Expanding maps of an interval with holes

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Abstract

We study a class of open chaotic dynamical systems. Consider an expanding map of an interval from which a few small open subintervals are removed (thus creating "holes"). Almost every point of the original interval then eventually escapes through the holes, so there can be no absolutely continuous invariant measures. We construct a so called conditionally invariant measure that is equivalent to the Lebesgue measure. Our measure is unique and naturally generates an invariant measure, which is singular. These results generalize early work by Pianigiani, Yorke, Collet, Martinez and Schmidt, who studied similar maps under an additional Markov assumption. We do not assume any Markov property here and use "bounded variation" techniques rather than Markov coding. Our results supplement those of Keller, who studied analytic interval maps with holes by using different techniques.

1 Introduction

Expanding maps of an interval are very popular in the theory of chaotic dynamical systems. A map $\hat{T}\colon [0,1] \to [0,1]$ is said to be *expanding* if it is piecewise smooth and its derivative satisfies the condition $\inf |\hat{T}'(x)| > 1$. Simple examples of expanding maps are:
(i) the so-called beta-transformations defined by $\hat{T}(x) = \beta x \pmod 1$ with $\beta > 1$, in particular the doubling map $\hat{T}(x) = 2x \pmod 1$;
(ii) the tent map defined by $\hat{T}(x) = 2x$ for $0 \le x \le 1/2$ and $\hat{T}(x) = 2(1-x)$ for $1/2 < x \le 1$.

Expanding maps have good ergodic and statistical properties. A. Lasota and J. Yorke [LaY] proved that C^2 expanding maps admit absolutely continuous invariant measures (a.c.i.m.'s), whose densities are of bounded variation. In fact, such maps have only finitely many ergodic a.c.i.m.'s, each of which is mixing up to a cycle [LiY, HK]. R. Bowen [Bo]

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showed that a mixing a.c.i.m. is Bernoulli. F. Hofbauer and G. Keller [HK] proved that mixing a.c.i.m.'s have good statistical properties, too - they enjoy exponential decay of correlations and satisfy the central limit theorem (for observables with bounded variation). Some extensions of these results may be found in [BK, Bu, K, KL, L, Ry].

Note that if an ergodic a.c.i.m. has a strictly positive density, then there is no other a.c.i.m.'s, because distinct ergodic measures are always mutually singular. In particular, beta-transformations with an integer $\beta \geq 2$ and the tent map preserve the Lebesgue measure, hence they have no other a.c.i.m.'s.

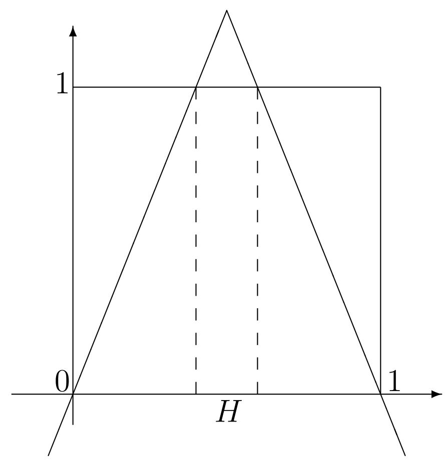

Figure 1: A modified tent map

Now we introduce the idea of *open systems* that have become popular in the studies of chaos and statistical mechanics in recent years [D, GD]. We do that by example. Consider a modification of the tent map defined by $\hat{T}(x) = cx$ for $x \leq 1/2$ and $\hat{T}(x) = c(1-x)$ for $x > 1/2$, with some $c > 2$. This is a map $\hat{T}\colon \mathbb{R} \to \mathbb{R}$ shown on Fig. 1. One can easily check that the points $x \in H\colon= (c^{-1}, 1-c^{-1})$ will be mapped out of the interval $[0,1]$ and eventually, under the iterations of $\hat{T}$, they will escape to $-\infty$. The same will happen to all the points $x \in \hat{T}^{-k}H$ for all $k \geq 1$. The set $\cup_k \hat{T}^{-k}H$ is open and dense in $[0,1]$ and has full Lebesgue measure. In other words, almost every point $x \in [0,1]$ eventually escapes through the "opening" H, which can be also regarded as a *hole* in the interval $[0,1]$. Note that the map $\hat{T}$ was used in [D] to illustrate the concept of open dynamics and escape through holes.

In physical applications, the process of escape of points from the unit interval is characterized by a conditionally invariant measure. A probability measure μ on $[0,1]$ is said to be *conditionally invariant* if there exists $\lambda > 0$ such that $\hat{T}_*\mu = \lambda\mu$. Here, as usual, $\hat{T}_*$ is a dual operator acting on measures as defined by $(\hat{T}_*\mu)(A) = \mu(\hat{T}^{-1}A)$ for all Borel sets $A \subset [0,1]$. The constant $\lambda > 0$ is called the eigenvalue of μ, see [C], and $-\ln \lambda$ is called the escape rate [D, GD].

In physical experiments, if one chooses N points in $[0,1]$ at random according to the distribution μ, then after n iterations of $\hat{T}$ there will be $\approx \lambda^n N$ points remaining on $[0,1]$, and they will be distributed according to the measure μ. Note that conditionally invariant measures are analogous to quasi-stationary distributions for Markov chains with absorbing states (or absorbing boundary conditions) - see [CMM, FKMP].

From the physicist point of view, only absolutely continuous conditionally invariant measures (a.c.c.i.m.'s) are relevant, and this is what we will study in this paper. It is also important that an a.c.c.i.m. attracts other smooth measures, i.e. there is a large class $\mathcal{C}$ of absolutely continuous measures on $[0,1]$ such that for each $\nu \in \mathcal{C}$ the sequence $c_n \hat{T}_*^n \mu$, where $c_n^{-1} = (\hat{T}_*^n \nu)([0,1])$ is the normalizing factor, converges to the a.c.c.i.m. μ (in the sense that the densities converge uniformly) . We will refer to this as the *convergence property* for μ (in the class $\mathcal{C}$). We note that the convergence property allows to choose N points in the above mentioned physical experiment according to any distribution in the class $\mathcal{C}$, with the same final result for large n.

The first results on a.c.c.i.m.'s were obtained by G. Pianigiani and J. Yorke:

Theorem 1.1 ([PY]) *Let $I \subset [0,1]$ be a finite union of disjoint closed intervals $I_1, \ldots, I_m$. Let $T\colon I \to [0,1]$ be a map that is C^2 smooth and expanding on each interval $I_j \subset I$ and satisfies the condition $T(I) \supset I$ and the Markov property $\operatorname{int} I \cap T(\partial I) = \emptyset$ (i.e. for each pair of intervals I_i, I_j we have either $T(I_i) \supset I_j$ or $T(I_i) \cap \operatorname{int} I_j = \emptyset$). Then an a.c.c.i.m. exists. If T satisfies an additional transitivity assumption, then an a.c.c.i.m. is unique in a class of absolutely continuous measures whose densities have positive upper and lower bounds. In that class, the a.c.c.i.m. has the convergence property.*

For example, let T be the restriction of the modified tent map $\hat{T}$ shown on Fig. 1 to the set $I\colon= [0,1] \setminus H$. Then T satisfies the conditions of Theorem 1.1 and hence admits an a.c.c.i.m., which is in this case just the (normalized) Lebesgue measure on I.

Remark. One would expect that under the conditions of Theorem 1.1 the a.c.c.i.m. is unique (and has the convergence property) even if we relax the requirement that the densities be bounded away from zero. Surprisingly, this is not true, even for the just mentioned modified tent map. Indeed, choose any $0 < \alpha < 1/2$ and define a density by setting it to α^k on the set $T^{-k}H$ for each $k \geq 1$. This defines the density almost everywhere on I, and the corresponding measure (after normalization) will be an a.c.c.i.m. with eigenvalue $\lambda = 2\alpha c^{-1}$. Moreover, the variation of this density is finite (less than $4/(1-2\alpha)$). This example shows that there is a mysterious difference between the properties of a.c.i.m.'s for ordinary expanding maps and those of a.c.c.i.m.'s for expanding maps with holes.

An a.c.c.i.m. naturally generates a $\hat{T}$-invariant measure $\bar{\mu}$ (which is singular). Its construction is described below, here we give its physical interpretation. Suppose one chooses N points at random according to any distribution in the class $\mathcal{C}$, then after n

iterations of $\hat{T}$ there will be $\approx \lambda^n N$ points remaining on $[0,1]$. We take their images after $m \ll n$ (instead of n) iterations of $\hat{T}$, then those will be distributed according to the measure $\bar{\mu}$.

Theorem 1.2 ([CMS1]) *Under the conditions of the previous theorem (including transitivity) there is a unique T-invariant measure associated with the a.c.c.i.m. μ.*

The purpose of this paper is to extend Theorems 1.1 and 1.2 to expanding maps with holes that do not necessarily satisfy the Markov property. In other words, we allow more generic holes than in [PY, CMS1]. Our approach is the following. We start with an ordinary expanding map $\hat{T}$ of the unit interval. For simplicity, we assume that it admits a mixing a.c.i.m. whose density is bounded away from zero. Let H be a finite union of small open intervals (holes in $[0,1]$), and T the restriction of $\hat{T}$ to $I = [0,1] \setminus H$. Here is our main theorem, stated a bit loosely:

Theorem 1.3 *Suppose that the total length of holes is small enough and they are in generic position (i.e. their images do not overlap for sufficiently many iterations of $\hat{T}$). Then the map $T\colon I \to [0,1]$ admits an a.c.c.i.m. μ, which is unique in a class of absolutely continuous measures whose densities have bounded variation. In that class, the a.c.c.i.m. μ has the convergence property. The measure μ generates a unique T-invariant measure $\bar{\mu}$.*

The precise conditions and claims of the theorem will be specified below.

Remark. We emphasize that even though we are making a rather strong assumption on the given expanding map $\hat{T}$ (that it has a mixing a.c.i.m. with a strictly positive density), the holes are allowed to be rather arbitrary, in fact, they must be in generic position (as explained in Section 3). We note that our assumption on $\hat{T}$ is not too restrictive either: if it fails, it is usually possible to find a finite union of subintervals $J := \cup J_i$ in $[0,1]$ and a higher iteration of the map, $\hat{T}_1 := \hat{T}^k$, such that $\hat{T}_1(J) \subset J$ and the map $\hat{T}_1\colon J \to J$ admits an a.c.i.m. that is mixing and has a strictly positive density on J. Then the problem can be reduced to the map $\hat{T}_1$ on J.

Remark. G. Keller (see Sect. 9C in [K]) obtained similar results for expanding interval maps with holes by a different approach. He assumed that the original interval map is piecewise analytic and used Fredholm theory to study the corresponding transfer operator. He proved the exponential escape of mass (he called this process *extinction*) through holes (which he called *traps*) and related the extinction rate to the topological pressure of the corresponding weight function. Keller also constructed an equilibrium state, which was an invariant measure. In the introduction to [K], he said that he used the heavy machinery of Fredholm theory because "it seems quite difficult to obtain rigorous results about extinction rates ... by studying transfer operators acting on spaces of function of bounded variation". This is exactly what we are doing in this paper, hence we supplement the results of [K].

Remark. As a referee pointed out to us, most of the results we obtain here can be derived from spectral properties of transfer operators and their perturbations studied recently in [BK] and [KL]. It is also possible to show, in addition to what we prove, that the invariant measure is ergodic and has rapidly decaying correlations. We do not apply the spectral operator technique and restrict ourselves to more elementary and direct arguments.

2 Notation and preliminary lemmas

Here we introduce our notation and collect necessary tools for the proof of the main theorem.

Let $\hat{T}: I \to I$ be a piecewise C^2 expanding map. We denote

$$s := \sup_x 1/|T'(x)| < 1$$

and

$$S := \sup_x |T'(x)| < \infty$$

We also put

$$\bar{C} = \sup_{x\in I} |T''(x)| \tag{2.1}$$

and

$$\tilde{C} = \frac{s^2 \bar{C}}{1-s} \tag{2.2}$$

For each $n \geq 1$, we denote by $\hat{I}_k^n$, $k \geq 1$, the intervals of monotonicity for the map $\hat{T}^n$. We put $\hat{J}_k^n = \hat{T}^n \hat{I}_k^n$.

Let δ_n be the length of the smallest interval $\hat{I}_k^n$, $k \geq 1$, that is

$$\delta_n = \min_k m(\hat{I}_k^n) \tag{2.3}$$

Here and on m stands for the Lebesgue measure on $[0,1]$.

Assumption on $\hat{T}$. We assume that the map $\hat{T}$ preserves a mixing measure with density $h(x)$ that is bounded away from zero: $h(x) \geq h_{\min} > 0$.

We make an important note. Under this assumption, the map $\hat{T}$ has the following *covering property* [L]: for every $n \geq 1$ there is a $K(n) < \infty$ such that, for every k, the set $\hat{T}^{K(n)} \hat{I}_k^n$ covers $[0,1]$, up to finitely many points. For a proof, see C. Liverani [L]. We note that Liverani [L] proved the converse: the covering property implies that the invariant density is bounded away from zero. It is also interesting to note that recent results by J. Buzzi [Bu] imply that if $\hat{T}$ is topologically mixing, then it has the covering property.

Holes. We now introduce holes in the interval $[0,1]$. Let $H = \cup_l H_l$ be a finite union of disjoint open subintervals H_l (holes). We denote by L the number of H_l (holes) and by

$$h = m(H) = \sum_{l=1}^{L} m(H_l)$$

their total length.

Let $I = [0,1] \setminus H$. Put $I^n = \cap_{i=0}^{n} \hat{T}^{-i} I$ and $H^n = [0,1] \setminus I^n$.

We now define the map T on I and its iterations. We denote by T^n the restriction of $\hat{T}^n$ to the set I^n. Note that $T^n(I^n) \subset I$. That is, the map T^n is the restriction of the map $\hat{T}$ to the set of points whose trajectories do not enter the holes (the set H) at times $0, 1, \ldots, n$.

We put $I_k^n = \hat{I}_k^n \cap I^n$. Note that T^n is well defined and monotonic on each set I_k^n. Put $J_k^n = T^n I_k^n$. Note that both I_k^n and J_k^n are finite unions of intervals. As the reader may have noticed, we attach "hats" to notation related to the original map $\hat{T}$, and remove hats when dealing with the map T generated by holes.

Note that the set H^n consists of points where the map T^n is not defined. We have a simple bound on its Lebesgue measure:

$$m(H^n) \le h \sum_{i=0}^{n} (sq)^i \tag{2.4}$$

where q is the number of intervals $\hat{I}_k^1$ of monotonicity of the map $\hat{T}$.

Perron-Frobenius operator. Since we study absolutely continuous measures on $[0,1]$, we will need the Perron-Frobenius operator that describes the transformation of densities under the map T.

Let $f \ge 0$ be the density function of a measure ν on I. The density of the measure $T_*^n \nu$ is given by

$$(P_{T^n} f)(x) = \sum_{y \in T^{-n} x} \frac{f(y)}{|(T^n)'(y)|} \tag{2.5}$$

and if $T^{-n}x = \emptyset$, we set $(P_{T^n} f)(x) = 0$ (here and on $(T^n)'$ stands for the derivative of the map T^n). This equation defines a linear operator P_{T^n} on $L^1(I)$, known as the Perron-Frobenius operator. We will denote it simply by P^n. One can easily verify that $P_{T^n} = \underbrace{P_T \circ \cdots \circ P_T}_{n}$, so this notation is consistent.

Note that $P^n f \ge 0$ if $f \ge 0$. Also, by a simple change of variable,

$$\|P^n f\|_1 = \int_{I^n} f(x)\, dx = \nu(I^n)$$

hence $\|P^n f\|_1 \le \|f\|_1 = \nu(I)$. This shows that, when holes are present, the Perron-Frobenius operator decreases the norm of measures. For this reason we also consider a *modified* (or *normalized*) Perron-Frobenius operator

$$P_1^n f = P^n f / \|P^n f\|_1 \tag{2.6}$$

defined only if $\|P^n f\|_1 \neq 0$. Now we have $\|P_1^n f\| = 1$. The operator P_1^n does preserve the norm of probability measures, but it is not a linear operator anymore.

We study the action of the Perron-Frobenius operator on functions of bounded variation. The variation of a function f on I is defined by

$$\operatorname{Var} f = \sup \left\{ \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| \colon\ a_0 < \cdots < a_n,\ a_k \in I \right\}$$

where the supremum is taken over all finite sequences in I. For each $b > 0$ we put

$$E_b = \left\{ f \in L^1(I) \colon\ f \geq 0,\ \|f\|_1 = 1,\ \operatorname{Var} f \leq b \right\}$$

It is straightforward to verify that E_b is a compact, convex subset of $L^1(I)$.

A crucial property of the Perron-Frobenius operator for ordinary expanding maps is that they reduce variation of densities, i.e. they satisfy the bound $\operatorname{Var} Pf \leq \alpha \operatorname{Var} f + \beta \|f\|_1$ for some constants $0 < \alpha < 1$ and $\beta > 0$, see [LaY, HK]. We now prove a similar property for our maps with holes:

Proposition 2.1 (Variation bound) *For every $n \geq 1$ and a nonnegative function f of bounded variation on I we have*

$$\operatorname{Var}_I (P^n f) \leq s^n (2L(n+1) + 3) \operatorname{Var}_I f + [\tilde{C} + (\delta_n - h)^{-1}] \int_I f\, dx \tag{2.7}$$

(the value of δ_n is defined in (2.3)).

We first prove an auxiliary bound on distortions. Let I_k^n be one of the intervals of the set I^n and $J_k^n = T^n I_k^n$. The map T^n is defined and monotonic, hence invertible, on the interval I_k^n. Denote by $\psi\colon J_k^n \to I_k^n$ the inverse of the map T^n restricted to I_k^n. For any points $x, y \in J_k^n$ we put $x' = \psi(x)$, $y' = \psi(y)$. Note that $|\psi'| \leq s^n$. Hence, $|x' - y'| \leq s^n \cdot |x - y|$, and for the same reason

$$|T^i(x') - T^i(y')| \leq s^{n-i} \cdot |x - y| \tag{2.8}$$

for all $0 \leq i \leq n$.

Lemma 2.2 (Distortion bounds) *In the above notation*

$$\Big| \ln |(T^n)'(x')| - \ln |(T^n)'(y')| \Big| \leq \tilde{C} \cdot |x - y|$$

Proof. By a simple application of the chain rule and then (2.1) and (2.8) we have

$$\begin{aligned} \Big| \ln |(T^n)'(x')| - \ln |(T^n)'(y')| \Big| &\leq \sum_{i=0}^{n-1} \Big| \ln |T'(T^i x')| - |\ln T'(T^i y')| \Big| \\ &\leq \sum_{i=0}^{n-1} s\, \bar{C}\, |T^i(x') - T^i(y')| \\ &\leq \bar{C}(s^2 + s^3 + \cdots + s^{n+1})\, |x - y| \end{aligned}$$

This proves the lemma. □

Lemma 2.2 implies

$$\left| \frac{(T^n)'(x')}{(T^n)'(y')} - 1 \right| \le \tilde{C} \cdot |x - y| \tag{2.9}$$

Proof of Proposition 2.1. Let $\hat{I}_k^n$ be an arbitrary interval of monotonicity of the map $\hat{T}^n$. Recall that $\hat{J}_k^n = \hat{T}^n \hat{I}_k^n$. Note that $\hat{T}^n \colon \hat{I}_k^n \to \hat{J}_k^n$ is a monotonic, hence invertible, map, and denote by $\hat{\psi}_k \colon \hat{J}_k^n \to \hat{I}_k^n$ the inverse of the map $\hat{T}^n$ restricted to $\hat{I}_k^n$. Note that $|\hat{\psi}_k'| \le s^n$.

The set $I_k^n = \hat{I}_k^n \cap I^n$ consists of some number, say $L_{n,k}$, of connected components (intervals). We claim that

$$L_{n,k} \le L(n+1) + 1 \tag{2.10}$$

To prove this claim, we note that the gaps between the above subintervals of I_k^n are made by the components of $H^n \cap \hat{I}_k^n$. The maps $\hat{T}^i$ for $1 \le i \le n$ are monotonic on $\hat{I}_k^n$, thus the set $\hat{T}^{-i}(H \cap \hat{T}^i \hat{I}_k^n)$ has no more than L connected components for every $i = 0, 1, \ldots, n$. So, the set $H^n \cap \hat{I}_k^n$ has no more than $L(n+1)$ connected components. Hence (2.10). Denote by $b_{k,1}^n, \ldots, b_{k,q_k}^n$ the endpoints of the intervals of which the set I_k^n consists. Note that $q_k \le 2L(n+1) + 2$ by (2.10).

It follows from the definition of the Perron-Frobenius operator P^n that

$$\begin{aligned} \mathrm{Var} P^n f &\le \sum_k \mathrm{Var}_{J_k^n} [(f \circ \hat{\psi}_k)\, |\hat{\psi}_k'|] + s^n \sum_k \sum_i f(b_{k,i}^n) \\ &\le \sum_k \mathrm{Var}_{J_k^n} [(f \circ \hat{\psi}_k)\, |\hat{\psi}_k'|] + s^n (2L(n+1)+2) \sum_k \max_{I_k^n} f(x) \end{aligned} \tag{2.11}$$

For each n and k we have

$$\begin{aligned} \mathrm{Var}_{J_k^n} [(f \circ \hat{\psi}_k)\, |\hat{\psi}_k'|] &\le \int_{J_k^n} \left| d[(f \circ \hat{\psi}_k)\, |\hat{\psi}_k'|] \right| \\ &\le \int_{J_k^n} |d(f \circ \hat{\psi}_k)| \cdot |\hat{\psi}_k'| + \int_{J_k^n} (f \circ \hat{\psi}_k) \cdot |\hat{\psi}_k''| \, dx \end{aligned}$$

Lemma 2.2 implies that

$$\sup_{x \in J_k^n} |\hat{\psi}_k''(x) / \hat{\psi}_k'(x)| = \sup_{x \in J_k^n} |(\ln |\hat{\psi}_k'(x)|)'| \le \tilde{C} \tag{2.12}$$

and we obtain

$$\begin{aligned} \mathrm{Var}_{J_k^n} [(f \circ \hat{\psi}_k)\, |\hat{\psi}_k'|] &\le s^n \int_{J_k^n} |d(f \circ \hat{\psi}_k)| + \tilde{C} \int_{J_k^n} |(f \circ \hat{\psi}_k)|\, |\hat{\psi}_k'| \, dx \\ &= s^n \, \mathrm{Var}_{I_k^n} f + \tilde{C} \int_{I_k^n} |f(x)| \, dx \end{aligned}$$

Now summing over k yields

$$\sum_k \mathrm{Var}_{J_k^n} [(f \circ \hat{\psi}_k)\, |\hat{\psi}_k'|] \le s^n \, \mathrm{Var}_{I^n} f + \tilde{C} \int_{I^n} |f(x)| \, dx \tag{2.13}$$

To estimate the second term in (2.11), we note that

$$\begin{aligned}\max_{I_k^n} f(x) &\le \max_{\hat{I}_k^n \setminus H} f(x) \\ &\le \min_{\hat{I}_k^n \setminus H} f(x) + \mathrm{Var}_{\hat{I}_k^n \setminus H} f \\ &\le [m(\hat{I}_k^n \setminus H)]^{-1} \int_{\hat{I}_k^n \setminus H} f(x)\, dx + \mathrm{Var}_{\hat{I}_k^n \setminus H} f \end{aligned} \tag{2.14}$$

Note that $m(\hat{I}_k^n \setminus H) \ge \delta_n - h$. Now summing over k in (2.14) and combining with (2.11) and (2.13) complete the proof of Proposition 2.1. □

3 Existence and uniqueness of a.c.c.i.m.

Fix $N \in \mathbb{N}$ such that

$$\alpha := s^N (2L(N+1)+3) < 1 \tag{3.1}$$

Proposition 3.1 *There exists a $b_{\min} > 0$ such that for every $b_{\max} \ge b_{\min}$ there is an $h_0 = h_0(b_{\min}, b_{\max}) > 0$ such that whenever $h < h_0$ the modified Perron-Frobenius operator P_1^N is well defined and on E_b and*

$$P_1^N(E_b) \subset E_b$$

for all $b \in (b_{\min}, b_{\max})$. Furthermore, for such b's we have $\mathrm{Var}\, P_1^N f < b$ for every $f \in E_b$.

Proof. First, we assume that $h_0 < \delta_N/2$. Then Proposition 2.1 implies

$$\mathrm{Var}\, P^N f \le \alpha\, \mathrm{Var}\, f + \beta \|f\|_1$$

with

$$\beta := \tilde{C} + 2/\delta_N$$

Now we put

$$b_{\min} = \frac{2\beta}{1-\alpha}$$

Let $b_{\max} > b_{\min}$ be given. Obviously,

$$\|f\|_\infty < b_{\max} + 2 \tag{3.2}$$

for all $f \in E_b$, $b \le b_{\max}$. Now we choose

$$h_0 = \min\left\{ \frac{1-\alpha}{2} \left((b_{\max}+2) \sum_{i=0}^{N} (sq)^i \right)^{-1}, \delta_N/2 \right\} \tag{3.3}$$

This implies, along with (2.4), that

$$\|P^N f\|_1 = \int_{I\setminus H^N} f\,dx > 1 - \frac{1-\alpha}{2} = \alpha + \frac{1-\alpha}{2} \tag{3.4}$$

for all $f \in E_b$, $b \le b_{\max}$. Then an easy calculation yields

$$\operatorname{Var} P_1^N f \le \|P^N f\|_1^{-1}(\alpha \operatorname{Var} f + \beta) < b$$

for all $f \in E_b$. □

It is easy to see that when the conditions of the above proposition hold, the operator P_1^N on E_b is continuous (in the L^1 metric). The Shauder-Tykhonov theorem then implies the existence of a fixed point:

Corollary 3.2 *Under the conditions of the previous proposition, the operator P_1^N preserves a density $f \in E_b$. Hence, $P^N f = \|P^N f\|_1 f$, so the density f defines an a.c.c.i.m. for the map T^N with eigenvalue $\lambda = \|P^N f\|_1$.*

Remark. Since $\operatorname{Var} P_1^N f < b$ for any $f \in E_b$, every fixed density $f \in E_b$ actually belongs in $E_{b_{\min}}$.

Next we will show that the above fixed density f is unique and defines an a.c.c.i.m. for the map T (not only for T^N). We first need to fix $b_{\max}$.

Lemma 3.3 *Let $b_{\min}$ be as in Proposition 3.1. Then there exists a $b_{\max} > b_{\min}$ such that whenever $h < h_0(b_{\min}, b_{\min})$, we have*

$$P_1(E_{b_{\min}}) \subset E_{b_{\max}}$$

whenever $h < h_0$.

Proof. For every $f \in E_{b_{\min}}$ we have, with the help of (3.4), that $\|Pf\|_1 \ge \|P^N f\|_1 \ge (1+\alpha)/2$. Now Proposition 2.1 implies

$$\operatorname{Var}(P_1 f) \le 2(1+\alpha)^{-1}[s(4L+3)b_{\min} + \tilde{C} + 2/\delta_N]$$

Hence it is enough to set $b_{\max}$ to the value on the right hand side. □

Now we fix a $b_{\max}$ as in this lemma. Assume that $h < h_0(b_{\min}, b_{\max})$. Proposition 3.1 ensures the existence of a fixed point $f \in E_b$ for the operator P_1^N in every space E_b, $b_{\min} \le b \le b_{\max}$. Assume, for the moment, that such a fixed point is unique.

Lemma 3.4 *If a fixed point $f \in E_{b_{\max}}$ for the operator P_1^N is unique, then there is a unique density $f \in E_{b_{\min}}$ such that $P_1 f = f$. Hence, f defines a unique a.c.c.i.m. for the map T.*

Proof. Indeed, if $g = P_1 f \neq f$, then $g \in E_{b_{\max}}$ by Lemma 3.3 and $P_1^N g = g$. This contradicts the uniqueness of a fixed point for P_1^N in $E_{b_{\max}}$. □

It is therefore enough to prove the uniqueness of a fixed point for P_1^N in $E_{b_{\max}}$. We do just that in the rest of this section.

Lemma 3.5 *For any $c \in (0,1)$ there exists a $\delta \in (0,1)$ such that for all $f \in E_{b_{\max}}$ we have*

$$m\{x\colon f(x) > c\} \geq \delta$$

Proof. Suppose, on the contrary, that there exists a $\tilde{c} \in (0,1)$ such that for any $\delta \in (0,1)$ we can find $f_\delta \in E_{b_{\max}}$ for which $m\{x\colon f_\delta(x) > \tilde{c}\} < \delta$. Choose $\delta < (1-\tilde{c})/(b_{\max}+2)$. Since $\|f\|_\infty < b_{\max} + 2$ for all $f \in E_{b_{\max}}$, we have

$$1 = \int_I f_\delta \, dm \leq \delta(b_{\max}+2) + (1-\delta)\tilde{c} < 1$$

a contradiction. □

Fix a $c_0 > 3/4$ and the corresponding δ_0 according to the above lemma. Let M denote a multiple of N satisfying

$$m(\hat{I}_k^M) < \frac{\delta_0}{2(2L+1)(b_{\max}+2)}$$

for all intervals $\hat{I}_k^M$. Note that the set $\{x\colon f(x) > c_0\}$ intersects at least $2(2L+1)(b_{\max}+2)$ intervals $\hat{I}_k^M$.

Since the map $\hat{T}^M$ has the same mixing invariant density h as the map $\hat{T}$, it also has the covering property, cf. Sect. 2. Therefore, there is a $K \geq 1$ such that the set $\hat{T}^{KM}\hat{I}_k^M$ covers $[0,1]$, up to finitely many points, for every k. Observe that now for every $x \in [0,1]$ (except finitely many) we have $\hat{T}^{-KM}x \cap \hat{I}_k^M \neq \emptyset$ for all intervals $\hat{I}_k^M$.

Recall that the Perron-Frobenius operator $P^n = P_{T^n}$ is defined by (2.5). If we assume $H = \emptyset$ (i.e., $I = [0,1]$), that definition would give us the ordinary Perron-Frobenius operator for the expanding map $\hat{T}^n$ (without holes). We denote it by $\hat{P}^n := P_{\hat{T}^n}$. It acts on $L^1([0,1])$.

For every function $f \in E_{b_{\max}}$ denote by $\hat{f}$ its extension to $[0,1]$ obtained by setting f to zero on the set $H = [0,1] \setminus I$. Denote the space of the so defined functions $\hat{f}$ by $\hat{E}_{b_{\max}}$. Such extension of f certainly increases its variation, but since $\|f\|_\infty < b_{\max}+2$, we have

$$\operatorname{Var} \hat{f} < b_{\max} + 2L(b_{\max}+2) < (2L+1)(b_{\max}+2) \tag{3.5}$$

Lastly, note that the operator P^n is well defined on $\hat{E}_{b_{\max}}$ for all $n \geq 1$.

Lemma 3.6 *For any $0 < c < 1/4$ there exists an $\varepsilon_c > 0$ such that if $\hat{f} \in \hat{E}_{b_{\max}}$ and $\hat{P}^{KM}\hat{f}(x) \leq \varepsilon_c$, then $\hat{f}(y) \leq c$ for all $y \in \hat{T}^{-KM}x$.*

Proof. Pick a $c < 1/4$, choose $\varepsilon_c < cS^{-KM}$, where $S = \sup |\hat{T}'|$, and suppose $\hat{f}(\tilde{y}) > c$ for some $\tilde{y} \in \hat{T}^{-KM}x$. Then

$$\hat{P}^{KM}\hat{f}(x) = \sum_{y \in \hat{T}^{-KM}x} \frac{\hat{f}(y)}{|(\hat{T}^{KM})'(y)|} > \frac{c}{S^{KM}} + \sum_{y \in \hat{T}^{-KM}x \setminus \tilde{y}} \frac{\hat{f}(y)}{|(\hat{T}^{KM})'(y)|} > \varepsilon_c$$

□

Lemma 3.7 *There exists an* $\varepsilon_0 > 0$ *such that* $\inf \hat{P}^{KM}\hat{f} \geq \varepsilon_0$ *for all* $\hat{f} \in \hat{E}_{b_{\max}}$.

Proof. Fix some $0 < c < 1/4$, and set $\varepsilon_0 = \varepsilon_c$ according to Lemma 3.6. Suppose $\hat{P}^{KM}f(x) < \varepsilon_0$ for some $\hat{f} \in \hat{E}_{b_{\max}}$ and $x \in [0,1]$. From Lemma 3.6 we obtain $\hat{f}(y) \leq c < 1/4$ for all $y \in \hat{T}^{-KM}x$. Since $\hat{T}^{KM}$ is covering, each interval $\hat{I}_k^M$ contains one such point y. Our choice of M implies that at least $2(2L+1)(b_{\max}+2)$ of those intervals contain a point z for which $\hat{f}(z) > c_0 > 3/4$. On each of the latter intervals, the variation of $\hat{f}$ exceeds $1/2$. Thus,

$$\operatorname{Var} \hat{f} > 2(2L+1)(b_{\max}+2)\frac{1}{2} = (2L+1)(b_{\max}+2)$$

which contradicts (3.5). □

Next, we will take a closer look at the holes. For technical reasons we will choose an even higher iterate of the map, $\hat{T}^{2KM}$. Let $x \in I$. Since $\hat{T}^{KM}$ is covering, we may expect x to have plenty of pre-images under T^{2KM} (and not only under $\hat{T}^{2KM}$). However, it may happen that too many of those preimages are "eaten up" by the holes, so that not enough are left in $T^{-2KM}x$. We need to prevent this from happening. Note that if $y \in \hat{T}^{-n}x \cap H$ for some $1 \leq n < 2KM$, then $\hat{T}^{n-2KM}y \cap T^{-2KM}x = \emptyset$, i.e. all the further preimages of the point y are excluded from the set $T^{-2KM}x$.

To ensure that sufficiently many preimages of each $x \in I$ survive the removal of the holes, we impose the condition that the holes are not only small, but in "generic" position:

Genericity Conditions on the Holes. The collection of holes H_l, $1 \leq l \leq L$, satisfies the following assumptions:

(G1) $T^{-1}x \neq \emptyset$ for each $x \in I$.

(G2) The images of the holes under the maps $\hat{T}^n$, $1 \leq n \leq 2KM$ do not overlap. In particular, we assume that the map $\hat{T}^{2KM}$ is one-to-one on each hole H_l and furthermore require that

$$\hat{T}^i(H_p) \cap \hat{T}^j(H_q) \neq \emptyset$$

for $1 \leq i, j \leq 2KM$ if and only if $i = j$ and $p = q$. Hence, for each $x \in I$ the set

$$\left(\cup_{n=1}^{2KM} \hat{T}^{-n}x\right) \cap H \tag{3.6}$$

consists of at most one point.

Remark. The reason why we call these conditions generic is clear from the following simple property: given $L \geq 1$, if we choose L points $y_l \in I$, $1 \leq l \leq L$, randomly (with respect to the Lebesgue measure on I), then with probability one each point y_l has a small neighborhood H_l, so that the above conditions hold. While this is quite obvious for (G2), it may not be immediately true for (G1). However, if we require that $\#\{\hat{T}^{-1}x\} \geq 2$ for each $x \in [0,1]$, then (G1) will hold as well. By the covering property, we can always fulfill the above requirement by replacing $\hat{T}$ with its higher iterate, before making holes in $[0,1]$. We do not do that because our version of (G1) is the weakest one we need.

Lemma 3.8 *Let $x \in I$. Then there exists a point $y \in T^{-KM}x$ such that y has a full set of pre-images under T^{KM}; i.e. $T^{-KM}y = \hat{T}^{-KM}y$.*

Proof. If $T^{-2KM}x = \hat{T}^{-2KM}x$, there is nothing to show. If not, the set (3.6) is not empty, and according to (G2) it consists of a single point, call it $z = \hat{T}^{-i}x \cap H$ with some $i = 1, \dots, 2KM$. If $i \leq KM$, then any point $y \in T^{-KM}x$ has a full set of preimages under T^{KM} (and the set $T^{-KM}x$ is not empty according to (G1)). If $i > KM$, then $T^{-KM}x = \hat{T}^{-KM}x$, and we know that $\#\{\hat{T}^{-KM}x\} \geq 2$ by the covering property. Now we have that $\hat{T}^{-i+KM}y \cap H \neq \emptyset$ for at most one point $y \in T^{-KM}x$, so all the others will have a full set of preimages. See an illustration to our argument in Fig. 2. □

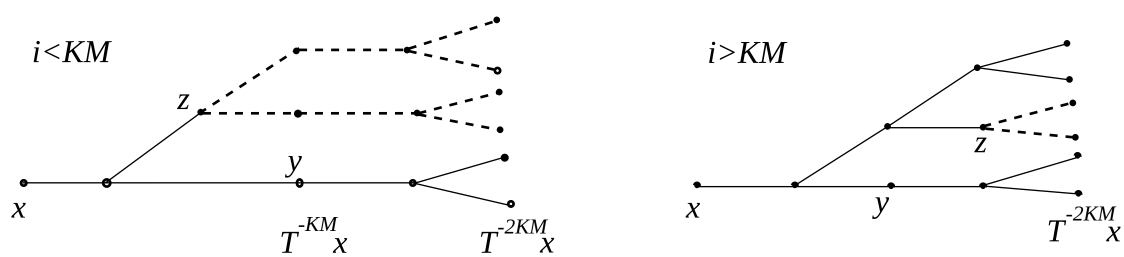

Figure 2: Pre-images of x under T. Dashed branches disappear into holes.

Lemma 3.9 *There exists an $\varepsilon_1 > 0$ such that $P_1^{2KM}f \geq \varepsilon_1$ for all $f \in E_{b_{\max}}$.*

Proof. Let $x \in I$. By the previous lemma there exists $y_1 \in T^{-KM}x$ such that y_1 has a full set of pre-images under T^{KM}. Then,

$$\begin{aligned}
(P_1^{2KM}f)(x) &= \|P^{2KM}f\|_1^{-1} \sum_{y \in T^{-2KM}x} \frac{f(y)}{|(T^{2KM})'(y)|} \\
&\geq \sum_{y \in \hat{T}^{-KM}y_1} \frac{f(y)}{|(\hat{T}^{KM})'(y_1)|\,|(\hat{T}^{KM})'(y)|} \\
&\geq S^{-KN0}(\hat{P}^{KM}f)(y_1) \\
&\geq \varepsilon_0 S^{-KM}
\end{aligned}$$

where we used Lemma 3.7 at the last step. We set $\varepsilon_1 = \varepsilon_0 S^{-KM}$ and complete the proof. $\square$

Proposition 3.10 *Assume that the holes satisfy the genericity conditions (G1) and (G2). Then there exists a unique $f \in E_{b_{\max}}$ such that $P_1^N f = f$.*

Proof. We only need to prove the uniqueness. Suppose P_1^N fixes two distinct densities f_1 and f_2 in $E_{b_{\max}}$. Since M is a multiple of N, we have $f_1 = P_1^{KM} f_1 \neq P_1^{KM} f_2 = f_2$. We distinguish two cases:

1. The densities f_1 and f_2 have equal eigenvalues, hence $\|P^{KM} f_1\|_1 = \|P^{KM} f_2\|_1$. For $s \in \mathbb{R}$ set $f_s = sf_1 + (1-s)f_2$. Then for all s, $\int_I f_s\, dm = 1$, and as long as $f_s \geq 0$ we have $P_1^N f_s = f_s$, hence $f_s \in E_{b_{\min}}$ by Remark after Corollary 3.2. Let $\sigma > 1$ such that $\inf_x f_\sigma(x) = 0$ for some $x \in I$. Since $f_\sigma = \lim_{s\to\sigma} f_s$, we have $f_\sigma \in E_{b_{\min}}$. Therefore, $P_1^{2KM} f_\sigma \geq \varepsilon_1$ by the previous lemma. But then $f_\sigma = P_1^{2KM} f_\sigma \geq \varepsilon_1$, a contradiction.

2. The eigenvalues are not equal. For example, let $\|P^{2KM} f_1\|_1 > \|P^{2KM} f_2\|_1$. Since $f_2 > \varepsilon_1$, there exists a $\beta > 0$ such that $\beta f_2 \geq f_1$. Therefore, $\beta P^{2KMn} f_2 \geq P^{2KMn} f_1$, and so $\beta \|P^{2KM} f_2\|_1^n \geq \|P^{2KM} f_1\|_1^n$ for all $n \geq 1$, a contradiction. $\square$

Hence we proved the following:

Theorem 3.11 *Suppose the holes satisfy the conditions $h < h_0(b_{\min}, b_{\max})$ and (G1)-(G2). Then there is a unique a.c.c.i.m. μ for the map T with a density $f_* \in E_{b_{\min}}$. Its eigenvalue is*

$$\lambda_* = \int_{I^1} f_*(x)\, dx$$

Note that the measure μ is equivalent to the Lebesgue measure on I, in fact $f_* \geq \varepsilon_1 > 0$.

4 The convergence property

Here we prove the following:

Theorem 4.1 *For any $f \in E_{b_{\max}}$ the sequence $\{P_1^n f\}$ converges to f_*, as $n \to \infty$, uniformly on I. Furthermore, there are constants $C > 0$ and $\theta \in (0,1)$ such that*

$$\|P_1^n f - f_*\|_\infty \leq C\, \theta^n$$

for all $f \in E_{b_{\max}}$.

Proof. By Lemma 3.9, it is enough to prove this fact for $f \in E^1_{b_{\max}}$, where

$$E^1_{b_{\max}} := \{f \in E_{b_{\max}} : \inf f \geq \varepsilon_1\}$$

For $f \in E^1_{b_{\max}}$ and $n \geq 1$ denote

$$\lambda_n(f) = \|P^n f\|_1$$

Note that $\lambda_n(f_*) = \lambda_*^n$.

Lemma 4.2 *For all $f \in E^1_{b_{\max}}$ and $n \geq 1$*

$$0 < C_1^{-1} \leq \lambda_n(f)/\lambda_*^n \leq C_1 < \infty$$

where C_1 is a constant (independent of f and n).

Proof. Set $C_1 = (b_{\max} + 2)/\varepsilon_1$. Then $f \leq C_1 f_*$ and $f_* \leq C_1 f$, and the lemma follows by the linearity of P. $\square$

Note that by the same argument

$$\|P^n f\|_\infty < C_2\, \lambda_*^n \tag{4.1}$$

with $C_2 = C_1\,(b_{\max} + 2)$.

Next we show that P_1 is "uniformly Lipschitzean" on $E^1_{b_{\max}}$ in the L^∞ metric. We denote by χ the function on I identically equal to $(1-h)^{-1}$. Note that $\chi \in E^1_{b_{\max}}$, and by the linearity of P we have

$$\|P^n f - P^n g\|_\infty \leq \|f - g\|_\infty \cdot \|P^n \chi\|_\infty \leq C_2\, \|f - g\|_\infty\, \lambda_*^n \tag{4.2}$$

Also, note that

$$|\lambda_n(f) - \lambda_n(g)| \leq \|P^n f - P^n g\|_1 \leq C_2\, \|f - g\|_\infty\, \lambda_*^n \tag{4.3}$$

Lemma 4.3 *For all $f, g \in E^1_{b_{\max}}$ and $n \geq 1$*

$$\|P_1^n f - P_1^n g\|_\infty \leq C_3\, \|f - g\|_\infty$$

where $C_3 > 0$ is a constant (independent of f, g and n).

Proof. For brevity, we write $\|\cdot\|$ for $\|\cdot\|_\infty$. By using (4.2) and (4.3), we have

$$\begin{aligned}
\|P_1^n f - P_1^n g\| &= \|P^n f/\lambda_n(f) - P^n g/\lambda_n(g)\| \\
&\leq \|(P^n f - P^n g)/\lambda_n(f)\| \\
&\quad + \|(P^n g)(\lambda_n(f) - \lambda_n(g))/(\lambda_n(f) \cdot \lambda_n(g))\| \\
&\leq C_2\, \|f - g\|\, (\lambda_*^n/\lambda_n(f) + \|P^n g\|\, \lambda_*^n/(\lambda_n(f) \cdot \lambda_n(g)))
\end{aligned}$$

Now the result follows from Lemma 4.2 and (4.1). $\square$

By this lemma, it is enough to prove that

$$\|P_1^{nR} f - f_*\|_\infty \leq C\, \theta^n \tag{4.4}$$

for any fixed $R \geq 1$ and all $f \in E^1_{b_{\max}}$ with some $\theta < 1$ and $C > 0$. We do this next.

Lemma 4.4 *There are constants $\kappa > 0$ and $\theta_1 < 1$ such that for every $f \in E^1_{b_{\max}}$*
(a) $f - \kappa f_* > 0$;
(b) *Put* $f_\kappa = (f - \kappa f_*)/\|f - \kappa f_*\|_1$, *then* $P_1^N f_\kappa \in E_{b_{\max}}$;
(c) $f - \kappa f_* \le \theta_1 f$.

Proof. The property (a) holds for all $\kappa < \varepsilon_1/(b_{\max}+2)$. For such κ, we have $\|f - \kappa f_*\|_1 = 1 - \kappa$, hence

$$\operatorname{Var} f_\kappa \le \frac{\operatorname{Var} f + \kappa \operatorname{Var} f_*}{1-\kappa} \le \frac{1+\kappa}{1-\kappa}\, b_{\max}$$

Now, by Proposition 2.1 and (3.1)-(3.4) we have

$$\operatorname{Var} P_1^N f_\kappa \le \frac{\alpha\,\frac{1+\kappa}{1-\kappa}\, b_{\max} + \beta}{(1+\alpha)/2} < b_{\max}$$

provided κ is small enough. The part (c) clearly holds with $\theta_1 = 1 - \kappa\varepsilon_1/(b_{\max}+2)$. □

We put $R\colon = 2KM + N$. By using Lemma 3.9 we obtain

Corollary 4.5 *For every $f \in E^1_{b_{\max}}$ there is a decomposition $P^R f = f_1 + f_{1*}$ such that*
(a) $f_{1*} = c_1 f_*$ *for some* $c_1 > 0$;
(b) $f_1 > 0$ *and* $f_1/\|f_1\|_1 \in E^1_{b_{\max}}$;
(c) $f_1 \le \theta_1 P^R f$.

Applying the corollary to the function $f_1/\|f_1\|_1$ and continuing in the same manner n times yields

Proposition 4.6 *For every $n \ge 1$ and $f \in E^1_{b_{\max}}$ there is a decomposition $P^{nR} f = f_n + f_{n*}$ such that*
(a) $f_{n*} = c_n f_*$ *for some* $c_n > 0$;
(b) $f_n > 0$ *and* $f_n/\|f_n\|_1 \in E^1_{b_{\max}}$;
(c) $f_n \le \theta_1^n P^{nR} f$.

Note that $\|P^{nR} f\|_1 = \lambda_{nR}(f) = O(\lambda_*^{nR})$ by Lemma 4.2, hence in the decomposition $P^{nR} f = f_n + f_{n*}$ we have

$$\sup f_n \le \mathrm{const} \cdot \theta_1^n \inf f_{n*}$$

i.e. the first term f_n is exponentially smaller than the second term f_{n*}.

This proves (4.4), and hence Theorem 4.1. □

Proposition 4.7 *For every $f \in E_{b_{\max}}$ there is a limit*

$$B_f\colon = \lim_{n\to\infty} \lambda_n(f)/\lambda_*^n$$

Hence,

$$\lim_{n\to\infty} \frac{P^n f}{\lambda_*^n} = B_f\, f_*$$

Proof. Note that for any $m < n$ we have $\lambda_{m+n}(f) = \lambda_n(P_1^m f) \cdot \lambda_m(f)$, hence it is enough to prove the proposition for $f \in E^1_{b_{\max}}$. Using $\lambda_n(f) = \lambda_1(P_1^{n-1} f) \cdot \lambda_{n-1}(f)$ gives

$$\ln \lambda_n(f) - \ln \lambda_*^n = \sum_{i=0}^{n-1} \Big(\ln \lambda_1(P_1^i f) - \ln \lambda_*\Big)$$

By Theorem 4.1 and (4.3) we obtain

$$\Big|\ln \lambda_1(P_1^i f) - \ln \lambda_*\Big| \leq \text{ const} \cdot \theta^i$$

hence the series is convergent. □

5 The invariant measure for T

Here we construct an invariant measure for the map T associated with the a.c.c.i.m. μ. Note first that any T-invariant measure must be supported on the Cantor set

$$\Lambda = \cap_{n \geq 1} I^n$$

so it is necessarily singular. A standard way of constructing an invariant measure associated with μ is [CMS1] to consider the conditional measures $\mu_n := \mu|_{I^n}$ for $n \geq 1$, i.e. the measures on I^n with densities

$$\frac{d\mu_n}{dm}(x) = \frac{f_*(x)}{\int_{I^n} f_*(y)\, dy} = \frac{f_*(x)}{\lambda_*^n} \qquad \text{for } x \in I^n$$

Since μ is conditionally invariant, we have $T_* \mu_n = \mu_{n-1}$ for all $n \geq 1$ (setting $\mu_0 = \mu$). We will prove that the sequence of the measures μ_n has a weak limit, $\bar{\mu}$, which is an invariant measure for the map T.

Proposition 5.1 *The sequence $\{\mu_n\}$ weakly converges to a probability measure $\bar{\mu}$.*

Proof. Let $J \subset [0, 1]$ be an interval (open or closed) and χ_J its indicator function: $\chi_J(x) = 1$ if $x \in J$ and $\chi_J(x) = 0$ otherwise. Then

$$\mu_n(J) = \lambda_*^{-n} \int_{I^n} f_*(x)\, \chi_J(x)\, dx = \frac{\mu(J)}{\lambda_*^n} \cdot \lambda_n \left(\frac{f_* \, \chi_J}{\mu(J)} \right)$$

If J is large enough, the function $f_* \, \chi_J / \mu(J)$ belongs in $E_{b_{\max}}$, and Proposition 4.7 implies that the sequence $\{\mu_n(J)\}$ has a limit. If J is small, then one can represent $J = J_1 \setminus J_2$ for some two large enough intervals $J_1 \supset J_2$, hence $\mu_n(J) = \mu_n(J_1) - \mu_n(J_2)$, and by the previous claim the limit of $\{\mu_n(J)\}$ exists again. □

Theorem 5.2 *The measure $\bar{\mu} = \lim_n \mu_n$ is T-invariant.*

Proof. On could hope to obtain the theorem immediately from Proposition 5.1 and the fact $T_*\mu_n = \mu_{n-1}$ by using the continuity of T_*. But the map T_* is actually discontinuous (because so is T), hence some more work is needed. Note that the map T is only discontinuous at finitely many points. Therefore, it is enough to show that the $\bar{\mu}$-measure of those points is zero. We will show a little more: the measure $\bar{\mu}$ has no atoms.

Lemma 5.3 *The measure $\bar{\mu}$ has no atoms.*

Proof. Suppose, on the contrary, that $\bar{\mu}$ has atoms, and x_0 is the "heaviest" one, i.e. $p_0 = \bar{\mu}(\{x_0\}) = \max_x \bar{\mu}(\{x\})$.

Next we use a standard argument in the study of expanding maps.

Sublemma 5.4 $s < \lambda_*$.

Proof. Note that by (3.1) we have $s^N < 1/3$ and by (3.3)

$$\lambda_*^N = 1 - \int_{I \setminus I^N} f_*(x)\, dx \geq 1 - \frac{1-\alpha}{2} > \frac{1}{2}$$

Hence $s^N < \lambda_*^N$. □

Now choose $m \geq 1$ so that

$$2(b_{\max} + 2)(s/\lambda_*)^m < \varepsilon_1/4 \tag{5.1}$$

Note that the map T^m, just like T, has finitely many discontinuity points. We take a tiny open interval $J \subset I$ containing x_0 such the map T^m has at most one discontinuity point on J. Then $TJ = J_1 \cup J_2$ where J_1, J_2 are two intervals (which may overlap).

For $n \geq 1$ denote by $\mu_n|_J$ the restriction of μ_n on J, and put $\nu_n = T_*^m(\mu_n|_J)$. Note that

$$\liminf_{n\to\infty} \nu_n(J_1 \cup J_2) = \liminf_{n\to\infty} \mu_n(J) \geq p_0 \tag{5.2}$$

for any open interval J containing x_0.

On the other hand, note that any point $x \in J_1 \cup J_2$ has at most two preimages under T^m on the interval J, call them y_1 and y_2, hence

$$\frac{d\nu_n}{dm}(x) = \frac{f_*(y_1)}{\lambda_*^n\, |(T^m)'(y_1)|} + \frac{f_*(y_2)}{\lambda_*^n\, |(T^m)'(y_2)|} \leq \frac{2(b_{\max}+2)s^m}{\lambda_*^n}$$

and by (5.1) we get

$$\frac{d\nu_n}{dm}(x) \leq \frac{\varepsilon_1}{4\lambda_*^{n-m}} \leq \frac{f_*(x)}{4\lambda_*^{n-m}}$$

Therefore,

$$\nu_n(J_1 \cup J_2) \leq \frac{1}{4}\, \mu_{n-m}(J_1 \cup J_2)$$

If the interval J is small enough, then so are J_1 and J_2, and then obviously

$$\limsup_{n\to\infty} \mu_{n-m}(J_1 \cup J_2) < 3p_0$$

which contradicts (5.2). Theorem 5.2 is proved. $\square$

We conclude with some open questions. It would be interesting to study the properties of the measure $\bar{\mu}$. We conjecture that it is ergodic, Bernoulli, and an equilibrium state for the potential function $-\ln|T'(x)|$. Under the assumptions of Theorem 1.1, these properties of $\bar{\mu}$ have been proved in [CMS1], along with a remarkable *escape rate formula*:

$$\chi(\bar{\mu}) = h_{\rm KS}(\bar{\mu}) + \gamma$$

Here $\chi(\bar{\mu}) = \int_\Lambda \ln|T'|\, d\bar{\mu}$ is the Lyapunov exponent, $h_{\rm KS}(\bar{\mu})$ is the Kolmogorov-Sinai entropy, and $\gamma = -\ln\lambda$ is the escape rate of the a.c.c.i.m. μ (recall that $\lambda = \mu(I^1)$ is its eigenvalue). The proofs in [CMS1, CMS2] were based on thermodynamic formalism and the symbolic representation of the system by a finite Markov chain (which existed because of the Markov property, see Theorem 1.1).

In our case, no finite Markov partition exists, hence one needs to develop a different approach. One way to do that is approximate the holes H by slightly larger holes that satisfy the Markov property, and then use the above results. This approach was employed in [CMT1, CMT2] where Anosov diffeomorphisms with small open holes were studied.

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References

[BK] V. Baladi, G. Keller, *Zeta functions and transfer operators for piecewise monotone transformations*, Commun. Math. Phys.**127** (1990), 459–477.

[Bo] R. Bowen, *Bernoulli maps of the interval*, Israel J. Math. **28** (1977), 161–168.

[Bu] J. Buzzi, *Specification on the interval*, Trans. Amer. Math. Soc. **349** (1997), 2737–2754.

[C] N. N. Cencova, *Statistical properties of smooth Smale horseshoes*, In: Mathematical problems of statistical mechanics and dynamics, 199–256, Math. Appl. (Soviet Ser.), 6, Reidel, Dordrecht-Boston, MA, 1986.

[CMT1] N. Chernov, R. Markarian, S. Troubetzkoy, *Conditionally invariant measures for Anosov maps with small holes*, Ergod. Th. Dynam. Syst. **18** (1998), 1049–1073.

[CMT2] N. Chernov, R. Markarian, S. Troubetzkoy, *Invariant measures for Anosov maps with small holes,* Ergod. Th. Dynam. Syst. (2000) to appear.

[CMM] P. Collet, S. Martnez, J. San Martn, *Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption*, Ann. Probab. **23** (1995), 1300–1314.

[CMS1] P. Collet, S. Martnez, B. Schmitt, *The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems*, Nonlinearity **7** (1994), 1437–1443.

[CMS2] P. Collet, S. Martnez, B. Schmitt, *The Pianigiani-Yorke measure for topological Markov chains*, Israel J. Math. **97** (1997), 61–70.

[D] J. R. Dorfman, *An introduction to chaos in nonequilibrium statistical mechanics.* Cambridge Lecture Notes in Physics, 14. Cambridge University Press, Cambridge, 1999.

[FKMP] P. A. Ferrari, H. Kesten, S. Martinez, P. Picco, *Existence of quasi-stationary distributions. A renewal dynamical approach* Ann. Probab. **23** (1995), 501–521.

[GD] P. Gaspard and J. R. Dorfman, *Chaotic scattering theory, thermodynamic formalism, and transport coefficients*, Phys. Rev. E **52** (1995), 3525–3552.

[HK] F. Hofbauer, G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982) 119–140.

[K] G. Keller, *Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems*, Trans. Amer. Math. Soc. **314** (1989), 433–497.

[KL] G. Keller, C. Liverani, *Stability of the Spectral Gap for transfer operators*, Annali della Scuola Normale di Pisa, Classe di Scienze (4) **XXVIII** (1999), pp. 141–152.

[LaY] A. Lasota, J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488

[LiY] T. Li, J. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. **235** (1978), 183–192.

[L] C. Liverani, *Decay of correlations for piecewise expanding maps*, J. Statist. Phys. **78** (1995), 1111–1129.

[PY] G. Pianigiani and J. Yorke, *Expanding maps on sets which are almost invariant. Decay and chaos*, Trans. Amer. Math. Soc. **252** (1979), 351–366.

[Ry] M. Rychlik, *Bounded variation and invariant measures*, Studia Math. **76** (1983), 69–80.