Improved estimates for correlations in billiards

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Abstract

We consider several classes of chaotic billiards with slow (polynomial) mixing rates, which include Bunimovich's stadium and dispersing billiards with cusps. In recent papers by Markarian and the present authors, estimates on the decay of correlations were obtained that were sub-optimal (they contained a redundant logarithmic factor). We sharpen those estimates by removing that factor.

Keywords: Billiards, decay of correlations, stadium.

1 Introduction

Here we sharpen estimates on mixing rates (i.e. the decay of correlations) for several classes of chaotic billiards, including the celebrated stadium introduced by Bunimovich [4, 5] and dispersing tables with cusps studied by Machta [15, 16]. In all our models the billiard map is known to be hyperbolic, as well as ergodic and Bernoulli, but its hyperbolicity is very non-uniform and consequently its mixing rates are slow (polynomial).

Physicists described this phenomenon as “intermittent chaos”. If you watch a typical trajectory of a billiard with polynomial mixing rates, then you observe that periods of truly chaotic behavior alternate with long regular-looking cycles when the orbit remains confined to a small and special part of phase space. That part acts as a ‘trap’ and that trap will play an important

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role in our analysis. The study of mixing rates in intermittent chaotic systems is more difficult than that of truly chaotic ones, and the resulting estimates may depend on delicate details of the dynamics in the traps.

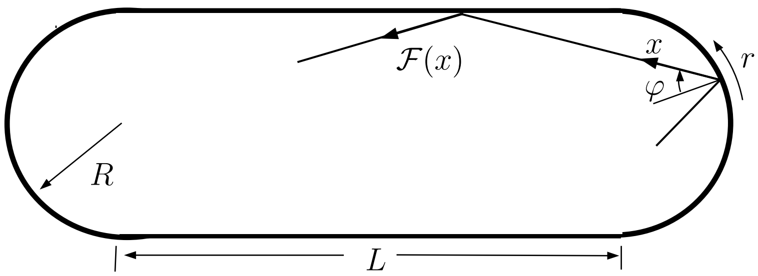

Figure 1: Bunimovich's (straight) stadium.

Our two main models are stadia. Bunimovich's original stadium is a convex billiard table bounded by two equal semicircles and two parallel straight lines, see Fig. 1. Due to its simplicity and geometric appeal it became popular in many theoretical and experimental studies, see for example [2, 8, 22, 26] and [25, Section 5.3]. We also discuss a 'skewed' stadium shown on Fig. 2, which is a convex domain bounded by two unequal circular arcs and two non-parallel lines; we call it 'drivebelt' due to its shape. Note that both types of stadia have C^1, but not C^2, boundary.

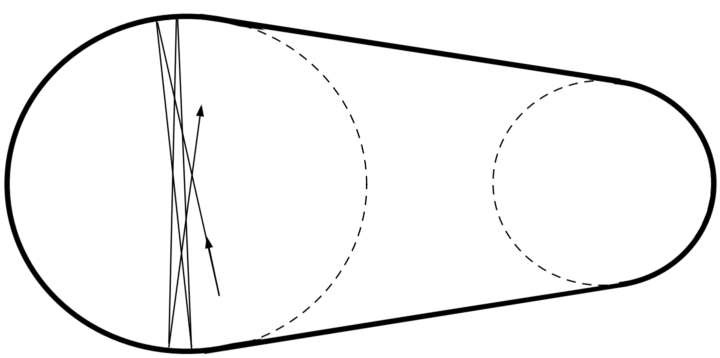

Figure 2: Skew stadium ('drivebelt' table).

Another interesting class of billiards with slow decay of correlations is made by tables with concave (dispersing) boundary that includes cusps (corner points with zero interior angle). For example, Fig. 3 shows a table studied by Machta [15, 16] made by three identical circular arcs. We also consider billiards in a square with a small fixed circular obstacle removed (Fig. 4). Such models are known as semi-dispersing billiards.

Let Γ denote the boundary of a billiard table and $\mathcal{M} = \Gamma \times [-\pi/2, \pi/2]$ the standard *collision space* whose canonical coordinates are r, φ, where r is the arc length parameter on Γ and $\varphi \in [-\pi/2, \pi/2]$ the angle of reflection, see Fig. 1. The collision map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ taking a collision point to the next collision, see Fig. 1, preserves smooth measure $d\mu = c\cos\varphi\, dr\, d\varphi$ on $\mathcal{M}$, here $c = (2|\Gamma|)^{-1}$ is normalizing constant.

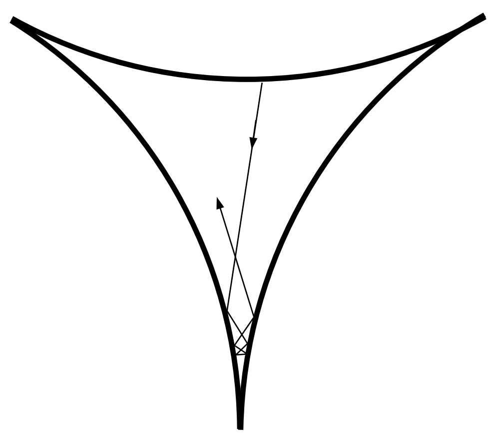

Figure 3: Machta's table with three cusps.

Let $f, g \in L^2_\mu(\mathcal{M})$ be two functions. *Correlations* are defined by

$$\mathcal{C}_n(f, g, \mathcal{F}, \mu) = \int_{\mathcal{M}} (f \circ \mathcal{F}^n)\, g\, d\mu - \int_{\mathcal{M}} f\, d\mu \int_{\mathcal{M}} g\, d\mu. \tag{1.1}$$

It is well known that $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is *mixing* if and only if

$$\lim_{n\to\infty} \mathcal{C}_n(f, g, \mathcal{F}, \mu) = 0 \qquad \forall f, g \in L^2_\mu(\mathcal{M}). \tag{1.2}$$

The rate of mixing of $\mathcal{F}$ is characterized by the speed of convergence in (1.2) for smooth enough functions f and g. We will always assume that f and g are Hölder continuous or piecewise Hölder continuous with singularities that

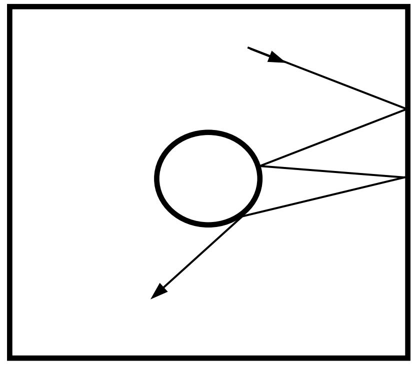

Figure 4: Semi-dispersing billiard.

coincide with those of the map $\mathcal{F}^k$ for some k. For example, the free path between successive reflections is one such function.

Bunimovich proved that under very general conditions (that easily hold for both types of stadia) billiards bounded by circular arcs and straight lines are hyperbolic, ergodic and K-mixing [3, 4, 5]. Due to other general results, these systems are also Bernoulli [10, 18]. Similar results were proved for semi-dispersing billiards and dispersing tables with cusps by Sinai [20], Reháček [19] and others.

It has been long expected bases on heuristic analysis [22, 15] that correlations decay as $\mathcal{O}(1/n)$, but rigorous estimates were obtained only recently: Markarian [17] proved that

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq \text{ const} \cdot (\ln n)^2/n \tag{1.3}$$

for the 'straight' Bunimovich stadia, the present authors [11] extended this result to the drivebelt stadia and semi-dispersing billiards, and lastly one of us with Markarian [13] derived the same bound on correlations for tables with cusps.

It was clear to all of us [17, 11, 13] that the logarithmic factor $(\ln n)^2$ was just an artifact of our method. Here we refine the method and remove that factor:

Theorem 1.1. *For both types of stadia, semi-dispersing billiards, and dispersing tables with cusps the correlations (1.1) for the billiard map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ and piecewise Hölder continuous functions f, g on $\mathcal{M}$ decay as $|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq$ const$/n$.*

We note that Bálint and Gouëzel very recently obtained a lower bound on correlations for stadia: they proved that $|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \geq \text{const}/n$ for infinitely many n's, see details in [1, Corollary 1.3]. Thus the speed of the decay of correlations for stadia is now completely determined.

2 General scheme

We start by briefly repeating the general scheme for the analysis of correlations for nonuniformly hyperbolic maps developed in [17, 11].

The first step is to localize places in the phase space $\mathcal{M}$ where the hyperbolicity of the map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ deteriorates (becomes non-uniform). In those places long sequences of iterations of the map $\mathcal{F}$ occur without exponential divergence of nearby trajectories. For example, in stadia, see [3, 4, 5] and [12, Chapter 8], hyperbolicity is ensured by the 'defocusing mechanism', which works when the billiard particle moves from one circular arc to the other, hence during long series of consecutive collisions with the same arc or bounces between two flat sides, the hyperbolicity weakens.

This suggests us to reduce the space $\mathcal{M}$ to a subset $M \subset \mathcal{M}$ on which the induced (first return) map $F\colon M \to M$ would be uniformly hyperbolic. For stadia [11] we define $M \subset \mathcal{M}$ as a set consisting of *first* (initial) collisions with every circular arc, that is,

$$M = \{x \in \mathcal{M} \colon x \text{ lies on an arc } C \subset \Gamma \text{ and } \mathcal{F}^{-1}x \notin C\}.$$

It is proven in [11] that indeed the first return map $F\colon M \to M$ is uniformly hyperbolic (more precisely, we proved uniform expansion and contraction of, respectively, unstable and stable tangent vectors). A similar reduction of the collision space was constructed for semi-dispersing billiards and tables with cusps [13].

It is also proven in [11, 13] that in all these cases the reduced map $F\colon M \to M$ enjoys exponential decay of correlations. This was done by constructing Young's tower [23] in M. That tower plays the role of a Markov partition of M; its full description is fairly complicated, but we only need two elements of it here.

The first element is the 'base of the tower' $\Delta_0 \subset M$, which Young calls a 'horseshoe with hyperbolic structure'. For us, its structure is irrelevant, we can just think of Δ_0 as a subset of M of positive measure. Then for a.e. point $x \in M$, its orbit $\{F^n x\}$ makes infinitely many returns to Δ_0, according to the Poincaré theorem. Young only counts 'proper returns' (or 'Markov returns'), as she defines in [23], but for us the exact meaning of proper returns is not relevant. Young proved that a.e. point $x \in M$ properly returns to Δ_0 infinitely many times. Let $R(x; F, \Delta_0)$ denote the time of the first proper return of x to Δ_0 (under F). The second important element of Young's tower construction is the exponential tail bound

$$\mu\big(x \in M\colon R(x; F, \Delta_0) > n\big) \leq \text{const} \cdot \theta^n \qquad \forall n \geq 1 \tag{2.1}$$

where $\theta < 1$ is a constant.

Next we turn back to the map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$. The tower in M can be easily (and naturally) extended to $\mathcal{M}$, thus we get a bigger tower (with the same base $\Delta_0 \subset M$); and a.e. point $x \in \mathcal{M}$ again properly returns to Δ_0 (now under $\mathcal{F}$) infinitely many times. For every $x \in \mathcal{M}$ let $R(x; \mathcal{F}, \Delta_0)$ denote the time of the first proper return of x to Δ_0 (under $\mathcal{F}$). Since $\mathcal{M}$ is larger than M, it takes typical points longer to return to Δ_0. In fact, only a polynomial tail bound on return times presumably holds:

$$\mu\big(x \in \mathcal{M}\colon R(x; \mathcal{F}, \Delta_0) > n\big) \leq \text{const} \cdot n^{-1} \qquad \forall n \geq 1. \tag{2.2}$$

Assuming that (2.2) holds, the bound on correlations in Theorem 1.1 immediately follows from Young's general result [24]. Our goal is to prove (2.2).

Consider the return times to M under $\mathcal{F}$, i.e.

$$R(x; \mathcal{F}, M) = \min\{r \geq 1 :\ \mathcal{F}^r(x) \in M\} \tag{2.3}$$

for $x \in \mathcal{M}$. The following estimate is standard for systems under consideration [7, 13, 11, 17, 22]:

$$\mu(x \in \mathcal{M} :\ R(x; \mathcal{F}, M) > n) \leq \text{const} \cdot n^{-1} \qquad \forall n \geq 1 \tag{2.4}$$

Equivalently,

$$\mu(x \in M :\ R(x; \mathcal{F}, M) > n) \leq \text{const} \cdot n^{-2} \qquad \forall n \geq 1 \tag{2.5}$$

The equivalence of (2.4) and (2.5) is proven in [11].

Now for every $n \geq 1$ and $x \in \mathcal{M}$ denote

$$r(x; n, M) = \#\{1 \leq i \leq n :\ \mathcal{F}^i(x) \in M\}$$

and

$$A_n = \{x \in \mathcal{M}\colon \;\; R(x; \mathcal{F}, \Delta_0) > n\},$$
$$B_{n,b} = \{x \in \mathcal{M}\colon \;\; r(x; n, M) > b \ln n\},$$

where $b > 0$ is a constant to be chosen shortly. By (2.1),

$$\mu(A_n \cap B_{n,b}) \leq \text{ const} \cdot n\, \theta^{b \ln n}.$$

Let us choose and fix $b > 0$ large enough so that $n\, \theta^{b \ln n} < n^{-1}$. It remains to prove the following:

Proposition 2.1. $\mu(A_n \setminus B_{n,b}) \leq \text{ const} \cdot n^{-1}$.

This proposition constitutes the main result of our paper and will be proven in the next sections.

Proposition 2.1 concludes the proof of (2.2). Now Theorem 1.1 readily follows from Young's general result [24].

To conclude this section, we recall how $\mu(A_n \setminus B_{n,b})$ is estimated in [17, 11], this should clarify our ideas. Since points $x \in A_n \setminus B_{n,b}$ return to M at most $b \ln n$ times during the first n iterates of $\mathcal{F}$, it is observed in [17, 11] that there are $\leq b \ln n$ time intervals between successive returns to M, and hence the longest such interval, we call it I, has length $\geq n/(b \ln n)$. Applying the bound (2.5) to the interval I gives

$$\mu(A_n \setminus B_{n,b}) \leq \text{ const} \cdot n\, (\ln n)^2\, n^{-2} \tag{2.6}$$

(the extra factor of n must be included because the interval I may appear anywhere within the longer interval $[1, n]$, and the measure μ is invariant). This gives us a weaker version of (2.2):

$$\mu(x \in \Delta\colon \;\; R(x; \mathcal{F}, \Delta_0) > n) \leq \text{ const} \cdot (\ln n)^2\, n^{-1} \qquad \forall n \geq 1.$$

Now Young's general result [24] implies the sub-optimal correlation bound (1.3), which is the main result of [17, 11].

But it is clear that the estimate (2.6) can only be sharp if most of the intervals between returns to M have length $\sim n/\ln n$. This is, however, the 'worst case scenario', which is extremely unlikely due to a special character of the dynamics between returns to M. We explore these special features to improve the estimate on $\mu(A_n \setminus B_{n,b})$ in this paper.

3 Analysis of the set $A_n \setminus B_{n,b}$

In this section we develop a strategy of the proof of Proposition 2.1 in the case of stadia. (Semi)dispersing billiards require a different approach that will be discussed in Section 5.

The set $A_n \setminus B_{n,b}$ consists of points $x \in \mathcal{M}$ whose images under n iterations of the map $\mathcal{F}$ satisfy two conditions: (i) they never return to the 'base' Δ_0 of Young's tower and (ii) they return to M at most $b \ln n$ times. Our goal is to show that $\mu(A_n \setminus B_{n,b}) = \mathcal{O}(n^{-1})$.

The following subsets of M

$$M_m = \{x \in M \colon R(x; \mathcal{F}, M) = m + 1\}$$

are called m-cells, $m \geq 0$. Here M_0 constitutes the 'bulk' of the space M and $M_1, M_2, \dots$ are (usually) small regions for which we make the following assumptions.

First, their measures decrease polynomially:

$$\mu(M_m) \leq C/m^r \tag{3.1}$$

where $r \geq 3$ and $C > 0$ are constants. (In all our models, $r = 3$, see [11, 13], but we adopt a more general assumption here.) Second, if $x \in M_m$ then $F(x) \in M_k$ with

$$\beta^{-1} m - C \leq k \leq \beta m + C \tag{3.2}$$

Here $\beta > 1$ is another constant. We will denote by $C > 0$ various constants whose exact values are not important.

Our next assumption concerns transition probabilities 'from cells to other cells':

$$p_{k,m_1,\dots,m_t} = \mu(M_k / F M_{m_1} \cap F^2 M_{m_2} \cap \cdots \cap F^t M_{m_t}) \tag{3.3}$$

where $k, m_1, \dots, m_t \geq 2$ are indices and $\mu(A/B) = \mu(A \cap B)/\mu(B)$ denotes the conditional measure. If we fix a sequence $m_1, \dots, m_t$, then $k = k_{m_1,\dots,m_t}$ becomes a random variable with probability distribution $\{p_{k,m_1,\dots,m_t}\}$. We will also use the random variable

$$\xi_{m_1,\dots,m_t} = \ln(k_{m_1,\dots,m_t}/m_1). \tag{3.4}$$

Note that $|\xi_{m_1,\dots,m_t}| \le \ln\beta + \mathcal{O}(1/k)$, due to (3.2). It is known that in the billiards under consideration the distribution of $\xi_{m_1,\dots,m_t}$ weakly converges to a fixed probability distribution on the interval $[-\ln\beta, \ln\beta]$, as $m_1,\dots,m_t \to \infty$. The reason is that the map F on the cells M_m with high indices m can be well approximated by a stationary random walk; this fact was already explored in [1, Section 4]. We only need a somewhat weaker property here, we state it as an assumption below and prove it in the next section.

We assume that for any $m \ge 2$ there exists a subset $\tilde{M}_m \subset M_m$ such that

$$\mu(M_m \setminus \tilde{M}_m)/\mu(M_m) \le C/m^d \tag{3.5}$$

for some $d > 0$, and for any $t \ge 1$ and $m_1,\dots,m_t$ the transition probabilities

$$\mu(M_k/F\tilde{M}_{m_1} \cap F^2\tilde{M}_{m_2} \cap \cdots \cap F^t\tilde{M}_{m_t}) \tag{3.6}$$

define a random variable $k = \tilde{k}_{m_1,\dots,m_t}$ so that the logarithmic variable

$$\tilde{\xi}_{m_1,\dots,m_t} = \ln\bigl(\tilde{k}_{m_1,\dots,m_t}/m_1\bigr)$$

satisfies

$$\tilde{\xi}_{m_1,\dots,m_t} \le \eta, \tag{3.7}$$

where η is a random variable supported on the interval $[-\ln\beta, \ln\beta + 1]$ and having a negative mean value

$$\bar{\eta} = \mathbb{E}(\eta) < 0. \tag{3.8}$$

We stress that the distribution of η is fixed (independent of $m_1,\dots,m_t$). Note that our approximation of (3.3) by (3.6) is only good as long as all $m_1,\dots,m_t$ are large, due to (3.5).

Proposition 3.1. *Under the above assumptions* $\mu(A_n \setminus B_{n,b}) = \mathcal{O}(n^{-r+2})$

The rest of this section is devoted to the proof of this proposition.

For every point $x \in A_n \setminus B_{n,b}$ we consider all its returns to M within the first n iterations of $\mathcal{F}$, i.e. all $0 \le i_1,\dots,i_{J_*} \le n$ such that $\mathcal{F}^{i_j}(x) \in M$. Note that the sequence $\{i_1,\dots,i_{J_*}\}$ and its length J_* depend on x and recall that $J_* \le b\ln n$. For every i_j we have $\mathcal{F}^{i_j}(x) \in M_{m_j}$ for some $m_j \ge 1$.

We fix a small $q > 0$ and say that m_j is *large* if $m_j \ge n^q$. We call an *island* a subinterval $I \subset [0,n]$ such that for all $k \in I$ the point $\mathcal{F}^k(x)$ either belongs to $\mathcal{M} \setminus M$ or lies in an m-cell M_m with a large m, i.e. $m \geq n^q$. Each island terminates at 0 or n or at a point k satisfying $\mathcal{F}^k(x) \in M_m$ with some 'small' $m < n^q$. Let $I_{\max} \subset [0, n]$ be the longest island (note that $I_{\max}$ depends on x).

Lemma 3.2. *There is a constant $\kappa = \kappa(b, \beta, q) > 0$ such that every $x \in A_n \setminus B_{n,b}$ we have $|I_{\max}| \geq \kappa n$.*

Proof. The point x will be fixed throughout the proof. Consider an arbitrary island I. If it does not terminate at 0 or n, then due to (3.2) we have

$$\#\{k \in I \colon \mathcal{F}^k(x) \in M\} \geq t$$

where t is the smallest integer satisfying $|I| \leq n^q(1 + \beta + \cdots + \beta^t)$, hence

$$t \geq \frac{\ln|I| - \ln n^q + \mathcal{O}(1)}{\ln \beta}.$$

Now it is easy to see that

$$\frac{\ln|I_{\max}| - \ln n^q + \mathcal{O}(1)}{\ln \beta} \times \frac{n}{|I_{\max}|} \leq b \ln n.$$

Suppose $q < 1/2$, then for large n we have

$$|I_{\max}| \geq \frac{n}{b \ln n} \geq n^{2q}$$

thus $\ln|I_{\max}| - \ln n^q \geq q \ln n$, which implies the lemma with any choice of $\kappa < q/(b \ln \beta)$. $\square$

Our further analysis is done within the maximal island $I_{\max} = [K_0, K_1]$. Given $x \in A_n \setminus B_{n,b}$, we call a subinterval $J \subset I_{\max}$ a *run* if $\mathcal{F}^k(x) \notin M$ for every $k \in J$. Let $J_{\max} = [n_0, n_1]$ be the longest run within $I_{\max}$. As the number of runs does not exceed $b \ln n$, we have

$$|J_{\max}| \geq \frac{|I_{\max}|}{b \ln n} \geq \frac{\kappa n}{b \ln n}.$$

Without loss of generality, we assume that $K_1 - n_1 \geq n_0 - K_0$, i.e. the right subinterval of $I_{\max} \setminus J_{\max}$ is at least as long as the left one (because the time reversibility of the billiard dynamics allows us to turn the time backwards).

Before going into further detail, we describe the idea of the proof. If $|J_{\max}| \sim n$, then due to (3.1) the measure of the corresponding points x is $\mathcal{O}(n^{-r})$. Summing over all possible n_0 and n_1 gives the measure bound $\mathcal{O}(n^{-r+2})$ as claimed by Proposition 3.1. The problem may arise when $|J_{\max}| = o(n)$, because the measure of such points x will be $\mathcal{O}(|J_{\max}|^{-r}) \gg n^{-r}$. But then we will show that, due to (3.8), for typical points $y \in M_{|J_{\max}|}$ we have $F^t(y) \in M_{m_t}$ where m_t decreases exponentially fast. Then if we add up all runs covered by the trajectory $\mathcal{F}^k(x)$ of x during the interval $n_1 \le k \le K_1$, we get $\mathcal{O}(|J_{\max}|)$. On the other hand, $K_1 - n_0 \ge \kappa n/2$. This will prove that $|J_{\max}| \ge cn$ for some $c > 0$, taking us back to the case $|J_{\max}| \sim n$, which is handled already.

Let $n_1 < \cdots < n_s \le K_1$ be all the moments such that $\mathcal{F}^{n_t}(x) \in M$. We need to estimate the measure of points x such that

(a) $\mathcal{F}^{n_0}(x) \in M_{|J_{\max}|}$;

(b) $\mathcal{F}^{n_t}(x) \in M_{m_t}$ for $n^q \le m_t \le m_0 = |J_{\max}|$ for all $t = 0, \dots, s$;

(c) $m_0 + \cdots + m_s \ge \kappa n/2$ and $s < b \ln n$.

We will use the assumptions (3.5)–(3.8). At each iteration of F we incur relative losses bounded by $\mathcal{O}(1/m_t^d) = \mathcal{O}(1/n^{qd})$ due to (3.5), thus the total losses are bounded by $r\mu(M_{J_{\max}})/n^{qd} = \mathcal{O}(n^{-r-qd} \ln n)$, which is $\ll n^{-r}$.

Next our assumptions (3.5)–(3.8) allow us to estimate the cell index m_t from above

$$m_t \le \tilde{m}_t = m_0\, e^{\eta_1 + \cdots + \eta_t} \tag{3.9}$$

where $\eta_1, \dots, \eta_t$ are independent random variables having the same distribution as η. The probability distribution here is induced by the measure on $\tilde{M}_{m_0}$, so we denote it by $\mathbb{P}_{m_0}$.

Of course it is possible that the random variable $\tilde{m}_t$ defined by (3.9) will exceed $|J_{\max}|$ or fall below n^q, thus violating the above restriction (b) on m_t. But this only means that our probabilistic estimates will exceed the actual measure of points satisfying (a)–(c). Since we are estimating our measures from above, this approach is logically consistent.

Lemma 3.3. *For every $\varepsilon > 0$ there are $C > 0$ and $\gamma \in (0,1)$ such that for all $t \ge 1$ we have the following probability estimate:*

$$\mathbb{P}_{m_0}\big(\eta_1 + \cdots + \eta_t > (\bar{\eta} + \varepsilon)t\big) \le C\gamma^t$$

Proof. This follows from the classical theorem on large deviations, see e.g. [14, Theorem 2.2.3]. $\square$

We fix $\varepsilon > 0$ such that $\bar{\eta} + \varepsilon < 0$, i.e. $\theta\colon = e^{\bar{\eta}+\varepsilon} < 1$ and then fix the corresponding $\gamma < 1$. Thus with an overwhelming probability we should have $\tilde{m}_t \le m_0\theta^t$ for all large enough t. This idea lies behind the following lemma.

Lemma 3.4. *There is $C > 0$ such that*

$$\sum_{m_0=n/(b\ln n)}^{n} \mu(M_{m_0})\,\mathbb{P}_{m_0}\big(\tilde{m}_t > m_0\theta^t \ \text{ for some } \ t > \kappa n/(10m_0)\big) \le Cn^{-r+1}.$$

Proof. Due to the previous lemma, our probability is bounded by

$$\sum_{m_0=\kappa n/(b\ln n)}^{n} \mu(M_{m_0})\,\frac{C\gamma^{\kappa n/10m_0}}{1-\gamma}.$$

Since $\mu(M_{m_0}) = \mathcal{O}(m_0^{-r})$, we can obtain an upper bound by integral estimation:

$$\begin{aligned}\sum_{m_0=\kappa n/(b\ln n)}^{n} \frac{\gamma^{\kappa n/(10m_0)}}{m_0^r} &\le \text{ const} \int_1^n x^{-r}\gamma^{\frac{\kappa n}{10x}}\,dx \\ &= \frac{\text{const}}{n^{r-1}} \int_1^n y^{r-2}\gamma^{\kappa y/10}\,dy\end{aligned}$$

where we used the change of variables $y = n/x$. It remains to note that $\int_1^\infty y^{r-2}\gamma^{\kappa y/10}\,dy < \infty$ since $\gamma < 1$. $\square$

Adding over n_0 gives an upper bound $\mathcal{O}(n^{-r+2})$ on the measure as required by Proposition 3.1.

Finally, if $m_t \le m_0\theta^t$ for all $t > \kappa n/(10m_0)$, then due to the above restrictions (b) and (c) we have

$$\frac{\kappa n}{2} \le m_0 + \cdots + m_r \le \frac{m_0\kappa n}{10m_0} + \sum_{t=1}^{\infty} m_0\theta^t \le \frac{\kappa n}{10} + \frac{m_0}{1-\theta},$$

hence $m_0 \ge (1-\theta)\kappa n/3$ and $\mu(M_{m_0}) = \mathcal{O}(n^{-r})$. This completes the proof of Proposition 3.1.

4 Dynamics in cells for stadia

It remains to prove our assumptions (3.5)–(3.8). They essentially follow from the Markovian character of the map F restricted to cells M_m with high indices m. First we do this for Bunimovich's 'straight' stadia, for which the structure of m-cells is well known [6, 7, 11, 17, 22] and the Markovian character of the map F is already explored in [1]. Here we just recall relevant facts.

First of all, the m-cell M_m for any large m is a union of domains of two types: one consists of points whose trajectories experience a long series of consecutive collisions at the same semicircle, and the other is made of points whose trajectories bounce between the two parallel flat lines. The first type domains are small, they have measure $\sim m^{-4}$, which is negligible. The second type domains have measure $\sim m^{-3}$ and we only consider them. Slightly abusing notation we will call those domains m-cells.

The set $M \subset \mathcal{M}$ is the union of two identical parallelograms in the r, φ coordinates, each one is constructed on one semicircle $\mathcal{C} \subset \Gamma$. Fig. 5 shows one of them ($CEDB$), and the m-cells make a nested structure of self-similar domains converging to the vertices C and D as $m \to \infty$.

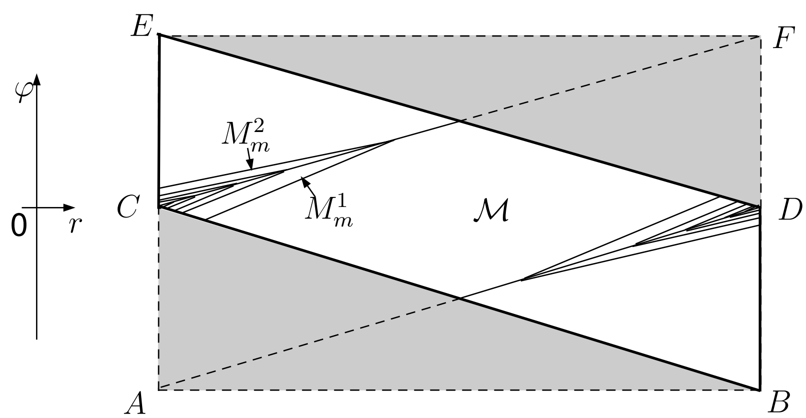

Figure 5: One half of the set M.

Each m cell consists of two identical parts, one near C and the other near D. Due to the obvious symmetry, we only describe the domain of M_m near C. It, in turn, is a union of two strips, one (M_m^1) below the line CF and the other (M_m^2) above the line CF. The domain M_m^1 contains points mapped (by

$\mathcal{F}$) directly to a straight side of the stadium; the domain M_m^2 contains points that are mapped by $\mathcal{F}$ onto the same arc (but to its almost diametrically opposite point) and then to a straight side.

Fig. 6 shows the image $F(M_m)$ of an m-cell. These images, too, make a nested structure of self-similar domains converging to the vertices C and D as $m \to \infty$. The intersection of k-cells with the image $F(M_m)$ is depicted on Fig. 6. Each $F(M_m)$ intersects M_k with

$$\tfrac{1}{3}m - \mathcal{O}(1) \leq k \leq 3m + \mathcal{O}(1). \tag{4.1}$$

Most of the intersections $F(M_m) \cap M_k$ are parallelogram-looking domains, but a few ($\leq$const) intersections near the ends of the strip $F(M_m)$ look like less regular polygons with 3, 4 or 5 sides.

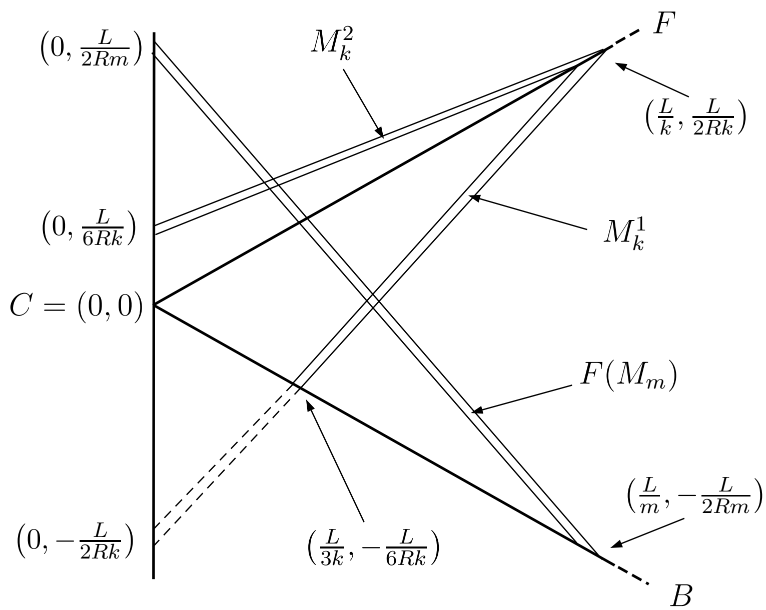

Figure 6: The domains M_k^1 and M_k^2 and the image $F(M_m)$ near the vertex C. The vertex C has coordinates $r = 0$ and $\varphi = 0$; other coordinates are shown, to the leading order.

To clarify our ideas, let us assume for a moment that the domains M_m^1 and M_m^2 are exact trapezoids which shrink homotetically as m growth. Also let the strip $F(M_m)$ be a perfect trapezoid that scales with m, i.e. shrinks homotetically as $m \to \infty$. Let the measure μ have constant density and the map F be linear within every m-cell. Lastly, let us ignore the irregular intersections $F(M_m) \cap M_k$, i.e. assume that all of these intersections are parallelograms, and (4.1) holds without the $\mathcal{O}(1)$ terms.

Under these ideal conditions, it is easy to see that the cells M_m would make a Markov partition, so the action of F would be equivalent to a discrete Markov chain. Moreover, the random variables (3.4) would be almost identically distributed, i.e. their distribution would not depend on $m_1, \dots, m_t$ and t, modulo a $\mathcal{O}(1/k)$ error term accounting for their discrete character (and of course, as long as all the indices are ≥ 3). Effectively, the transitions between cells could be described by a sequence of independent random variables.

It is known, see [1, Eq. (35)], that

$$\mu\big(M_k/F(M_m)\big) = \frac{3m}{8k^2} + \mathcal{O}\Big(\frac{1}{m^2}\Big) \tag{4.2}$$

for $k \in [m/3 - C, 3m + C]$. Under our ideal assumptions, the same estimate would hold for multistep transition probabilities (3.3). Observe that

$$\sum_{k=m/3}^{3m} \frac{3m}{8k^2} \ln\frac{k}{m} \sim \int_{m/3}^{3m} \frac{3m}{8x^2} \ln\frac{x}{m}\, dx = 1 - \frac{5}{4} \ln 3 < 0. \tag{4.3}$$

Thus one can easily find a random variable η satisfying (3.7) and (3.8).

Now in reality cell boundaries are curvilinear, the density of μ is not constant, and the map F is nonlinear. We just need to estimate the effect of nonlinearity in order to prove our assumptions (3.5)–(3.8). This was in fact done in [1, Section 4], we only outline the argument here.

For one-step transition probabilities, i.e. for $t = 1$ in (3.6), this can be done by a direct and fairly elementary analysis. The measure μ has density $\cos\varphi$, and $|\varphi| = \mathcal{O}(m^{-1})$ on M_m, so the density variation over M_m is just $\mathcal{O}(m^{-2})$. The derivative of the map $F\colon (r,\varphi) \mapsto (r_1, \varphi_1)$ is given by (see [12, Chapter 2])

$$DF = \frac{-1}{\cos\varphi_1} \begin{bmatrix} R^{-1}\tau + \cos\varphi & \tau \\ R^{-2}\tau + R^{-1}\cos\varphi_1 + R^{-1}\cos\varphi & \tau R^{-1} + \cos\varphi_1 \end{bmatrix},$$

where R is the radius of the semicircles bounding the stadium (Fig. 1). Within the m-cell, we have

$$\tau = \sqrt{4R^2m^2 + L^2} + \mathcal{O}(m^{-2}),$$

where L is the length of the flat lines bounding the stadium. Thus the derivative of F varies by $\mathcal{O}(m^{-2})$ within every m-cell. Lastly, the curve separating M_m^1 from M_{m-1}^1 has equation

$$r = R\varphi + R\sin^{-1}\frac{R\sin\varphi}{\sqrt{(2m-1)^2R^2+L^2}} + R\tan^{-1}\frac{L}{(2m-1)R},$$

i.e. its slope is $dr/d\varphi = R + \mathcal{O}(m^{-1})$. The curve separating M_m^2 from M_{m-1}^2 has similar equation

$$r = 3R\varphi + R\sin^{-1}\frac{R\sin\varphi}{\sqrt{(2m-1)^2R^2+L^2}} + R\tan^{-1}\frac{L}{(2m-1)R},$$

i.e. its slope is $dr/d\varphi = 3R + \mathcal{O}(m^{-1})$. So the curves separating neighboring m-cells can be approximated in the C^1 metric by parallel straight lines, up to some $\mathcal{O}(1/m)$ error terms. It is now easy to check that the width of the domain M_m^i, $i = 1, 2$, is $c_i m^{-2} + \mathcal{O}(m^{-3})$, where $c_1, c_2 > 0$ are constants.

Thus our map, measure, and cells admit good linear approximations: the non-linearity only affects higher-order terms in all relevant parameters. Also, the irregular intersections $F(M_m) \cap M_k$, see above, have to be thrown out (of M_m), but their relative measure is $\mathcal{O}(1/k)$. We conclude that (3.5)–(3.8) hold for $t = 1$, we can even afford a luxury to set $d = 1$ in (3.5).

For multi-step transition probabilities, i.e. for $t \geq 2$, such a direct analysis is hardly feasible, so one needs a more sophisticated argument. In [1, pages 488–490], multi-step transition probabilities are estimated by foliating each m-cell M_m by unstable curves $\{W^u\}$ which stretch completely across M_m, i.e. terminate on its 'long' sides (separating M_m from M_{m-1} and M_{m+1}). Such a foliation can be chosen smooth enough so that conditional measures on the fibers W^u are nearly uniform. To analyze multistep transition probabilities (3.3), we would like our foliations to be F-invariant under relevant iterations of F, i.e. until the image of a fiber either falls into an irregular intersection $F(M_m) \cap M_k$, see above, or lands in the 'bulk' M_0. Such a 'limited' invariance can be ensured with a little extra work [1, pages 488–490].

We describe here an alternative approach that gives the invariance of the foliation 'for free'. Let us foliate m-cells by unstable manifolds of the map F, so that in each cell M_m only unstable manifolds that stretch across M_m completely (terminating on its long sides) are used. The invariance of this foliation under F is then automatic. A little price to pay for this convenience is to deal with 'gaps' between unstable manifolds.

Indeed, since arbitrarily short unstable manifolds are dense in M, our foliation is 'holey' (it covers a Cantor-like set), there infinitely many gaps in M_m where unstable manifolds fail to reach one of the two long sides of M_m. We need to estimate the relative measure of gaps in M_m. For $x \in M$, let $r^u(x)$ denote the distance from x to the nearer endpoint of the unstable manifold passing through x (if none exists, we put $r^u(x) = 0$). Then for any stable curve $W \subset M$ and $\varepsilon > 0$ we have

$$\mathbf{m}(x \in W \colon r^u(x) < \varepsilon) < C\varepsilon$$

where $C > 0$ is a constant (independent of W) and $\mathbf{m}$ denotes the Lebesgue measure on W. (For the proof, see [12]: it is shown in [12, Section 5.12] that this estimate follows from the so-called first growth lemma in the case of dispersing billiards, and the argument applies to stadia without change; and the first growth lemma for stadia is proved in [12, Section 8.14].)

Thus the union of all gaps has relative measure $\mathcal{O}(1/m)$ in M_m, so it can be simply excluded from M_m. After all bad parts of M_m are removed, as described above, we obtain the desired subset $\tilde{M}_m \subset M_m$.

The conditional measure on each unstable manifold W is smooth and its density ρ satisfies

$$\left|\frac{d}{dx} \ln \rho(x)\right| \leq \frac{C}{|W|^{1/2}},$$

where $C > 0$ is a constant, see [12, Section 8.12]. Hence the fluctuations of the density on our fibers in $F(M_m)$ are bounded by

$$\frac{\max_W \rho - \min_W \rho}{\min_W \rho} \leq C|W|^{1/2} \leq Cm^{-1/2}.$$

The distortions of unstable manifolds under the map F are also estimated in [12, Section 8.12]: if W is an unstable manifold on which F^n is smooth, then

$$\frac{\max_W \mathcal{J}_W F^n - \min_W J_W F^n}{\min_W J_W F^n} \leq C|W|^{1/2} \leq Cm^{-1/2},$$

where $\mathcal{J}_W F^n$ is the Jacobian of the map F^n restricted to W, and the constant $C > 0$ is independent of W and n.

These facts imply that the transformation of the conditional measures on unstable manifolds can be tightly approximated by a Markov chain as non-linearity only affects higher order terms.

Now in order to handle multistep transition probabilities (3.6) we note that $F\tilde{M}_{m_1} \cap F^2\tilde{M}_{m_2} \cap \cdots \cap F^t\tilde{M}_{m_t}$ is a union of unstable manifolds in $F\tilde{M}_{m_1}$, each stretching completely across the strip $F\tilde{M}_{m_1}$. Thus it is enough to show that for any two such unstable manifolds $W_1, W_2 \subset FM_{m_1}$ and the conditional measures ν_{W_1}, ν_{W_2} on them we have

$$\left|\nu_{W_1}(M_k) - \nu_{W_2}(M_k)\right| \leq Cm_1^{-1/2},$$

for every $k \in [m_1/3 + C, 3m_1 - C]$. This readily follows from standard estimates on the Jacobian of the holonomy map, see [12, Section 8.13].

In summary, we obtain (3.5)–(3.8) with $d = 1/2$. This concludes our analysis of the 'straight' Bunimovich stadium.

The dynamics in the 'drivebelt' billiard table is studied in [11, Section 9]. The m-cells M_m there consist only of points experiencing long series of consecutive collisions at the same arc (as the number of possible consecutive bounces off the two nonparallel flat sides is limited). But there are two very different types of series of collisions with the same arc. First, there are 'sliding' trajectories (where $\varphi \approx \pm\pi/2$), just like in the straight stadium, which make a small set of measure $\sim m^{-4}$, and so they are negligible. Second, there are trajectories bouncing off within the bigger circle almost orthogonally to its boundary (i.e. with $\varphi \approx 0$), see Fig. 2. It was shown in [11, Section 9] that they make a set of measure $\sim m^{-3}$, thus they are of interest to us.

The structure of m-cells in the drivebelt stadium are described in [11, Section 9]. It is a sequence of self-similar domains accumulating at two corner points of the space M (which again consists of two parallelograms in the $r\varphi$ coordinates). The images of the m-cells are also self-similar domains accumulating at the same corner points of M. More precisely, each $F(M_m)$ intersects M_k with

$$\tfrac{1}{7}m - \mathcal{O}(1) \leq k \leq 7m + \mathcal{O}(1), \tag{4.4}$$

which is similar to (4.1), but now $\beta = 7$ instead of $\beta = 3$. Accordingly, the transition probabilities are

$$\mu\big(M_k/FM_m\big) = \frac{7m}{48k^2} + \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

for $k \in [m/7 - C, 7m + C]$, which is an analogue of (4.2). Here $7/48$ is just a normalizing factor resulting from the requirement $\sum_k \mu\big(M_k/FM_m\big) = 1$. A simple calculation analogous to (4.3) gives

$$\sum_{k=m/7}^{7m} \frac{7}{48k^2} \ln\frac{k}{m} \sim \int_{m/7}^{7m} \frac{7m}{48x^2} \ln\frac{x}{m}\, dx = 1 - \frac{50}{48} \ln 7 < 0.$$

Thus we get a necessary Markov approximation in this case, too. We note that the billiard dynamics in the drivebelt region is very similar to that in Bunimovich's stadia, but because the latter has been popular for a long time, its dynamical properties have been investigated in great detail (see, for example in [12, Chapter 8]). These include sharp estimates on distortion bounds, conditional densities on unstable manifolds, and the Jacobian of the holonomy map. For the drivebelt region, such estimates are obtained, in a weaker form, in [7, Appendix 1.5]. We plan to publish separately a detailed investigation of the drivebelt stadia along the lines of [12, Chapter 8].

5 Cell dynamics for (semi-)dispersing tables

Lastly we turn to the semi-dispersing billiards (Fig. 4) and dispersing billiards with cusps (Fig. 3). They turn out to be much easier than the stadia.

For semi-dispersing billiards, we define M to consist of all collisions with the circular obstacle. Now the return map $F\colon M \to M$ is equivalent to the well studied Lorentz gas billiard map without horizon [11]. Then m-cells are made of points colliding with the sides of the square exactly m times before returning to the obstacle.

The structure of m cells in the semi-dispersing billiards is described in the literature [6, 7, 12]. In particular, we still have $\mu(M_m) = \mathcal{O}(m^{-3})$, as for the stadia. But there is a crucial difference: the bound (3.2) fails, and instead the image $F(M_m)$ of the m-cell intersects other cells M_k with

$$\mathcal{O}(m^{1/2}) < k < \mathcal{O}(m^2). \tag{5.1}$$

Moreover, typical points $x \in M_m$ land in cells M_k with $k \ll m$, in fact the average value of k is $E(k) = \mathcal{O}(m^{1/2})$. Thus the majority of points $x \in M_m$ 'escape' from 'high cells' into M_0 much faster than they do in the case of stadia. This is good, but our method used for the stadia (based on Lemma 3.2) will no longer apply, so a different strategy must be employed.

A crucial estimate is proved in [21, Lemma 16]:

Lemma 5.1 ([21]). *There are constants $p, q > 0$ such that for any large $b > 0$ there is a subset $\tilde{M}_m \subset M_m$ such that*

$$\mu(M_m \setminus \tilde{M}_m) \le Cm^{-p}\mu(M_m),$$

where $C = C(b) > 0$ is a constant, and for every $x \in \tilde{M}_m$ the images $F^i(x)$ for $i = 1, \dots, b \ln m$ never appear in cells M_k with $k > m^{1-q}$.

Lemma 16 in [21] is stated and proved for the iterations of T^{-1}, but the time reversibility of the billiard dynamics makes it applicable to T as well. For the sake of completeness, we outline the proof here. Examining the cell structure it is easy to see that the one-step transition probabilities are

$$\mu\big(M_k/F(M_m)\big) \asymp (m+k)/k^3, \tag{5.2}$$

where $A \asymp B$ means that $0 < c_1 < A/B < c_2$ for two constants c_1, c_2 and k satisfies (5.1). Thus for any small $e > 0$

$$\mu\Big(\cup_{k=m^{1/2+e}}^{m^2} M_k/M_m\Big) = \mathcal{O}\big(1/m^{2e}\big). \tag{5.3}$$

Hence we can neglect points $x \in M_m$ such that $F(x) \in M_k$ with $k > m^{\frac{1}{2}+e}$ and those for which $F^2(x) \in M_j$ with $j > m^{\frac{1}{4}+3e}$. It remains to estimate the probability that points $y \in M_j$ with $j \le m^{\frac{1}{4}+3e}$ will come up to M_i, $i \ge m^{1-q}$, within $\mathcal{O}(\ln m)$ iterations of F. The key observation is that the cells M_j are long (their length is $l_j \sim j^{-1/2}$) and the cells M_i are very short (because $l_i \sim i^{-1/2} = \mathcal{O}(l_j^3)$).

Actually we need to deal with homogeneous sections of m-cells, in which distortion bounds can be enforced [12, Chapter 5]. Every cell M_j has length $\sim j^{1/2}$, and it is divided into homogeneous sections of length k^{-3} for $k \ge j^{1/4}$. We will only keep homogeneous sections with $k \le j^{1/4+e}$, as the union of the rest has measure $\mathcal{O}(j^{-3-4e}) = \mathcal{O}\big(\mu(M_j)/j^{4e}\big)$, which is negligible.

Just as we indicated in the previous section, we can foliate each homogeneous section of M_m by smooth unstable curves, then their images in other cells will be homogeneous unstable curves stretching completely across homogeneous sections in those cells (with negligible exceptions caused by irregular intersections at the ends of homogeneous sections). Then we consider an arbitrary homogeneous unstable curve $W \subset M_j$, $j \le m^{\frac{1}{4}+3e}$, in the k-th section, where $k \le j^{1/4+e}$. Its length is $\mathbf{m}_W(W) \sim k^{-3} \ge j^{-3/4-3e} \ge m^{-\frac{3}{16}-3e-9e^2}$, where $\mathbf{m}_W$ denotes the Lebesgue measure on W.

Next we use the key estimate of the 'growth lemma' [9, Theorem 3.1], which says that there are constants $\alpha \in (0,1)$ and $\beta > 0$ such that for every homogeneous unstable curve W, every $n \geq 1$ and $\varepsilon > 0$

$$\mathbf{m}_W(r_n < \varepsilon) \leq (\alpha\Lambda)^n\, \mathbf{m}_W(r_0 < \varepsilon/\Lambda^n) + \beta\varepsilon\mathbf{m}_W(W). \tag{5.4}$$

Here $\Lambda > 1$ is the hyperbolicity constant and $r_n(x)$ is a function on W equal to the distance from $F^n(x)$ to the nearest endpoint of the component of $F^n(W)$ that contains x, we refer to [9] for details. A crucial observation is that if $F^n(x) \in M_k$, then because the length of the largest homogeneous section in M_k is $\mathcal{O}(k^{-3/4})$, we have $r_n(x) = \mathcal{O}(k^{-3/4})$. So applying (5.4) with $\varepsilon = m^{-3(1-q)/4}$ (note that $\varepsilon \ll \mathbf{m}_W(W)$) completes the proof of the lemma.

We now turn back to our analysis of the set $A_n \setminus B_{n,b}$ in the end of Section 2. Let

$$C_n = \{x \in A_n \setminus B_{n,b} \colon |I| > n/10\},$$

where I is again the longest time interval, within $[1,n]$, between successive returns to M. It is immediate that $\mu(C_n) = \mathcal{O}(n^{-1})$, because $\mu(M_m) = \mathcal{O}(m^{-3})$, just like in the proof of Proposition 3.1. On the other hand, the above lemma implies that $\mu(A_n \setminus B_{n,b} \setminus C_n)$ is even smaller: it can be bounded, say, by $n^{-1-q/2}$.

Finally we deal with dispersing billiards with cusps. Here the hyperbolicity of the map $\mathcal{F}$ is weak during long series of reflections deep in a cusp, see [13] and Fig. 3. We define $M \subset \mathcal{M}$ to consist of points that do not belong to series of $N \geq N_0$ reflections is a cusp, where N_0 is a large constant. A cell M_m is then made by points whose trajectories enter a cusp and come out of it after m bounces.

The cell structure is described in [13] in detail, and it is surprisingly similar to the cell structure of semi-dispersing billiards treated above. In particular, the bound (5.1) holds. Lemma 5.1 carries over, and its proof is essentially the same. For example, the equation (5.2) takes form

$$\mu\big(M_k/F(M_m)\big) \sim m^{2/3}/k^{7/3},$$

hence for any small $e > 0$

$$\mu\Big(\cup_{k=m^{1/2+e}}^{m^2} M_k/M_m\Big) = \mathcal{O}\big(1/m^{4e/3}\big),$$

which is similar to (5.3). The rest of the argument goes word for word, requiring only changes in the values of some constants.

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