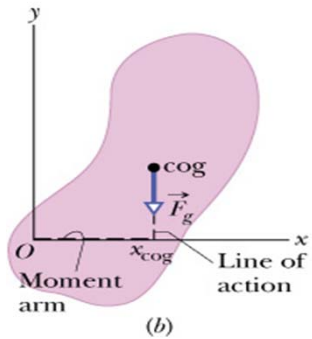
The center of Gravity (cog)

The gravitational force acting on an extended body is the vector sum of the gravitational forces acting on the individual elements of the body. The gravitational force \vec{F}_g on a body effectively acts at a single point known as the center of gravity of the body. Here "effectively" has the following meaning: If the individual gravitational forces on the elements of the body are turned off and replaced by \vec{F}_g acting at the center of gravity, then the net force and the net torque about any point on the body does not change. We shall prove that if the acceleration of gravity \vec{g} is the same for all the elements of the body then the **center of gravity** coincides with the **center of mass**. This is a reasonable approximation for objects near the surface of the earth because \vec{g} changes very little.



Consider the extended object of mass M shown in fig.a. In fig.a we also show the i -th element of mass m_i . The gravitational force on m_i is equal to $m_i \vec{g}_i$ where \vec{g}_i is the acceleration of gravity in the vicinity of m_i . The torque τ_i on m_i is equal to $F_{gi} x_i$. The net torque

$$\tau_{net} = \sum_i \tau_i = \sum_i F_{gi} x_i \quad (\text{eqs.1})$$



Consider now fig.b in which we have replaced the forces F_{gi} by the net gravitational force F_g acting at the center of gravity. The net torque τ_{net} is equal to :

$$\tau_{net} = x_{cog} F_g = x_{cog} \sum_i F_{gi} \quad (\text{eqs.2})$$

If we compare equation 1 with equation 2 we get: $x_{cog} \sum_i F_{gi} = \sum_i F_{gi} x_i$

We substitute $m_i g_i$ for F_{gi} and we have: $x_{cog} \sum_i m_i g_i = \sum_i m_i g_i x_i$

If we set $g_i = g$ for all the elements $\rightarrow x_{cog} = \frac{\sum_i m_i x_i}{\sum_i m_i} = x_{com}$

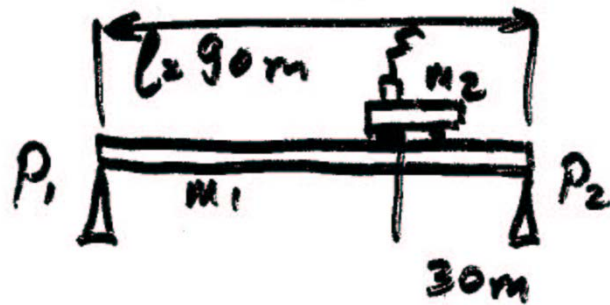
Statics Problem Recipe

1. Draw a force diagram. (Label the axes)
2. Choose a **convenient** origin O. A good choice is to have one of the unknown forces acting at O
3. Sign of the torque τ for each force:
 - If the force induces clockwise (CW) rotation
 - + If the force induces counter-clockwise (CCW) rotation
4. Equilibrium conditions:

$$F_{net,x} = 0 \quad F_{net,y} = 0$$
$$\tau_{net,z} = 0$$
5. Make sure that:
numbers of unknowns = number of equations

Examples of Static Equilibrium

A locomotive of 90,000 kg is one third of the way across a bridge 90 m long. The bridge consists of a uniform iron girder of 900,000 kg, which rests at two piers. What is the load on each pier?



Given:

$$M_1 = 900,000 \text{ kg}$$

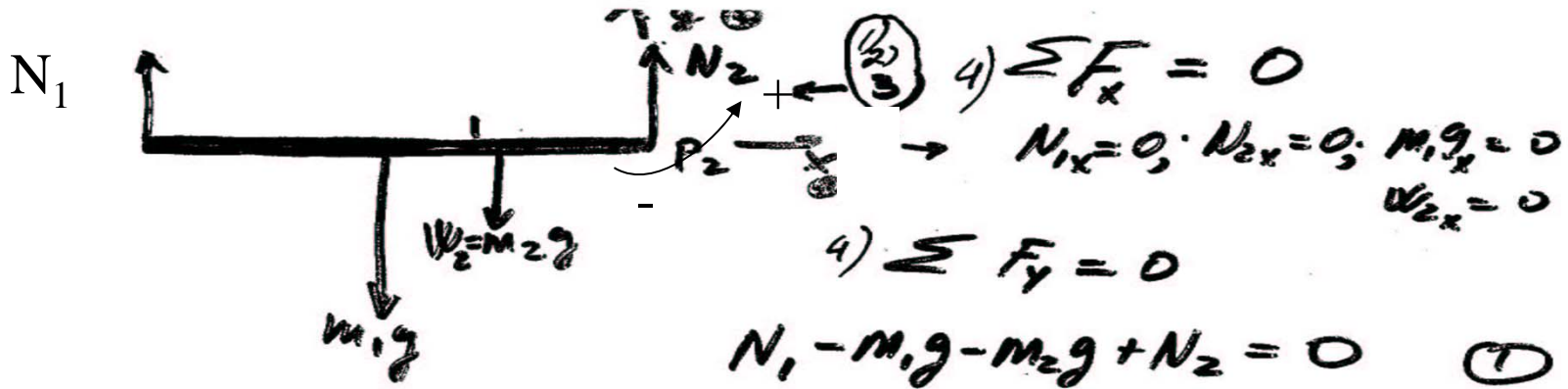
$$M_2 = 90,000 \text{ kg}$$

$$L_1 = 90 \text{ m}$$

$$L_2 = 30 \text{ m}$$

Method:

1. Select the body which is to obey the equilibrium condition
2. List all the external forces that act on this body and display them on a free body diagram
3. Choose coordinate axes and resolve the forces into x and y components. (+ or – with respect to positive direction that you have chosen)
4. Apply equilibrium condition: $\Sigma F_x = 0$
 $\Sigma F_y = 0$
5. Make a choice of axis rotation. Choose positive direction of rotation. Apply the static equilibrium condition
6. Solve the equations in steps 4 and 5 for the desired unknown quantities.



5) Torque about point P_2 .

$$\sum \tau = 0$$

$$-N_1 l + m_1 g \frac{l}{2} + m_2 g \frac{l}{3} = N_2 (0) = 0 \quad (2)$$

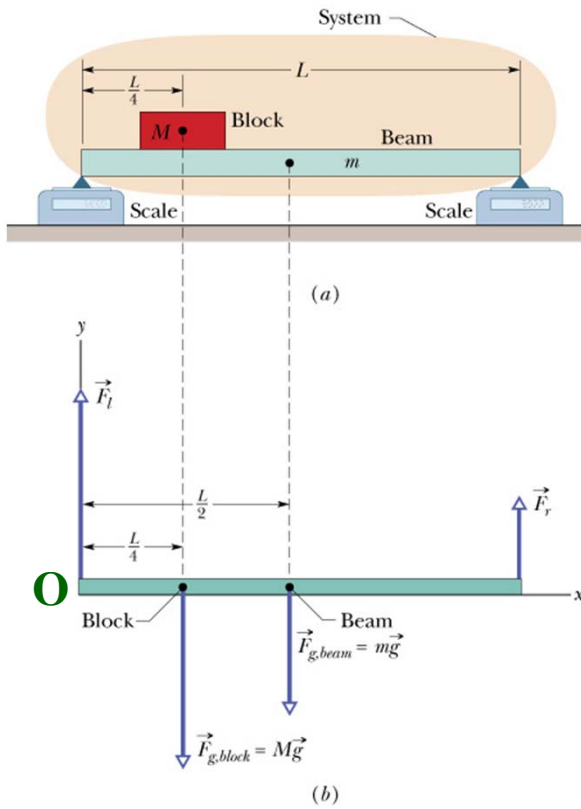
$$N_1 = \frac{m_1 g}{2} + \frac{m_2 g}{3} = \frac{900,000 \times 9.81}{2} + \frac{90,000 \times 9.81}{3}$$

$$= \boxed{4.7 \times 10^6 \text{ N}}$$

$$N_2 = m_1 g + m_2 g - N_1 = (900,000 \times 9.81) + (90,000 \times 9.81) - 4.7 \times 10^6 = \boxed{5 \times 10^6 \text{ N}}$$

$$W_1 = -N_1$$

$$W_2 = -N_2$$



Sample Problem 12-1. A uniform beam of length L and mass $m = 1.8$ kg is at rest on two scales.

A uniform block of mass $M = 2.7$ kg is at rest on the beam at a distance $L/4$ from its left end.

Calculate the scales readings

$$F_{net,y} = F_\ell + F_r - Mg - mg = 0 \quad (\text{eqs.1})$$

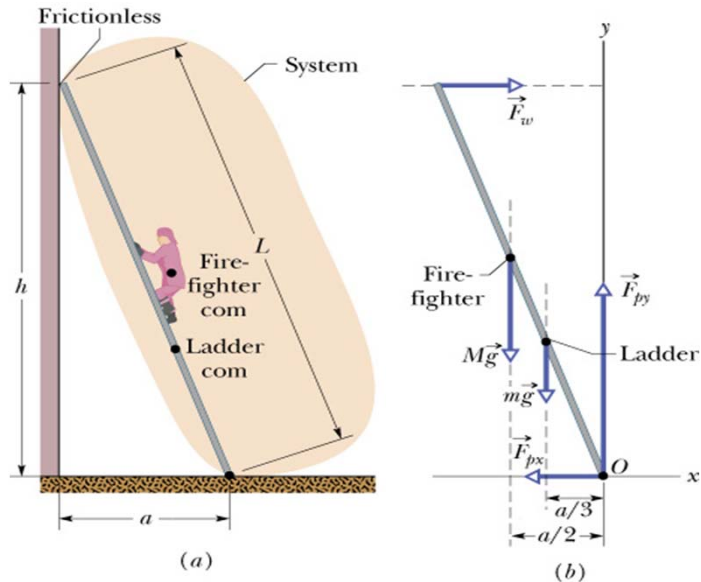
We choose to calculate the torque with respect to an axis through the left end of the beam (point O).

$$\tau_{net,z} = -\left(\frac{L}{4}\right)(Mg) - \left(\frac{L}{2}\right)(mg) + (L)(F_r) = 0 \quad (\text{eqs.2})$$

From equation 2 we get: $F_r = \frac{Mg}{4} + \frac{mg}{2} = \frac{2.7 \times 9.8}{4} + \frac{1.8 \times 9.8}{2} = 15.44 \approx 15$ N

We solve equation 1 for $F_\ell \rightarrow F_\ell = Mg + mg - F_r = (2.7 + 1.8) \times 9.8 - 15.44 = 28.66$ N

$F_\ell \approx 29$ N



Sample Problem 12-2: A ladder of length $L = 12$ m and mass $m = 45$ kg leans against a frictionless wall. The ladder's upper end is at a height $h = 9.3$ m above the pavement on which the lower end rests. The com of the ladder is $L/3$ from the lower end. A firefighter of mass $M = 72$ kg climbs half way up the ladder. Find the forces exerted on the ladder by the wall and the pavement. Distance $a = \sqrt{L^2 - h^2} = 7.58$ m

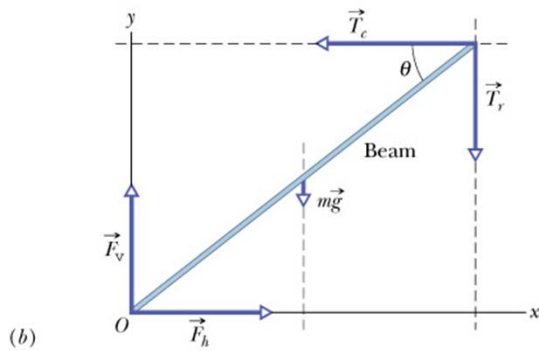
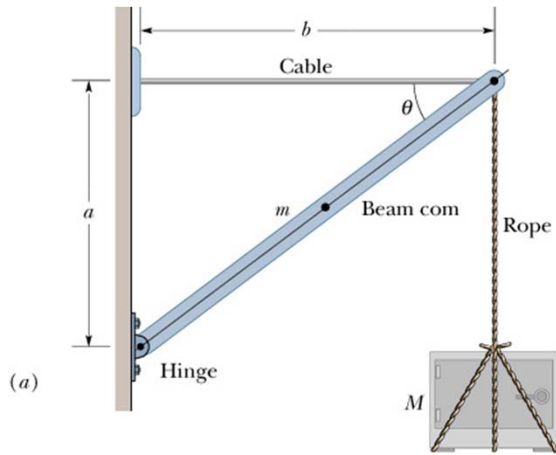
We take torques about an axis through point O.

$$\tau_{net,z} = -(h)(F_w) + \left(\frac{a}{3}\right)(mg) + \left(\frac{a}{2}\right)(Mg) = 0$$

$$\rightarrow F_w = \frac{ga\left(\frac{M}{2} + \frac{m}{3}\right)}{h} = \frac{9.8 \times 7.58 \times (72/2 + 45/3)}{9.3} = 407 \text{ N} \approx 410 \text{ N}$$

$$F_{net,x} = F_w - F_{px} = 0 \rightarrow F_{px} = F_w = 410 \text{ N}$$

$$F_{net,y} = F_{py} - Mg - mg = 0 \rightarrow F_{py} = Mg + mg = 9.8 \times (72 + 45) = 1146.6 \text{ N} \approx 1100 \text{ N}$$



Sample Problem 12-3: A safe of mass $M = 430$ kg hangs by a rope from a boom with dimensions $a = 1.9$ m and $b = 2.5$ m. The beam of the boom has mass $m = 85$ kg. Find the tension T_c in the cable and the magnitude of the net force F exerted on the beam by the hinge.

We calculate the net torque about an axis normal to the page that passes through point O.

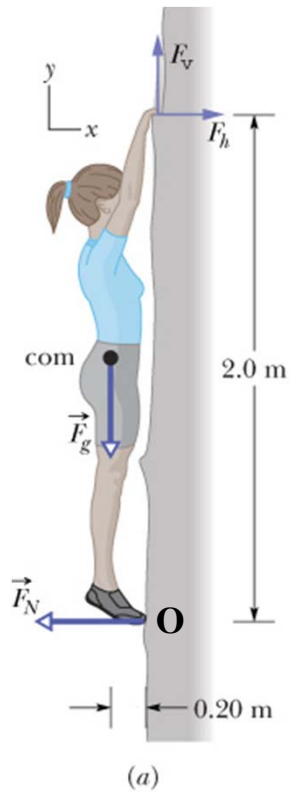
$$\tau_{net,z} = (a)(T_c) - (b)(T_r) - \left(\frac{b}{2}\right)(mg) = 0 \rightarrow$$

$$T_c = \frac{gb \left(M + \frac{m}{2} \right)}{a} = \frac{9.8 \times 2.5 (430 + 85/2)}{1.9} \approx 6100 \text{ N}$$

$$F_{net,x} = F_h - T_c = 0 \rightarrow F_h = T_c = 6093 \text{ N}$$

$$F_{net,y} = F_v - mg - T_r = 0 \rightarrow F_v = mg + T_r = g(m + M) = 9.8 \times (85 + 430) = 5047 \text{ N}$$

$$F = \sqrt{F_h^2 + F_v^2} = \sqrt{(6093)^2 + (5047)^2} \approx 7900 \text{ N}$$



Sample Problem 12-4: A 70 kg rock climber hangs by the crimp hold of one hand. Her feet touch the rock directly below her fingers. Assume that the force from the horizontal ledge supporting her fingers is equally shared by the four fingers. Calculate the horizontal and vertical components F_h and F_v of the force on each fingertip.

$$F_{net,x} = -F_N + 4F_h = 0$$

$$F_{net,y} = 4F_v - mg = 0 \rightarrow F_v = \frac{mg}{4} = \frac{70 \times 9.8}{4} = 171.5 \text{ N} \approx 170 \text{ N}$$

We calculate the net torque about an axis that is perpendicular to the page and passes through point O.

$$\tau_{net,z} = (0)F_N + (0.2)(mg) - (2.0)(4F_h) + (0)(4F_v) = 0 \rightarrow$$

$$F_h = \frac{0.20 \times 70 \times 9.8}{4 \times 2.0} = 17.15 \text{ N} \approx 17 \text{ N}$$

Problem (static equilibrium): The drawing shows an A-shaped ladder. Both sides of the ladder are equal in length. This ladder standing on a frictionless horizontal surface, and only the crossbar (which has a negligible mass) of the "A" keeps the ladder from collapsing. The ladder is uniform and has a mass of 20.0 kg. Determine the tension in the crossbar of the ladder.

4.2. - Select the body which is to obey the equil. condition
- free body diagram

③ Choose coordinate axes and resolve the forces into x and y components
Apply equilibrium condition: $\Sigma F_x = 0$
 $\Sigma F_y = 0$

$\Sigma F_x = 0: T - R = 0$ ① ; $R = T$

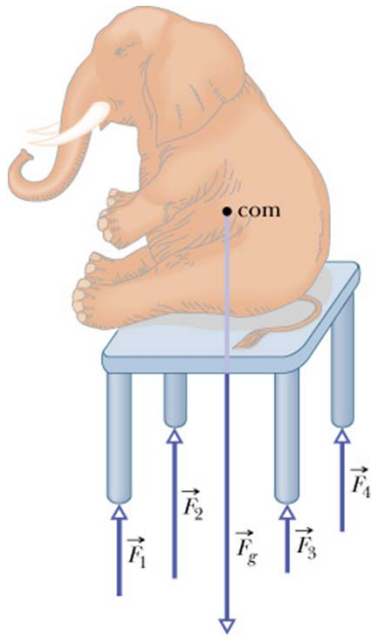
$\Sigma F_y = 0: N - W = 0$ ② ; $N = W = \frac{Mg}{2} = (10.0 \text{ kg})(9.80 \text{ m/s}^2) = 98.0 \text{ N}$

⑤ Choice of axis of rotation. \oplus
Apply the static equilibrium condition
 $\Sigma \tau = 0$

⑥ $\Sigma \tau = -T[(1.00 \text{ m}) \sin 75.0^\circ] - W[(2.00 \text{ m}) \cos 75.0^\circ] + R[(4.00 \text{ m}) \sin 75.0^\circ] = 0$

$T[(3.00 \text{ m}) \sin 75.0^\circ] = (98.0 \text{ N})(2.00 \text{ m}) \cos 75.0^\circ$

$T = \frac{(98.0 \text{ N})(2.00 \text{ m}) \cos 75.0^\circ}{(3.00 \text{ m}) \sin 75.0^\circ} = \boxed{17.5 \text{ N}}$



Indeterminate Structures.

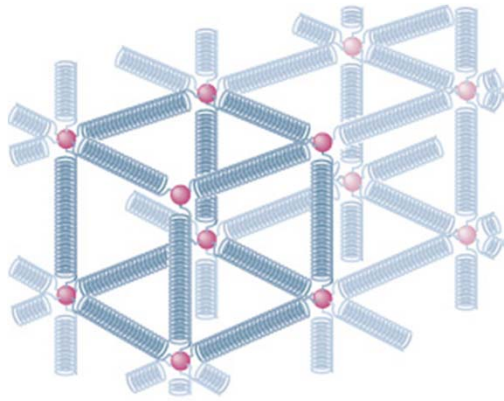
For the problems in this chapter we have the following three equations at our disposal:

$$F_{net,x} = 0 \quad F_{net,y} = 0 \quad \tau_{net,z} = 0$$

If the problem has more than three unknowns we cannot solve it

We can solve a statics problem for a table with three legs but not for one with four legs. Problems like these are called **indeterminate**

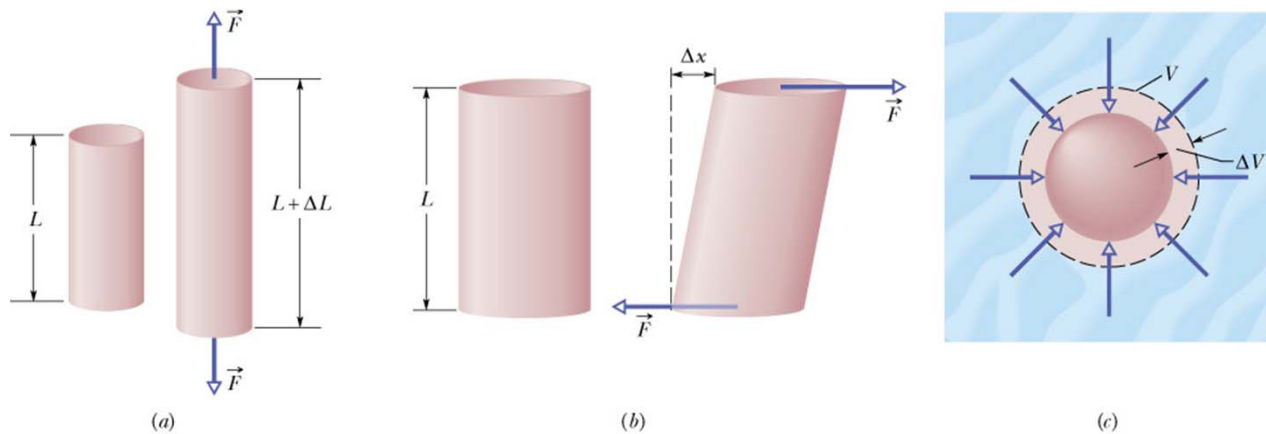
An example is given in the figure. A big elephant sits on a wobbly table. If the table does not collapse it will deform so that all four legs touch the floor. The upward forces exerted on the legs by the floor assume definite and different values. How can we calculate the values of these forces? To solve such an indeterminate equilibrium problem we must supplement the three equilibrium equations with some knowledge of **elasticity**, the branch of physics and engineering that describes how real bodies deform when forces are applied to them.



Elasticity

Metallic solids consist of a large number of atoms positioned on a regular three-dimensional lattice as shown in the figure. The lattice is repetition of a pattern (in the figure this pattern is a cube)

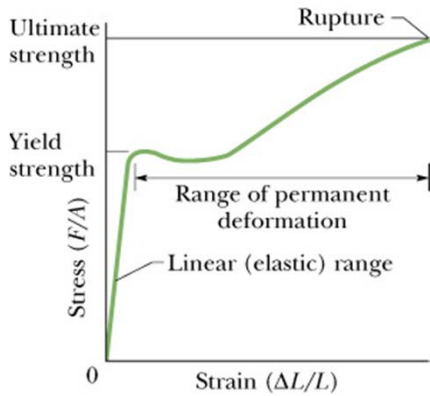
Each atom of the solid has a well-defined equilibrium distance from its nearest neighbors. The atoms are held together by interatomic forces that can be modeled as tiny springs. If we try to change the interatomic distance the resulting force is proportional to the atom displacement from the equilibrium position. The spring constants are large and thus the lattice is remarkably rigid. Nevertheless all "rigid" bodies are to some extent elastic, which means that we can change their dimensions slightly by pulling, pushing, twisting or compressing them. For example, if you suspend a subcompact car from a steel rod 1 m long and 1 cm in diameter, the rod will stretch by only 0.5 mm. The rod will return to its original length of 1 m when the car is removed. If you suspend two cars from the rod the rod will be permanently deformed. If you suspend three cars the rod will break.



$$\text{stress} = \text{modulus} \times \text{strain}$$

In the three figures above we show the three ways in which a solid might change its dimensions under the action of external deforming forces. In fig.a the cylinder is stretched by forces acting along the cylinder axis. In fig.b the cylinder is deformed by forces perpendicular to its axis. In fig.c a solid placed in a fluid under high pressure is compressed uniformly on all sides. All three deformation types have **stress** in common (defined as deforming force per unit area).

These stresses are known as **tensile/compressive** for fig.a, **shearing** for fig.b, and **hydraulic** for fig.c. The application of stress on a solid results in strain, or unit deformation. The stresses and strains take different forms, but over the range of engineering usefulness stress and strain are proportional to each other: **stress = modulus × strain**



Tensile stress is defined as the ratio $\frac{F}{A}$ where A is the solid area

Strain (symbol S) is defined as the ratio $\frac{\Delta L}{L}$ where ΔL

is the change in the length L of the cylindrical solid.

Stress is plotted versus strain in the upper figure.

For a wide range of applied stresses the

stress-strain relation is linear and the solid returns to its original

length when the stress is removed. This is known as the

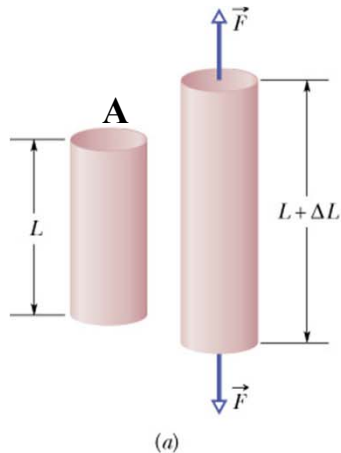
elastic range. If the stress is increased beyond a maximum value

known as the yield strength S_y the cylinder becomes permanently

deformed. If the stress continues to increase the cylinder breaks

at a stress value known as ultimate strength S_u

$$\frac{F}{A} = E \frac{\Delta L}{L}$$



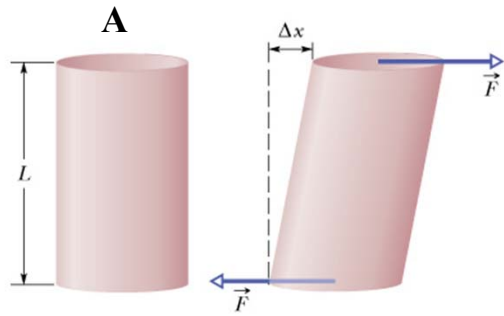
For stresses below S_y (elastic range) stress and strain are connected via the equation

$$\frac{F}{A} = E \frac{\Delta L}{L}$$

The constant E (modulus) is known as: **Young's modulus**

Note: Young's modulus is almost the same for tension and compression

The ultimate strength S_u maybe different



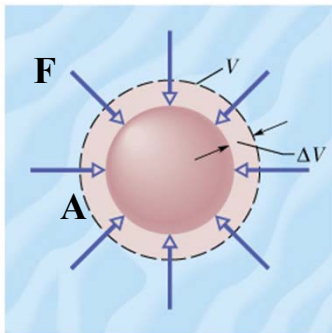
(b)

$$\frac{F}{A} = G \frac{\Delta x}{L}$$

Shearing: In the case of shearing deformation strain is defined as the dimensionless ratio $\frac{\Delta x}{L}$. The stress/strain equation has the form:

$$\frac{F}{A} = G \frac{\Delta x}{L}$$

The constant G is known as the shear modulus



(c)

$$p = B \frac{\Delta V}{V}$$

Hydraulic Stress. The stress in this case is the pressure $p = \frac{F}{A}$ the surrounding fluid exerts on the immersed object. Here A is the area of the object. In this case strain is defined as the dimensionless ratio $\frac{\Delta V}{V}$ where V is the volume of the object and ΔV the change in the volume due to the fluid pressure. The stress/strain equation has the form: $p = B \frac{\Delta V}{V}$ The constant B is known as the bulk modulus of the material

Problem 43. A horizontal aluminum rod 4.8 cm in diameter projects 5.3 cm from a wall. A 1200 kg object is suspended from the end of the rod. The shear modulus of aluminum is $3.0 \times 10^{10} \text{ N/m}^2$. Neglecting the rod's mass, find (a) the shear stress on the rod and (b) the vertical deflection of the end of the rod.

(a) The shear stress is given by F/A , where F is the magnitude of the force applied parallel to one face of the aluminum rod and A is the cross-sectional area of the rod. In this case F is the weight of the object hung on the end: $F = mg$, where m is the mass of the object. If r is the radius of the rod then $A = \pi r^2$. Thus, the shear stress is

$$\frac{F}{A} = \frac{mg}{\pi r^2} = \frac{(1200 \text{ kg})(9.8 \text{ m/s}^2)}{\pi(0.024 \text{ m})^2} = 6.5 \times 10^6 \text{ N/m}^2.$$

(b) The shear modulus G is given by

$$G = \frac{F / A}{\Delta x / L}$$

where L is the protrusion of the rod and Δx is its vertical deflection at its end. Thus,

$$\Delta x = \frac{(F / A)L}{G} = \frac{(6.5 \times 10^6 \text{ N/m}^2)(0.053 \text{ m})}{3.0 \times 10^{10} \text{ N/m}^2} = 1.1 \times 10^{-5} \text{ m}.$$