# Contents

**Preface** **xi**

**Symbols and notation** **xix**

**List of Figures** **xxiii**

**List of Tables** **xxix**

**1 Introduction and historic overview** **1**
1.1 Classical regression 1
1.2 Errors-in-variables (EIV) model 4
1.3 Geometric fit 6
1.4 Solving a general EIV problem 10
1.5 Nonlinear nature of the "linear" EIV 14
1.6 Statistical properties of the orthogonal fit 16
1.7 Relation to total least squares (TLS) 19
1.8 Nonlinear models: General overview 20
1.9 Nonlinear models: EIV versus orthogonal fit 22

**2 Fitting lines** **25**
2.1 Parametrization 25
2.2 Existence and uniqueness 27
2.3 Matrix solution 30
2.4 Error analysis: Exact results 32
2.5 Asymptotic models: Large $n$ versus small $\sigma$ 34
2.6 Asymptotic properties of estimators 36
2.7 Approximative analysis 40
2.8 Finite-size efficiency 42
2.9 Asymptotic efficiency 45

**3 Fitting circles: Theory** **47**
3.1 Introduction 47
3.2 Parametrization 48
3.3 (Non)existence 53

3.4 Multivariate interpretation of circle fit 55
3.5 (Non)uniqueness 57
3.6 Local minima 59
3.7 Plateaus and valleys 61
3.8 Proof of Two Valley Theorem 64
3.9 Singular case 67

**4 Geometric circle fits 69**
4.1 Classical minimization schemes 70
4.2 Gauss-Newton method 72
4.3 Levenberg-Marquardt correction 74
4.4 Trust region 76
4.5 Levenberg-Marquardt for circles: Full version 79
4.6 Levenberg-Marquardt for circles: Reduced version 80
4.7 A modification of Levenberg-Marquardt circle fit 82
4.8 Späth algorithm for circles 84
4.9 Landau algorithm for circles 88
4.10 Divergence and how to avoid it 91
4.11 Invariance under translations and rotations 93
4.12 The case of known angular differences 96

**5 Algebraic circle fits 99**
5.1 Simple algebraic fit (Kåsa method) 100
5.2 Advantages of the Kåsa method 101
5.3 Drawbacks of the Kåsa method 104
5.4 Chernov-Ososkov modification 107
5.5 Pratt circle fit 109
5.6 Implementation of the Pratt fit 111
5.7 Advantages of the Pratt algorithm 114
5.8 Experimental test 117
5.9 Taubin circle fit 121
5.10 Implementation of the Taubin fit 124
5.11 General algebraic circle fits 127
5.12 A real data example 130
5.13 Initialization of iterative schemes 132

**6 Statistical analysis of curve fits 137**
6.1 Statistical models 138
6.2 Comparative analysis of statistical models 140
6.3 Maximum Likelihood Estimators (MLE) 141
6.4 Distribution and moments of the MLE 146
6.5 General algebraic fits 150
6.6 Error analysis: A general scheme 153

6.7 Small noise and "moderate sample size" 156
6.8 Variance and essential bias of the MLE 159
6.9 Kanatani-Cramer-Rao lower bound 162
6.10 Bias and inconsistency in the large sample limit 163
6.11 Consistent fit and adjusted least squares 166

**7 Statistical analysis of circle fits 171**
7.1 Error analysis of geometric circle fit 171
7.2 Cramer-Rao lower bound for the circle fit 174
7.3 Error analysis of algebraic circle fits 177
7.4 Variance and bias of algebraic circle fits 179
7.5 Comparison of algebraic circle fits 182
7.6 Algebraic circle fits in natural parameters 185
7.7 Inconsistency of circular fits 190
7.8 Bias reduction and consistent fits via Huber 193
7.9 Asymptotically unbiased and consistent circle fits 195
7.10 Kukush-Markovsky-van Huffel method 197
7.11 Renormalization method of Kanatani: 1st order 200
7.12 Renormalization method of Kanatani: 2nd order 202

**8 Various "exotic" circle fits 207**
8.1 Riemann sphere 207
8.2 Simple Riemann fits 210
8.3 Riemann fit: the SWFL version 212
8.4 Properties of the Riemann fit 215
8.5 Inversion-based fits 217
8.6 The RTKD inversion-based fit 222
8.7 The iterative RTKD fit 224
8.8 Karimäki fit 228
8.9 Analysis of Karimäki fit 231
8.10 Numerical tests and conclusions 236

**Bibliography 241**

**Index 255**