FINAL EXAM

Duration 140 min, Max. Points: 40 (+ 4 Bonus).

Make sure to show all your work and underline your final results. Write on these sheets or use extra paper if needed. Each problem is worth 4 points. Good luck!

1. Determine whether the given series is convergent. Be sure to justify the validity of the tests you use. (6 points)

$$(a) \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \dots$$

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n^2}{2^n}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{2}{n^3+1}$

2. Find the radius and the interval of convergence. (6 points)

(a) $\displaystyle\sum_{n=1}^{\infty}(2n)!x^n$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)}(x-1)^n$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{7n^3}(x+2)^n$

3. Find the Maclaurin series of the function $f(x)$ and its interval of convergence.

(a) $\quad f(x) = \dfrac{x}{1+x^2}$

(b) $\quad f(x) = \dfrac{1-\cos(x)}{x}$

(6 points)

4. Find the Taylor series for $f(x) = \ln(x)$ about $x = 1$. Hint: First show that

$$f^{(n)}(x) = \frac{(n-1)!}{x^n}(-1)^{n-1}, \qquad \text{for } n \geq 1.$$

(4 points)

5. Find the largest number d for which Taylor's inequality guarantees that the approximation

$$e^{-x} \simeq 1 - x + x^2/2, \qquad \text{for } 0 \leq x \leq d$$

is accurate to within 0.01. (4 points)

6. Make a sketch of the graphs of $y = \sqrt{x}$ and $y = x$, and shade the region bounded by these graphs. Then find the volume of the solid obtained by rotating this region about the x-axis. (4 points)

7. Let $\vec{a} = \langle 3, 1, -1\rangle$ and $\vec{b} = \langle 4, 0, 2\rangle$. (4 points)
(a) Compute the dot product of $\vec{a}$ and $\vec{b}$.
(b) Find the angle between $\vec{a}$ and $\vec{b}$.

8. Evaluate two of the following four integrals. (If you do more than two, label those you want credit for.) (6 points)

$(a)\quad \displaystyle\int_0^1 \frac{x}{1+x^2}\,dx \qquad\qquad (c)\quad \displaystyle\int x^3 \ln(x)\,dx$

$(b)\quad \displaystyle\int \frac{\cos(1/t)}{t^2}\,dt \qquad\qquad (d)\quad \displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

Bonus. The parametric equations (4 points)

$$x(t) = e^{-t}\cos(t)$$
$$y(t) = e^{-t}\sin(t)$$

with $0 \le t < \infty$ describe a spiral. Find the length of this spiral. Is it finite or infinite? (In the picture the scale is changed to show more rotations.)