Problem (1.1)

In a dispersive medium the refractive index varies with wavelength. We can define a group refractive index by the relation

$$n_g = n - \lambda \frac{dn}{d\lambda}$$

(i) Prove that $n_g = n + \nu \frac{dn}{d\nu}$

$$\lambda = \frac{c}{\nu}$$

$$n_g = n - \frac{c}{\nu} \frac{dn}{d\frac{c}{\nu}} = n - \frac{\cancel{c}}{\nu} \frac{dn}{\cancel{c}(-\frac{1}{\nu^2}d\nu)} =$$

$$= n + \frac{\nu\, dn}{d\nu}$$

(ii) Prove that if a black-body cavity is filled with such a dispersive material then the radiation mode density, $\rho(\nu)$, satisfies

$$\rho(\nu) = \frac{8\pi \nu^2 n^2 n_g}{c_0^3}$$

See (1.34) - (1.32) $\quad N_\nu = \frac{8\pi \nu^3}{3c^3} L^3 \quad ; \quad V = L^3$

$$\rho(\nu) = \frac{1}{V} \frac{dN_\nu(\nu)}{d\nu} = \frac{1}{L^3} \frac{d\left(\frac{8\pi \nu^3 n^3 L^3}{3c_0^3}\right)}{d\nu} = \frac{8\pi L^3}{L^3 \cdot 3c_0^3} \cdot \left[\frac{\nu^3 \cdot 3n^2 dn + n^3 3\nu^2 d\nu}{d\nu}\right]$$

$$\boxed{\begin{array}{l} d(a \times b) = a\, db + b\, da \\ a = \nu^3 \\ b = n^3 \end{array}}$$

$$= \frac{8\pi \cancel{L^3}}{\cancel{L^3} \cdot \cancel{3} c_0^3} \cdot \cancel{3} \nu^2 n^2 \underbrace{\left[\frac{\nu\, dn}{d\nu} + n\right]}_{n_g} =$$

$$= \boxed{\frac{8\pi \nu^2 n^2 n_g}{c_0^3}}$$

(iii) Prove also that in such a situation

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3 n^2 n_g}{C_0^{\,3}}$$

$$\rho(\nu) = \frac{8\pi \nu^2 n^2 n_g}{C_0^{\,3}} \times h\nu \times \left(\frac{1}{e^{h\nu/KT}-1}\right)$$

$$0 = \frac{dN_2}{dt} = -N_2 B_{21}\, \rho(\nu) - A_{21} N_2 + N_1 B_{12}\, \rho(\nu)$$

$$N_2\left[B_{21} \cdot \frac{8\pi h\nu^3 n^2 n_g}{C_0^{\,3}\left(e^{h\nu/KT}-1\right)} + A_{21}\right] = N_1\left[B_{12}\, \frac{8\pi h\nu^3 n^2 n_g}{C_0^{\,3}\left(e^{h\nu/KT}-1\right)}\right]$$

$$\frac{N_2}{N_1} = e^{-\frac{h\nu}{KT}} \Rightarrow N_1 = N_2 \cdot e^{\frac{h\nu}{KT}}$$

$$\cancel{N_2 : e^{\frac{h\nu}{KT}}} \qquad A_{21} = \frac{8\pi h\nu^3 n^2 n_g}{C_0^{\,3}\left(e^{h\nu/KT}-1\right)}\left[B_{12} \cdot e^{\frac{h\nu}{KT}} - B_{21}\right]$$

$$\frac{A_{21}}{B_{12}\, e^{h\nu/KT} - B_{21}} = \frac{8\pi h\nu^3 n^2 n_g}{C_0^{\,3}\left(e^{h\nu/KT}-1\right)}$$

$$\Rightarrow B_{12} = B_{21}$$

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3 n^2 n_g}{C_0^{\,3}\cancel{\left(e^{h\nu/KT}-1\right)}} \cdot \cancel{\left(e^{h\nu/KT}-1\right)} = \boxed{\frac{8\pi h\nu^3 n^2 n_g}{C_0^{\,3}}}$$

Problem 1.2.

Calculate the photon flux (photons $\cdot m^{-2} \cdot s^{-1}$) in a plane monochromatic wave of intensity $100\ W/m^2$ at a wavelength of

(a) 100 nm

(b) 100 μm

$$N_{photons} = \frac{I(\nu_{\perp})}{h\nu} \quad \left[\frac{photons}{m^2 \cdot s} \right]$$

$$\nu = \frac{c}{\lambda}$$

(a) $$\nu = \frac{c}{\lambda} = \frac{3\times10^{8}\ m/s}{100\times10^{-9}\ m} = 3\times10^{15}\ Hz$$

$$h\nu = 6.62\times10^{-34}\ J\cdot s \times 3\times10^{15}\ Hz = 2\times10^{-18}\ J$$

$$N_{photons} = \frac{100\ \frac{J}{s\cdot m^2}}{2\times10^{-18}\ J} = \boxed{5\times10^{19}\ \frac{photons}{s\cdot m^2}}$$

(b) $$\nu = \frac{c}{\lambda} = \frac{3\times10^{8}\ m/s}{100\times10^{-6}\ m} = 3\times10^{12}\ Hz$$

$$h\nu = 6.62\times10^{-34}\ J\cdot s \times 3\times10^{12}\ Hz = 2\times10^{-21}\ J$$

$$N_{photons} = \frac{100\ \frac{J}{s\cdot m^2}}{2\times10^{-21}\ J} = \boxed{5\times10^{22}\ \frac{photons}{m^2\cdot s}}$$

Problem 1.3

What is the total # of modes per unit volume for visible light?

1) Visible light range is $\lambda = 400 - 700\ nm$

it corresponds to frequency range $\nu = \frac{c}{\lambda}\Big|_{700}^{400} =$

$$= \frac{(3\times10^{8} m/s)}{(400\times10^{-9} m)} - \frac{(3\times10^{8} m/s)}{(700\times10^{-9} m)} = \underbrace{7.5\times10^{14}}_{\nu_2} - \underbrace{4.29\times10^{14}}_{\nu_1}\ Hz$$

unit

2) The # modes per unit volume per ν frequency interval $\rho(\nu) = \frac{8\pi\nu^2}{c^3}$

Total # of modes per unit volume in frequency interval from ν_1 to ν_2 is

$$\int_{\nu_1}^{\nu_2} \rho(\nu)\,d\nu = \int_{\nu_1}^{\nu_2} \frac{8\pi\nu^2}{c^3}\,d\nu =$$

$$= \frac{8\pi}{c^3}\int_{\nu_1}^{\nu_2} \nu^2\,d\nu = \frac{8\pi}{c^3}\frac{\nu^3}{3}\Bigg|_{4.29\times10^{14}}^{7.5\times10^{14}} =$$

$$= \frac{8\pi}{(3\times10^{8} m/s)^3}\cdot\left[(7.5\times10^{14})^3 - (4.29\times10^{14})^3\right] =$$

$$= \frac{8\pi(10^{14})^3}{(3\times10^{8} m/s)^3}\left[(7.5^3 - 4.29^3)\,Hz^3\right] =$$

$$= \frac{8\pi\cdot10^{42}\cdot343}{27\times10^{24}}\left[m^{-3}\right] = 3.2\times10^{20}\ \frac{modes}{m^3}$$