A stretched exponential bound on time correlations for billiard flows

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Abstract

We construct Markov approximations to the billiard flows and establish a stretched exponential bound on time-correlation functions for planar periodic Lorentz gases (also known as Sinai billiards). Precisely, we show that for any (generalized) Hölder continuous functions F, G on the phase space of the flow the time correlation function is bounded by $\text{const} \cdot e^{-a\sqrt{|t|}}$, here $t \in \mathbb{R}$ is the (continuous) time and $a > 0$.

Keywords: Sinai billiards; Lorentz gas; decay of correlations.

1 Introduction

A billiard is a mechanical system in which a point particle moves freely (by inertia) at constant (unit) speed in a compact domain $\mathcal{D}$ and bounces off its boundary $\partial\mathcal{D}$ according to the classical law "the angle of incidence is equal to the angle of reflection". The dynamical properties of billiards are determined by the shape of $\partial\mathcal{D}$, and they may vary from completely regular (integrable) to strongly chaotic. The main class of chaotic billiards was introduced by Ya. Sinai in 1970, see [Si3], who considered containers defined by

$$\mathcal{D} = \text{Tor}^2 \setminus \cup_{i=1}^{p} \mathbb{B}_i, \tag{1.1}$$

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where Tor^2 is the unit 2-torus and $\mathbb{B}_i \subset \mathrm{Tor}^2$ disjoint strictly convex domains (scatterers) with C^3 smooth boundary whose curvature nowhere vanishes.

By lifting the billiard table $\mathcal{D}$ from Tor^2 to its universal cover $\mathbb{R}^2$ one gets a billiard in an unbounded table where the particle bounces off a periodic array of fixed obstacles (scatterers); this system is known as a periodic Lorentz gas. If the free path between collisions is uniformly bounded, then the system is said to have *finite horizon*. We always assume finite horizon in this paper.

The phase space of the billiard system is the compact 3D manifold $\mathcal{M} = \mathcal{D} \times S^1$, and the billiards dynamics generates a flow $\Phi^t\colon \mathcal{M} \to \mathcal{M}$. It is a Hamiltonian (contact) flow, and it preserves Liouville (uniform) measure μ on Ω.

At every reflection the velocity vector changes by the rule $v^+ = v^- - 2\langle v, n\rangle n$, where v^+ and v^- refer to the postcollisional and precollisional velocities, respectively, n denotes the inward unit normal vector to $\partial\mathcal{D}$ at the reflection point $q \in \partial\mathcal{D}$, and $\langle\cdot\rangle$ denotes the scalar product. The family of postcollisional velocity vectors with footpoints on $\partial\mathcal{D}$ makes a 2D manifold called the *collision space*:

$$\Omega = \{x = (q, v)\colon\ q \in \partial\mathcal{D},\quad \langle v, n\rangle \geq 0\}.$$

The billiard flow induces the return map $T\colon \Omega \to \Omega$ called the billiard map or collision map.

Standard coordinates on Ω are the arc length parameter r on the boundary $\partial\mathcal{D}$ and the angle $\varphi \in [-\pi/2, \pi/2]$ between the vectors v and n. Note that $\langle v, n\rangle = \cos\varphi$. The map $T\colon \Omega \to \Omega$ preserves smooth measure $d\nu = c_\nu \cos\varphi\, dr\, d\varphi$, where c_ν is the normalizing factor.

For $x \in \Omega$ denote by $\tau(x)$ the distance of the free path between the collision points x and $T(x)$. The flow Φ^t can be represented as a suspension flow over the map $T\colon \Omega \to \Omega$ under the ceiling function $\tau(x)$. In the suspension flow, every point $y \in \mathcal{M}$ is a pair $y = (x, t)$, where $x \in \Omega$ is the latest collision (in the past) along the trajectory of y and $t \in (0, \tau(x))$ is the time elapsed since that collision. There is a natural projection $\pi_\Omega\colon \mathcal{M} \to \Omega$ defined by $\pi_\Omega(x, t) = x$. Our finite horizon assumption means that $\tau(x)$ is bounded above.

Sinai [Si3] proved that the billiard map T is hyperbolic. We denote by $W^u(x)$ and $W^s(x)$ the unstable and stable manifolds through the point $x \in \Omega$, respectively. These are smooth curves, so we will call them fibers. The derivatives of the map T are unbounded (they blow up near grazing collisions, where $\varphi = 0$), and Sinai proposed [BSC2] a refinement of stable and unstable fibers to enforce distortion bounds (we define it in Section 3). The resulting (shorter) fibers are said to be *homogeneous*. We always consider homogeneous stable and unstable fibers, unless stated otherwise.

Each fiber has a finite (and uniformly bounded) length, and there are plenty of arbitrarily short fibers, but the following tail bound holds. For every $x \in \Omega$ denote by $r^u(x)$ and $r^s(x)$ the distance from x to the nearest endpoints of the (longest) unstable fiber $W^u(x)$ and stable fiber $W^s(x)$, respectively. Then for some constants $C, a > 0$ and all $\varepsilon > 0$

$$\nu\big(x \in \Omega\colon r^u(x) < \varepsilon\big) \leq C\varepsilon^a, \tag{1.2}$$

and a similar estimate holds for $r^s(x)$. Such bounds are standard for hyperbolic maps with singularities [KS]. For our billiards, in fact, (1.2) holds with $a = 1$, see [CM, Section 4.12]. Moreover, for any unstable fiber (or more generally, unstable curve [CM]) $W^u \subset \Omega$ we have

$$\ell_{W^u}\{x \in W^u\colon r^s(x) < \varepsilon\} \leq C\varepsilon \tag{1.3}$$

where ℓ_{W^u} denotes the Lebesgue measure on W^u, see [CM, Section 5.12].

The partition of Ω into unstable fibers is measurable, and the conditional measures on unstable fibers are absolutely continuous and called u-SRB measures (here SRB stands for Sinai, Ruelle, and Bowen who studied such measures for Axiom A diffeomorphisms [Bo1, R, Si4]), see a detailed presentation in [CM, Chapter 5]. Similarly, we call the conditional measures on stable fibers s-SRB measures.

The flow Φ^t is also hyperbolic. We denote by $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ the unstable and stable fibers through the point $x \in \mathcal{M}$. (Generally, we will use 'script' characters $\mathcal{M}$, $\mathcal{W}$, etc., to denote objects related to the flow Φ^t and regular Latin characters for objects related to the map T.) The partitions of $\mathcal{M}$ into unstable and stable fibers are measurable, and the corresponding conditional measures on those fibers are called u-SRB and s-SRB measures, respectively. Estimates similar to (1.2) and (1.3) hold for the flow as well.

Sinai proved [Si3] that the map T and hence the flow Φ^t are ergodic, mixing and K-mixing, see [CM] for a recent presentation of his results. The mixing property of the flow is equivalent to the decay of correlations. Given two functions $F, G\colon \mathcal{M} \to \mathbb{R}$ the time correlation function is defined by

$$\mathbf{C}_{F,G}(t) = \int_{\mathcal{M}} (F \circ \Phi^t)\, G\, d\mu - \int_{\mathcal{M}} F\, d\mu \cdot \int_{\mathcal{M}} G\, d\mu.$$

The flow is mixing if and only if $\mathbf{C}_{F,G}(t) \to 0$ as $t \to \infty$ for all $F, G \in L^2_\mu(\mathcal{M})$.

The rate of the decay of correlation (the speed of convergence of $\mathbf{C}_{F,G}(t)$ to zero) is an important characteristic of the flow and plays a role in physics applications. For arbitrary observables this speed cannot be controlled, but for reasonably smooth observables (and Hölder continuity of F, G is usually sufficient) a certain rate can be guaranteed. One expects that correlations for the billiard flow decay exponentially fast [CY, BV], but currently we are unable to prove this. We obtain a weaker (stretched exponential) bound on correlations.

The reason why the flow correlations are hard to study is the (natural) lack of hyperbolicity in the flow direction (the time one map Φ^1 is only partially hyperbolic). Even for smooth hyperbolic flows bounds on correlations have been established fairly recently [C3, L1, D1, D2, L2]. In [C3], Markov approximations were used and a (suboptimal) stretched exponential bound on correlations was derived for Anosov flows on 3D manifolds. This method was improved in [L1], where the same suboptimal bound was extended to multidimensional Anosov flows. In the later papers [D1, D2, L2] operator techniques were applied and an optimal exponential bound was obtained, though under some more stringent conditions.

In the context of singular flows (including billiard flows) much less is achieved so far. An exponential decay of correlations was proven for a very special case of 'open flows' [St], where the particle bounces off finitely many scatterers in the open plane with 'no eclipse' condition (the latter effectively eliminates the influence of singularities). For generic Sinai billiards it is only shown that correlations decay faster than any power function (this property is often referred to as a 'rapid mixing' or 'super-polynomial decay of correlations'), see [M]. This last result only applies to functions that are smooth in the direction of the flow, which is not the case for some physically interesting functions, such as position and velocity of the particle.

As opposed to the flow, the billiard map $T\colon \Omega \to \Omega$ is known to enjoy exponential decay of correlations [Y1, C4]. This fact implies certain statistical properties of the flow Φ^t, such as Bernoulliness [GO], Central Limit Theorem (CLT), Weak Invariance Principle (WIP), and well as Almost Sure Invariance Principle (ASIP), see [BSC2, MN, C5]. But bounds on correlations for the flow Φ^t cannot be derived from those for the map T.

Here we obtain a stretched exponential bound on correlations for the billiard flow Φ^t; our bound is perhaps less than optimal (as it is widely believed that correlations decay exponentially), but it is stronger that the super-polynomial bound of [M] and it holds for a much larger class of observables: the so called generalized Hölder continuous functions defined next. Given $F\colon \mathcal{M} \to \mathbb{R}$, $x \in \mathcal{M}$, and $r > 0$ we put $\operatorname{osc}_r(F, x) = \sup_B F - \inf_B F$, where $B = B_r(x)$ is the ball of radius r centered on x. Now F is said to be generalized Hölder continuous if there is $\alpha > 0$ (generalized Hölder exponent) such that

$$\|F\|_\alpha \colon = \sup_r r^{-\alpha} \int_{\mathcal{M}} \operatorname{osc}_r(F, x)\, d\mu(x) < \infty.$$

Every Hölder continuous and piecewise Hölder continuous function is generalized Hölder continuous [C3]. Lastly we put

$$\operatorname{var}_\alpha(F) = \|F\|_\alpha + \sup_{\mathcal{M}} F - \inf_{\mathcal{M}} F.$$

Here is our main result:

Theorem 1.1. *Let $\Phi^t \colon \mathcal{M} \to \mathcal{M}$ be a Lorentz gas billiard flow (with finite horizon) and $F, G\colon \mathcal{M} \to \mathbb{R}$ two generalized Hölder continuous functions. Then*

$$|\mathbf{C}_{F,G}(t)| \le c \operatorname{var}_\alpha(F) \operatorname{var}_\alpha(G)\, e^{-a\sqrt{|t|}}.$$

Here $c, a > 0$ depend on α and the billiard flow only.

As a consequence one gets the following *equidistribution property*. Let $\mathcal{W}^u \subset \mathcal{M}$ be a smooth unstable curve of length $|\mathcal{W}^u|$ and $\mathbf{m}$ the uniform probability measure on it. Denote by $\mathbf{m}_t = \Phi^t \mathbf{m}$ its image at time t.

Corollary 1.2. *Let $F\colon \mathcal{M} \to \mathbb{R}$ be a Hölder continuous function. Then for all $t > 0$*

$$\left| \int_{\mathcal{M}} F\, d\mathbf{m}_t - \int_{\mathcal{M}} F\, d\mu \right| \le |\mathcal{W}^u|^{-1} c' \operatorname{var}_\alpha(F)\, e^{-a'\sqrt{t}},$$

where $c', a' > 0$ depend on α and the billiard flow only.

Proof. Let $\mathcal{U} = \mathcal{U}_\varepsilon(\mathcal{W}^u)$ denote the ε-neighborhood of $\mathcal{W}^u$. We can foliate $\mathcal{U}$ by stable manifolds $\mathcal{W}^s$ of length $\sim 2\varepsilon$, except for a subset $\mathcal{U}^* \subset \mathcal{U}$ where the stable manifolds happen to be too short; we have $\mu(\mathcal{U}^*) = \mathcal{O}(\varepsilon^3)$ due to the estimate (1.3) for the flow (the value of ε will be determined later). Now we fix a smooth function $G \ge 0$ whose support is a slightly larger domain than $\mathcal{U}$ and such that $\int_{\mathcal{M}} G\, d\mu = 1$ and its integral over every stable curve $\mathcal{W}^s$ foliating $\mathcal{U} \setminus \mathcal{U}^*$ does not depend on $\mathcal{W}^s$. It is easy to choose G so that $\sup G = \mathcal{O}\big(\varepsilon^{-2}|\mathcal{W}^u|^{-1}\big)$ and $\|G\|_\alpha = \mathcal{O}(\varepsilon^{-\alpha})$, so that $\operatorname{var}_\alpha(G) =$

$\mathcal{O}\big(\varepsilon^{-2}|\mathcal{W}^u|^{-1}\big)$. Now we apply Theorem 1.1 and note that the set $\Phi^t(\mathcal{U}\setminus\mathcal{U}^*)$ will be in the ε-neighborhood of $\Phi^t(\mathcal{W}^u)$, thus

$$\left|\int_{\mathcal{M}} F\, d\mathbf{m}_t - \int_{\mathcal{M}} F\, d\mu\right| \leq c\, \mathrm{var}_\alpha(F)\, \mathrm{var}_\alpha(G)\, e^{-a\sqrt{|t|}} + C\varepsilon|\mathcal{W}^u|^{-1}\mathrm{var}_\alpha(F)$$

where the last term accounts for the loss of the measure due to $\mathcal{U}\setminus\mathcal{U}^*$. Now we choose $\varepsilon = e^{-\frac{1}{3}a\sqrt{t}}$ and complete the proof. □

This corollary can also be extended to measures $\mathbf{m}$ that are absolutely continuous and have the so called 'dynamically Hölder continuous density' [CD, CM]. We also note that the equidistribution itself implies certain bounds on correlations [CD, CM].

One may wonder why we only get a stretched exponential bound on correlations, instead of an exponential (optimal) one. There are two types of techniques used to estimate correlations for hyperbolic systems. One is based on 'coarse-graining' where the phase space is partitioned into coarse 'atoms' and the dynamics is approximated by a Markov chain. These techniques are inherently too crude to produce optimal bounds on correlations, see also [BSC2, C1]. There exist much finer techniques using functional analysis tools (Perron-Frobenius operator) that can achieve optimal bounds on correlations, but they are very sensitive to little details and often unable to cope with various unpleasant features (singularities) of the dynamics. It appears that the fine techniques cannot handle billiard flows yet, but the crude 'coarse-graining' methods (popular in physics [NMT, CELS]) are just flexible enough for this purpose. The 'only' price we pay is the non-optimality of the correlation bounds.

A related issue is the difference between smooth (Anosov and Axiom A) flows and singular billiard flows. In the smooth case, finite Markov partitions exist and they do a fine job – many constructions are relatively simple and elegant. In billiard flows, singularities are a major source of trouble – in many cases we have to divide their vicinities into fractal-like necklaces, thus our constructions become cumbersome and overcomplicated. We combine here the methods of other papers [BSC2, C3, CD, C5, CM] to handle billiard singularities, in all cases we are trying to suppress billiard-specific technical details but describe our ideas clearly.

2 H-structure

Here we describe our main construction. Its idea is derived from a classical proof, due to Sinai [Si3], of the K-mixing property for billiard flows. Take an unstable fiber, $\mathcal{W}^u \subset \mathcal{M}$, for the billiard flow Φ^t. The union $S = \cup_{x\in\mathcal{W}^u}\mathcal{W}^s(x)$ of stable fibers intersecting $\mathcal{W}^u$ has measure zero, but the union $U = \cup_{y\in S}\mathcal{W}^u(y)$ of the unstable fibers crossing S has a positive measure, and this fact implies the K-mixing property by a general argument [Si3], see also [CM, Chapter 6].

We will refine this construction as follows. First, we can take only sufficiently long stable fibers lying close to each other. We take stable fibers $\mathcal{W}^s(x)$ for points $x \in \mathcal{W}^u$ in a small ball-like neighborhood around some point $x_0 \in \mathcal{W}^u$, and, likewise, unstable fibers $\mathcal{W}^u(y)$ for points y in a small ball around some point $y_0 \in \mathcal{W}^s(x_0)$. Let R_H be the distance between x_0 and y_0 along $\mathcal{W}^s(x_0)$. Denote by B_1 and B_2 the above balls around x_0 and y_0, respectively, and by $r_H \ll R_H$ their common radius. In addition, we can only take stable fibers $\mathcal{W}^s(x)$ that stick out of those two balls by at least L_H, and unstable fibers $\mathcal{W}^u(y)$ that stick out of the ball B_2 by at least L_H in both directions, here $L_H \gg R_H$ is a constant.

Second, we run our construction 'backwards'. Fix one of the above fibers $\mathcal{W}^u(y)$. For every $z \in \mathcal{W}^u(y) \cap B_2$ we take the stable fiber $\mathcal{W}^s(z)$ that sticks out of B_2 by at least L_H in both directions; then for every $w \in \mathcal{W}^s(z) \cap B_1$ we take the unstable fiber $\mathcal{W}^u(w)$ that sticks out of B_1 by at least L_H in both directions (of course, if all these fibers exist). Denote by V_y the union of the fibers $\mathcal{W}^u(w)$. We mark the fiber $\mathcal{W}^u(y)$ if $\mu(V_y) > 0$. We can assume that the union of our marked fibers $\{\mathcal{W}^u(y)\}$ has positive measure, too.

Lastly, we can find a subset of marked fibers $\mathcal{W}^u(y)$, whose union we denote by V_2, such that (i) the intersection $V_1 = \cap_{y\in V_2} V_y$ has positive measure, and (ii) the union of the fibers $\mathcal{W}^u(y) \subset V_2$ has positive measure, too.

Since our measure μ is proportional to the volume in $\mathcal{M}$ and stable (unstable) fibers make measurable partitions of $\mathcal{M}$, it can be shown by standard measure-theoretic arguments that the sets V_1 and V_2 will have positive measure for some points x_0, y_0 and some small R_H and r_H.

Summarizing the above properties, we obtain:

Proposition 2.1 (H-structure)**.** *There are two (uncountable) families of unstable fibers, $\mathcal{W}^u_\alpha$, $\alpha \in \mathcal{A}$, and $\mathcal{W}^u_\beta$, $\beta \in \mathcal{B}$, such that*

(H1) *their unions* $V_1 = \cup \mathcal{W}^u_\alpha$ *and* $V_2 = \cup \mathcal{W}^u_\beta$ *are measurable sets of positive measure;*

(H2) *for every* $\mathcal{W}^u_\alpha$ *and* $\mathcal{W}^u_\beta$ *there exists a stable fiber,* $\mathcal{W}^s_{\alpha\beta}$*, that intersects both* $\mathcal{W}^u_\alpha$ *and* $\mathcal{W}^u_\beta$*;*

(H3) *the points* $x_{\alpha\beta} = \mathcal{W}^u_\alpha \cap \mathcal{W}^s_{\alpha\beta}$ *and* $x_{\beta\alpha} = \mathcal{W}^u_\beta \cap \mathcal{W}^s_{\alpha\beta}$ *lie in two small balls (*B_1 *and* B_2*, respectively) of radius* r_H*; all the fibers* $\mathcal{W}^u_\alpha$*,* $\mathcal{W}^u_\beta$*, and* $\mathcal{W}^s_{\alpha\beta}$ *stick out of* B_1 *and* B_2 *by at least* L_H *in both directions.*

Here $r_H \ll R_H \ll L_H$*, and* R_H *is the distance between the centers of the balls* B_1 *and* B_2*.*

This structure resembles the letter 'H' with two thin bundles of parallel unstable fibers joined by a thin bundle of stable fibers. The crucial property of this structure is that *every* $\mathcal{W}^u_\alpha$ is joined (coupled) with *every* $\mathcal{W}^u_\beta$.

We can provide the following 'high density' of unstable fibers of the H-structure, by reducing the sets V_1 and V_2, if necessary:

(H4) *There is a small constant* $s_H > 0$ *such that for every* $\alpha \in \mathcal{A}$ *there is an* $s_\alpha \in [0, s_H]$ *such that on the surface* $\Phi^{[s_\alpha - s_H, s_\alpha]}(\mathcal{W}^u_\alpha \cap B_1)$ *the points belonging to the set* V_1 *make a subset of 'high density' – its area at least* 0.99 *times the area of the entire surface (in the inner Riemannian metric on it). The same holds for every* $\beta \in \mathcal{B}$*.*

Here 0.99 can be changed to any number below 1. We used Bowen's notation $\Phi^{[a,b]}(A) = \cup_{t=a}^b \Phi^t(A)$ for $a < b$ and $A \subset \mathcal{M}$.

For any α, β and any point $x \in \mathcal{W}^u_\alpha$ close to $x_{\alpha\beta} = \mathcal{W}^u_\alpha \cap \mathcal{W}^s_{\alpha\beta}$ we denote by u the distance between x and $x_{\alpha\beta}$ and by $\boldsymbol{\tau}_{\alpha\beta}(u)$ the *temporal distance* between the fibers $\mathcal{W}^u_\beta$ and $\mathcal{W}^s(x)$, i.e. the (unique) small number satisfying $\Phi^{\boldsymbol{\tau}_{\alpha\beta}(x)}(\mathcal{W}^u_\beta) \cap \mathcal{W}^s(x) \neq \emptyset$. Of course, the function $\boldsymbol{\tau}_{\alpha,\beta}(u)$ is defined at u only if $\mathcal{W}^s(x)$ is long enough to intersect the surface $\Phi^{[-r_H, r_H]}\mathcal{W}^u_\beta$, hence its domain is a Cantor-like subset of $\mathbb{R}$. The following is a standard fact, see [KM, Lemma 5.1] and [CM, Section 6.11]:

Lemma 2.2 (Lipschitz regularity of the temporal distance)**.** *There are positive constants* $0 < \underline{d} < \bar{d} < \infty$ *such that*

$$\underline{d}u \leq |\boldsymbol{\tau}_{\alpha,\beta}(u)| \leq \bar{d}u \tag{2.1}$$

for all α, β *and all* u *where the function* $\boldsymbol{\tau}_{\alpha\beta}(u)$ *is defined.*

The fact that $\boldsymbol{\tau}_{\alpha,\beta}(u) \neq 0$ ensures the K-mixing property of the billiard flow, see [CM, Chapter 6]. The linear bounds (2.1) (especially the lower bound) are essential for our estimates on correlations.

We can guarantee an abundance of points $x \in \mathcal{W}_\alpha^u$ near $x_{\alpha\beta}$ for which $\mathcal{W}^s(x)$ crosses the surface $\Phi^{[-r_H,r_H]}\mathcal{W}_\beta^u$ (i.e., where the function $\boldsymbol{\tau}_{\alpha\beta}$ is defined) by further reducing the sets V_1 and V_2, if necessary. More precisely, let $\mathcal{W}_\alpha^{u0} \subset \mathcal{W}_\alpha^u$ denote a subset of points for which the stable fiber $\mathcal{W}^s(x)$ extends by at least $R_H + L_H$ in the direction of B_2 and by at least L_H in the opposite direction.

(H5) *There is a small constant $\varepsilon_H > 0$ such that for every $\varepsilon < \varepsilon_H$ and every pair $\mathcal{W}_\alpha^u$, $\mathcal{W}_\beta^u$*

$$\ell\big(\mathcal{W}_\alpha^{u0} \cap B_\varepsilon(x_{\alpha\beta})\big) / \ell\big(\mathcal{W}_\alpha^u \cap B_\varepsilon(x_{\alpha\beta})\big) \geq 0.99,$$

and this ratio approaches 1 as $\varepsilon \to 0$. Here $B_\varepsilon(x_{\alpha\beta})$ denotes the ball of radius ε centered on the point $x_{\alpha\beta} = \mathcal{W}_\alpha^u \cap \mathcal{W}_{\alpha\beta}^s$ and ℓ the Lebesgue measure on $\mathcal{W}_\alpha^u$.

This property can be easily ensured with the help of Lebesgue density points of the subsets $\{x_{\alpha\beta} \colon \beta \in \mathcal{B}\} \subset \mathcal{W}_\alpha^u$.

Let $\{\mathcal{W}_\gamma^s\}$, $\gamma \in \mathcal{G}$, denote the family of all stable fibers passing through both balls B_1 and B_2 and extending beyond them by at least L_H. The property (H5) ensures that each fiber $\mathcal{W}_{\alpha\beta}^s$ is a sort of 'density point' in the family $\{\mathcal{W}_\gamma^s\}$.

We denote by $\mathbf{H}_0$ the above H-structure (it consists of two families of unstable fibers, $\{\mathcal{W}_\alpha^u\}$ and $\{\mathcal{W}_\beta^u\}$, and a family of stable fibers $\{\mathcal{W}_\gamma^s\}$). Since the image of each fiber under the flow Φ^τ, at least for small τ, is also a fiber, then $\mathbf{H}_\tau = \Phi^\tau(\mathbf{H}_0)$ for any small τ, say $|\tau| \leq r_H$, is also an H-structure, it has all the same properties as $\mathbf{H}_0$, with the balls $B_k^{(\tau)} = \Phi^\tau(B_k)$ replacing B_k, $k = 1, 2$. Thus we get a one-parameter family of H-structures.

We also denote by $\mathbf{H}_0^1 = V_1$ and $\mathbf{H}_0^2 = V_2$ the unions of unstable fibers in the two families $\{\mathcal{W}_\alpha^u\}$ and $\{\mathcal{W}_\beta^u\}$. Then $\mathbf{H}_\tau^k = \Phi^\tau(\mathbf{H}_0^k)$, for $k = 1, 2$, will be the unions of the corresponding unstable fibers in the H-structure $\mathbf{H}_\tau$.

In the H-structure $\mathbf{H}_\tau$ every unstable fiber $W_1^u \subset \mathbf{H}_\tau^1$ is coupled with every unstable fiber $W_2^u \subset \mathbf{H}_\tau^2$ by a stable fiber. But all these fibers are located in a tiny part of the phase space $\mathcal{M}$. Next we will use the H-structures to couple images of arbitrary unstable fibers $\mathcal{W}^u \subset \mathcal{M}$.

Let $\mathcal{W}^u \subset \mathcal{M}$ be an unstable fiber and ν^u denote the u-SBR probability measure on it. For any $t > 0$ let ν^u_t be the image of ν^u on $\mathcal{W}^u_t = \Phi^t(\mathcal{W}^u)$. The set $\mathcal{W}^u_t$ is a finite or countable union of unstable fibers, $\mathcal{W}^u_{t,i}$, $i \geq 1$, and the measure ν^u_t conditioned on each $\mathcal{W}^u_{t,i}$ coincides with the u-SBR measure on $\mathcal{W}^u_{t,i}$.

One may expect (based on the K-mixing property of Φ^t) that the set $\mathcal{W}^u_t$ is asymptotically dense in $\mathcal{M}$, as $t \to \infty$, and the measure ν^u_t weakly converges to the invariant measure μ (this is actually true, but we will not use this fact here). In particular, some components $\mathcal{W}^u_{t,i}$ may come arbitrary close to the set $\mathbf{H}^k_\tau$ (here $k = 1$ or 2 and $|\tau| < r_H$) of our H-structure $\mathbf{H}_\tau$. We are interested in the components $\mathcal{W}^u_{t,i}$ such that

(W1) $\operatorname{dist}(\mathcal{W}^u_{t,i}, \mathbf{H}^k_\tau) \leq C_\Phi \lambda^t_\Phi$;

(W2) the curve $\mathcal{W}^u_{t,i}$ sticks out of the ball $B^{(\tau)}_k$ by at least L_H in both directions.

Here C_Φ and λ_Φ are hyperbolic constants, i.e. such that for any points x, y on the same stable fiber $\operatorname{dist}(\Phi^t x, \Phi^t y) \leq C_\Phi \lambda^t_\Phi$ for all $t > 0$. The union of such components $\mathcal{W}^u_{t,i}$ we denote by $\mathcal{W}^u_t(\mathbf{H}^k_\tau)$.

Remark 2.3. *The conditions (W1) and (W2) imply that $\mathcal{W}^u_{t,i}$ is close to $\mathbf{H}^k_\tau$ 'all the way' in the ball $B^{(\tau)}_k$, i.e. there exists an unstable fiber $\mathcal{W} \subset \mathbf{H}^k_\tau$ such that the curves $\mathcal{W}^u_{t,i} \cap B^{(\tau)}_k$ and $\mathcal{W} \cap B^{(\tau)}_k$ are $(C\lambda^t_\Phi)$-close in the Hausdorff metric for some constant $C > 0$ (we can put $C = 2C_\Phi$).*

The existence (and abundance) of $W^u_{t,i}$ for large t is guaranteed by the following:

Proposition 2.4. *Given the family of H-structures $\{\mathbf{H}_\tau\}$ described above, there are positive constants $a_H > 0$, $b_H > 0$, and $d_H > 0$ such that for any $|\tau| < r_H$, $k = 1, 2$, any unstable fiber $\mathcal{W}^u \subset \mathcal{M}$, and any*

$$t > t_0(W^u) \colon = a_H \big| \ln |\mathcal{W}^u| \big| + b_H$$

there is a measurable set $S(\mathcal{W}^u_t, \mathbf{H}^k_\tau) \subset [0, s_H]$ such that

(i) *for any $s \in S(\mathcal{W}^u_t, \mathbf{H}^k_\tau)$ we have $\nu^u_{t+s}\big(\mathcal{W}^u_{t+s}(\mathbf{H}^k_\tau)\big) \geq d_H$;*

(ii) *$\ell\big(S(\mathcal{W}^u_t, \mathbf{H}^k_\tau)\big) \geq 0.98 s_H$, where ℓ stands for the Lebesgue measure on the interval $[0, s_H]$.*

Here 0.98 can be changed to any number below 1 (but this might require replacing 0.99 in (H4) with a constant closer to 1).

Remark 2.5. *Obviously, we have*

$$S(\mathcal{W}_t^u, \mathbf{H}_\tau^k) = S(\mathcal{W}_{t-\tau}^u, \mathbf{H}_0^k)$$

and for every $s \in S(\mathcal{W}_t^u, \mathbf{H}_\tau^k) = S(\mathcal{W}_{t-\tau}^u, \mathbf{H}_0^k)$ *we have*

$$\mathcal{W}_{t+s}^u(\mathbf{H}_\tau^k) = \mathcal{W}_{t-\tau+s}^u(\mathbf{H}_0^k)$$

for small τ.

According the the above remark, it is enough to prove Proposition 2.4 for $\tau = 0$, and its proof is given in Appendix.

Next let $\mathcal{W}_1^u$ and $\mathcal{W}_2^u$ be a pair of unstable fibers. By Proposition 2.4, if t is large enough, then some components of $\mathcal{W}_{t+s_1,1}^u = \Phi^{t+s_1}(\mathcal{W}_1^u)$ for $s_1 \in S(\mathcal{W}_{t,1}^u, \mathbf{H}_0^1)$ will be close to $\mathbf{H}_0^1$ and some components of $\mathcal{W}_{t+s_2,2}^u = \Phi^{t+s_2}(\mathcal{W}_2^u)$ for $s_2 \in S(\mathcal{W}_{t,2}^u, \mathbf{H}_0^2)$ will be close to $\mathbf{H}_0^2$. Since both sets $S(\mathcal{W}_{t,1}^u, \mathbf{H}_0^1)$ and $S(\mathcal{W}_{t,2}^u, \mathbf{H}_0^2)$ have high density on the interval $[0, s_H]$, we can pick $s \in S(\mathcal{W}_{t,1}^u, \mathbf{H}_0^1) \cap S(\mathcal{W}_{t,2}^u, \mathbf{H}_0^2)$.

Corollary 2.6. *Let* $\mathcal{W}_1^u, \mathcal{W}_2^u \subset \mathcal{M}$ *be two unstable fibers. Then for every*

$$t > \max\{t_0(\mathcal{W}_1^u), t_0(\mathcal{W}_2^u)\} \tag{2.2}$$

there exists $s \in [0, s_H]$ *such that*

$$\nu_{t+s,k}\big(\mathcal{W}_{t+s,k}^u(\mathbf{H}_0^k)\big) \geq d_H \tag{2.3}$$

for both $k = 1, 2$. *Equivalently,*

$$\nu_{t,k}\big(\mathcal{W}_{t,k}^u(\mathbf{H}_{-s}^k)\big) \geq d_H \quad \text{for} \quad k = 1, 2 \tag{2.4}$$

In other words, given two unstable fibers of size $\geq \varepsilon$, their future images at any time $t > a_H|\ln\varepsilon| + b_H$ contain a certain fraction (measured by $d_H > 0$) of components such that *every* component of the image of the first fiber can be 'almost' joined (coupled) with *every* component of the image of the second fiber by a stable fiber (meaning that the coupling stable fiber misses our components by less than $C\lambda_\Phi^t$).

The advantage of (2.4) over (2.3) is that the coupling time t is arbitrary, whereas $t + s$ in (2.3) depends on the pair $\mathcal{W}_1^u, \mathcal{W}_2^u$.

Lemma 2.2 also ensures a certain stability of the connecting stable fiber W^s_γ joining two components $W^u_{t,1,i} \subset \Phi^t(W^u_1)$ and $W^u_{t,2,j} \subset \Phi^t(W^u_2)$ under small perturbations: if we replace W^s_γ with another stable fiber $W^s_{\gamma'}$ that crosses $W^u_{t,1,i}$ a distance δ from W^s_γ, then $W^s_{\gamma'}$ will miss $W^u_{t,2,j}$ by at most $\mathcal{O}(\delta)$.

3 Solid and Cantor rectangles

Markov partitions (and their variations) are useful in the studies of general smooth hyperbolic maps [Si1, Si2, Si4, Bo1, R] and specific hyperbolic billiards [BSC1, BSC2, Y1]. Atoms (building blocks) of such partitions are called *rectangles* (or parallelograms).

For linear automorphisms of a 2D torus, these atoms are true rectangles or parallelograms [AW]. For Anosov maps in 2D they are open domains, each bounded by two unstable and two stable fibers [Si1, Si2] (in higher dimension their boundary is necessarily very irregular [Bo2]). For generic Axiom A diffeomorphisms and billiard maps these atoms are complicated Cantor-like objects. In this section we recall necessary definitions and facts.

A solid rectangle $Q \subset \Omega$ is a closed domain bounded by two unstable fibers and two stable fibers (here we allow non-homogeneous fibers). We call these fibers *u-sides* and *s-sides* of Q, respectively. If an unstable (stable) fiber W crosses both s-sides (resp., u-sides) of Q, we say that W *fully crosses* Q. Any closed set $R \subset \Omega$ with the property

$$x, y \in R \quad \Longrightarrow \quad \emptyset \neq [x, y]\colon = W^s(x) \cap W^u(y) \in R$$

is called a (Cantor) rectangle; it is always a closed nowhere dense (Cantor-like) set. Let $z \in R$ and $C = W^u(z) \cap R$ and $D = W^s(z) \cap R$. Then

$$R = [C, D] = \{[x, y] \colon x \in C,\ y \in D\},$$

and for every $w \in R$ there is a unique representation $w = [x, y]$, where $x \in C$ and $y \in D$. Thus R has a direct-product structure.

Given a rectangle $R \subset \Omega$, we denote by $Q(R)$ the minimal solid rectangle containing R (we call it the *hull* of R). Given a solid rectangle Q, we denote by $R(Q)$ the maximal rectangle contained in Q (it is made by points of intersection of all unstable fibers fully crossing Q with all stable fibers fully crossing Q).

We will only deal with rectangles of positive measure. We have $\nu(R) > 0$ if and only if for any (and hence, for every) point $z \in R$ we have $\nu^u(W^u(z) \cap R) > 0$ and $\nu^s(W^s(z) \cap R) > 0$ (as usual, ν^u and ν^s are the u-SRB and s-SRB measures on the corresponding curves). We call

$$\rho^u(R) = \inf_{x \in R} \frac{\nu^u\big(W^u(x) \cap R\big)}{\nu^u\big(W^u(x) \cap Q(R)\big)}$$

the (minimal) *u-density* of R. Similarly the (minimal) *s-density* $\rho^s(R)$ is defined and we call

$$\rho(R) = \min\{\rho^u(R), \rho^s(R)\} \tag{3.1}$$

the (minimal) *density* of the rectangle R. Note that $0 \leq \rho^{u,s}(R) < 1$; if $\rho(R)$ is close to one, then the rectangle R is "very dense", i.e. it occupies nearly the entire available area of its hull $Q(R)$. In particular, we have

$$1 - \frac{\nu(R)}{\nu(Q(R))} \geq \text{ const } \big(1 - \rho(R)\big). \tag{3.2}$$

The abundance of rectangles in Ω is guaranteed by the bound (1.2).

Let R be a rectangle. A rectangle $R_1 \subset R$ is called a *u-subrectangle* if $W^u(x) \cap R = W^u(x) \cap R_1$ for any point $x \in R_1$. Similarly, $R_2 \subset R$ is an *s-subrectangle* if $W^s(x) \cap R = W^s(x) \cap R_2$ for any $x \in R_2$.

Given a rectangle R, its image $T^n(R)$ is a finite or countable union of rectangles $\{R_i\}$. For $n > 0$, their preimages $T^{-n}(R_i)$ are s-subrectangles in R. For $n < 0$, they are u-subrectangles in R.

Given two rectangles R_1 and R_2 and $n \geq 1$, we say that $T^n(R_1)$ intersects R_2 *properly* if the set $T^n(R_1) \cap R_2$ is a u-subrectangle in R_2 and the set $R_1 \cap T^{-n}(R_2)$ is an s-subrectangle in R_1. Proper intersection is characteristic for atoms of Markov partitions.

We recall that homogeneous fibers are defined by using the so called homogeneity strips in Ω, whose boundaries consist of countably many parallel lines

$$\mathbb{S} = \cup_{k \geq k_0} \{(r, \varphi) \colon |\varphi| = \pi/2 - k^{-2}\}$$

(here $k_0 \geq 1$ be a large constant), see [BSC2, C4, C5, CM]. Technically, the space Ω is divided along these lines into countably many strips and the map T becomes discontinuous at points mapped onto $\mathbb{S}$ (and the map T is naturally discontinuous at points mapped onto $\mathcal{S}_0 = \partial\Omega = \{|\varphi| = \pi/2\}$). Now $\mathbb{S}_n = \cup_{k=0}^{n} T^{-k}(\mathbb{S} \cup \mathcal{S}_0)$ is the set of points where the map T^n is discontinuous.

The following fact proven in [BSC2, Section 3.2] shows that rectangles have a direct product structure not only in a topological sense, but (approximately) in a measure-theoretic sense:

Proposition 3.1. *Let R be a rectangle such that $Q(R) \cap (\mathbb{S}_n \cup \mathbb{S}_{-n}) = \emptyset$, i.e. such that the maps $T^{\pm n}$ are continuous on $Q(R)$. Then there exists a probability measure ν_R on R such that*

(a) *it is almost uniform with respect to the measure ν restricted to R:*

$$\left| \frac{d\nu_R}{d\nu} - \nu(R) \right| \leq c\theta^n \tag{3.3}$$

for some constants $c > 0$ and $\theta \in (0,1)$.

(b) *ν_R is a product measure, i.e. for any u-subrectangle $R_1 \subset R$ and s-subrectangle $R_2 \subset R$ we have $\nu_R(R_1 \cap R_2) = \nu_R(R_1)\, \nu_R(R_2)$.*

In what follows we have many exponential bounds similar to (3.3), and we will denote by $c_i > 0$ and $\theta_i \in (0,1)$ various constants whose values are not important (they depend on the billiard table $\mathcal{D}$ alone).

For any curve $W \subset \Omega$ we denote by ℓ_W the (non-normalized) Lebesgue measure on W and by $\mathcal{J}_W T^n(x)$ the Jacobian (the 'expansion factor') of the map T^n restricted to W at the point $x \in W$. The following bound is proved in [CD, Lemma A.6] for any unstable curve $W \subset \Omega$ and $n > 0$

$$\int_W |\mathcal{J}_W T^n(x)|^{1/3}\, d\ell_W \leq c_1 \theta_1^{-n}. \tag{3.4}$$

Lemma 3.2. *The ε-neighborhood of the set $\mathbb{S}_n \cup \mathbb{S}_{-n}$ has measure*

$$\nu\big(\mathcal{U}_\varepsilon(\mathbb{S}_n \cup \mathbb{S}_{-n})\big) \leq c_2 \theta_2^{-n} \varepsilon^{1/4}.$$

Proof. It is enough to prove this for the ε-neighborhood of $\mathbb{S}_n$. For any curve $S \subset \mathbb{S}_n$ there exists a unique $k \in [0,n]$ such that $T^k(S)$ is either a singularity curve for T or a homogeneity line. Fix a $k \in [0,n]$ and denote by $\mathbb{S}^{(k)} \subset \mathbb{S}_n$ the union of all the corresponding curves. Let $\{\tilde{W}^u\}$ be a smooth foliation of Ω by unstable curves; it induces a smooth foliation of ε-neighborhood $\mathcal{U}_\varepsilon(\mathbb{S}^{(k)})$ by unstable curves W_α^u of length $\mathcal{O}(\varepsilon)$ terminating on (but not crossing) $\mathbb{S}^{(k)}$.

Note that the map T^k expands each W^u_α almost uniformly (due to the distortion bounds), and we denote $\mathcal{J}_{W^u_\alpha} T^k = \max_{x \in W^u_\alpha} \mathcal{J}_{W^u_\alpha} T^k(x)$. Integrating the estimate (3.4) over the chosen foliation $\{\tilde{W}^u\}$ of Ω gives

$$\nu\big(\cup W^u_\alpha \colon \mathcal{J}_{W^u_\alpha} T^k \geq B\big) \leq \text{const} \cdot \theta_1^{-k} B^{-1/3} \tag{3.5}$$

for any $B > 0$. We set $B = \varepsilon^{-3/4} \theta_1^{-3k/4}$, then the right hand side of (3.5) is $\mathcal{O}(\varepsilon^{1/4} \theta_1^{-3k/4})$. The curves W^u_α where $\mathcal{J}_{W^u_\alpha} T^k < B$ are mapped by T^k into the (εB)-neighborhood of the union of the singularity curves for T and the homogeneity lines. That neighborhood has measure $\mathcal{O}(\varepsilon B) = \mathcal{O}(\varepsilon^{1/4} \theta_1^{-3k/4})$, see [CM, Section 5.5]; now summing up over $k = 0, 1, \ldots, n$ proves the lemma with $\theta_2 = \theta_1^{3/4}$. □

Corollary 3.3. *Let $\theta_3 = \theta_2^8$; then $\nu\big(\mathcal{U}_{\theta_3^n}(\mathbb{S}_n \cup \mathbb{S}_{-n})\big) \leq c_2 \theta_2^n$.*

4 Special collections of solid rectangles

Here we construct a finite collection of solid rectangles in Ω that will be a basis for subsequent Markov approximations to the billiard map (and flow).

Proposition 4.1. *For any large $n \geq 1$ there exists a finite collection of solid rectangles $\Upsilon_n = \{Q_1, \ldots, Q_J\}$ such that*

(a) *we have* $\operatorname{int} Q_j \cap \operatorname{int} Q_{j'} = \emptyset$ *for any $j \neq j'$;*

(b) $\operatorname{diam}(Q_j) \leq \theta_3^n$ *for every $j = 1, \ldots, J$ and $J \leq \theta_4^{-n}$;*

(c) *the image of each u-side (s-side) of every rectangle Q_j under T^{-1} (resp., under T) lies either on a u-side (resp., an s-side) of another rectangle $Q_{j'}$, or outside their union $U_n = \cup_j Q_j$.*

(d) $U_n \cap (\mathbb{S}_n \cup \mathbb{S}_{-n}) = \emptyset$ *and $\nu(\Omega \setminus U_n) \leq c_2 \theta_2^n$;*

Proof. In fact, there are plenty of such collections and their construction is quite flexible. For any large $m \geq 1$, and small $\varepsilon < \varepsilon_0(m)$, the so-called pre-Markov partition ξ of Ω was constructed in [BSC1, Section 3] and [BSC2, Section 4.2] with the following properties:

(M1) Atoms of ξ are closed curvilinear polygons, each bounded by some stable fibers, some unstable fibers (both may be non-homogeneous), as well as some pieces of singularity curves of $\mathcal{S}_m \cup \mathcal{S}_{-m}$; accordingly we divide the union of all their boundaries $\partial\xi = \cup_{A\in\xi}\partial A$ into three parts: $\partial\xi = \partial^s\xi \cup \partial^u\xi \cup \partial^{\rm sing}\xi$; in fact $\partial^{\rm sing}\xi = \mathcal{S}_m \cup \mathcal{S}_{-m}$; and atoms that are not adjacent to $\mathcal{S}_m \cup \mathcal{S}_{-m}$ are solid rectangles;

(M2) we have $T(\partial^s\xi) \subset \partial^s\xi$ and $T^{-1}(\partial^u\xi) \subset \partial^u\xi$;

(M3) The diameters of the atoms do not exceed ε;

(M4) The number of atoms of this partition does not exceed ε^b for some constant $b > 0$.

It remains to take a pre-Markov partition ξ with some fixed $m \geq 1$ (independent of n) and $\varepsilon = \theta_3^n$ and remove the atoms of ξ whose closure intersects the set $\mathbb{S}_n \cup \mathbb{S}_{-n}$. Observe that $U_n = \cup_j Q_j$ covers $\Omega \setminus \mathcal{U}_{\theta_3^n}(\mathbb{S}_n \cup \mathbb{S}_{-n})$, so we can use Corollary 3.3 to complete the proof of (d). $\square$

Remark 4.2. *We can assume that $\theta_4 < \theta_2$ and eliminate all $Q_j \in \Upsilon_n$ such that $\nu(Q_j) < \theta_4^{2n}$. Their total measure is $\leq \theta_4^n$, due to clause (b), hence they will make an insignificant addition to the measure bound in clause (d). For the remaining rectangles, each u-side and s-side is longer than* $\mathrm{const}\cdot\theta_4^{2n}$.

As we said, there are plenty of collections Υ_n and their construction is quite flexible. Now we specify one that 'agrees' with the flow Φ^t so that the images of 'typical' stable fibers of the H-structures $\mathbf{H}_\tau$ (Section 2) at time $t = gn$ (here $g > 0$ is a constant) are comparable (in size) to the solid rectangles Q_j that their orbits are crossing at that time. More precisely, for any $|\tau| < r_H$ there exists a subset of stable fibers $\mathbf{W}_{n,\tau} = \{\mathcal{W}_\gamma^s\}$ in the H-structure $\mathbf{H}_\tau$ such that

$$\mu\big(\cup\mathcal{W}_\gamma^s\colon \mathcal{W}_\gamma^s \notin \mathbf{W}_{n,\tau}\big) \leq c_5\theta_5^n \tag{4.1}$$

and for every fiber $\mathcal{W}_\gamma^s \in \mathbf{W}_n$ we have

$$c|Q_j|_s \leq |\pi_\Omega\big(\Phi^{gn}(\mathcal{W}_\gamma^s)\big)| \leq c^{-1}|Q_j|_s \tag{4.2}$$

whenever $Q_j \cap \pi_\Omega\big(\Phi^{gn}(\mathcal{W}_\gamma^s)\big) \neq \emptyset$. Here $|Q_j|_s$ is the maximal length of stable fibers in Q_j and $c > 0$ is a small constant.

We first observe that stable fibers $\mathcal{W}^s_\gamma \in \mathbf{H}_\tau$ contract under the flow at variable speed, so that when the slowest ones are shrunk by the flow to the size of the solid rectangles Q_j's which they cross, the faster ones may be already much smaller, in fact their size may be an exponentially small fraction of the size of Q_j's that they cross. In order to ensure (4.2) we will take an arbitrary collection Υ_n satisfying Proposition 4.1 and partition some Q_j's (which are crossed by faster stable fibers) into smaller subrectangles.

For smooth Anosov flows [C3] the collection $\Upsilon_n = \{Q_1, \dots, Q_J\}$ was a Markov partition of Ω and its refinement along the above lines was constructed in [C3, Section 15] based on symbolic dynamics. Here we 'translate' the 'symbolic' argument of [C3] into geometric terms. We only sketch the procedure suppressing some technical details.

First, since $\mathbf{H}_\tau = \Phi^\tau(\mathbf{H}_0)$, it will be enough to deal with $\mathbf{H}_0$ only. If we project all fibers $\mathcal{W}^s_\gamma \in \mathbf{H}_0$ onto Ω, we get a collection of stable fibers for the map T, call them $\{W^s_\gamma\}$. They have an approximately constant length, say $|W^s_\gamma| \approx L'_H$. The orbits of points $x \in \mathcal{W}^s_\gamma$ during the time interval $(0,t)$, i.e. $\{\Phi^s x\}_{s=0}^t$, cross the base Ω a certain number of times, $m_{x,t}$, which satisfies $t/\tau_{\max} \le m_{x,t} \le t/\tau_{\min}$. We fix g so that $\lambda_T^{g/\tau_{\max}} = \theta_4^2$. Then the projection of the image $\Phi^{gn}(\mathcal{W}^s_\gamma)$ onto the base Ω has length $\le 2C_T L'_H \theta_4^{2n}$. We can assume that L'_H is small enough so that all those projections will be shorter than the s-side of the smallest solid rectangle $Q_j \in \Upsilon_n$, according to Remark 4.2.

Now denote $m = [gn/\tau_{\min}]+1$ and $d = gn/m$. Consider the time moments $t_i = id$ for $i = 0, \dots, m$. Observe that $d < \tau_{\min}$, so that every trajectory crosses Ω at most once during each time interval $[t_i, t_{i+1}]$. We will construct a sequence of collections $\Upsilon_n^{(i)}$ of solid rectangles, starting with $\Upsilon_n^{(0)} = \Upsilon_n$, inductively, so that the last one, $\Upsilon_n^{(m)}$, will be the one for which (4.1)–(4.2) will hold. Each collection $\Upsilon_n^{(i)}$ will be a refinement of the previous one $\Upsilon_n^{(i-1)}$.

Suppose $\Upsilon_n^{(i)}$ is already constructed so that the fibers $\Phi^{t_i}\mathcal{W}^s_\gamma$ satisfy the lower bound in (4.2). Moreover, assume that the length of the projection of each fiber $\Phi^{t_i}\mathcal{W}^s_\gamma$ onto Ω is at least $\frac{1}{2}|Q_j|_s$ whenever it crosses Q_j. (For $i=0$ this is obviously true, as our fibers in $\mathbf{H}_0$ have length $\mathcal{O}(1)$ and the rectangles $Q_j \in \Upsilon_n$ are shorter than θ_3^n.) Now move all the fibers $\Phi^{t_i}\mathcal{W}^s_\gamma$ further under the map Φ^d (this will produce the fibers $\Phi^{t_{i+1}}\mathcal{W}^s_\gamma$).

Let $\mathcal{W}_i = \Phi^{t_i}\mathcal{W}^s_\gamma$ be one such fiber and $\mathcal{W}_{i+1} = \Phi^{t_{i+1}}\mathcal{W}^s_\gamma$ its image. Let the projection $W_i = \pi_\Omega(\mathcal{W}_i)$ cross a solid rectangle $Q \in \Upsilon_n^{(i)}$. If there is no collisions during the time interval $[t_i, t_{i+1}]$ on the trajectory of $\mathcal{W}^s_\gamma$, then the projection $W_{i+1} = \pi_\Omega(\mathcal{W}_{i+1})$ will coincide with W_i, and we will not refine Q

(at this step). If there is a collision, then W_{i+1} may be (much) smaller than W_i (as stable fibers do contract!), and it will cross another solid rectangle $Q' \in \Upsilon_n^{(i)}$. Due to Proposition 4.1 (c), $T(Q) \cap Q'$ is a u-subrectangle in Q'. Now if W_{i+1} is smaller than half of the s-side of Q', we divide Q' into two or three u-subrectangles along the u-sides of $T(Q) \cap Q'$. Then W_{i+1} will be at least half the s-size of the new, smaller solid rectangle $T(Q) \cap Q'$. We do this refinement for every fiber $\Phi^{t_i}\mathcal{W}_\gamma^s$ and thus obtain a collection $\Upsilon_n^{(i+1)}$.

Observe that solid rectangles $Q \in \Upsilon_n^{(i+1)}$ have s-sides lying on the s-sides of $Q \in \Upsilon_n^{(i)}$ (hence, on the s-sides of $\Upsilon_n^{(0)}$, by induction), but their u-sides may be inside some elements of $\Upsilon_n^{(i)}$; however those u-sides are the images of some of the u-sides of $Q \in \Upsilon_n^{(i)}$ under T. Therefore the collection $\Upsilon_n^{(i+1)}$ will satisfy the 'Markov condition' (c) of Proposition 4.1 if so does $\Upsilon_n^{(i)}$. Obviously, our refinement will ensure the lower bound in (4.2).

The upper bound in (4.2) will be ensured automatically, as we explain next. First recall that by our choice of g at time t_m the image of every stable fiber will be shorter than the smallest s-side of the original solid rectangles $Q_j \in \Upsilon_n = \Upsilon_n^{(0)}$. Now if the upper bound in (4.2) fails, then a fiber $\mathcal{W} = \Phi^{t_m}\mathcal{W}_\gamma^s$ would have a projection $W = \pi_\Omega(\mathcal{W})$ onto Ω that would cross a rectangle $Q \in \Upsilon_n^{(m)}$ whose s-side is much smaller than $|W|$. But such a rectangle would have been created at some time $t_i \le t_m$ during our refinement procedure. Thus some stable fiber $\mathcal{W}_i = \Phi^{t_i}\mathcal{W}_{\gamma'}^s$ would have a projection $W_i = \pi_\Omega(\mathcal{W}_i)$ that would cross Q and at the same time been shorter than half the s-side of Q. In that case $|W_i| \ll |W|$, hence $|\mathcal{W}_i| \ll |\mathcal{W}|$. On the other hand, if we pull both fibers back under the map Φ^{-t_i}, then the smaller fiber $\mathcal{W}_i$ returns to $\mathbf{H}_0$, i.e. it recovers its size to about the constant value L'_H. But the pre-images of both fibers move next to each other (because their projections W and W_i are linked by some unstable fibers in Q). Hence both fibers are expanded under Φ^{-t_i} by about the same factor due to the distortion bounds. Therefore $|\Phi^{-t_i}\mathcal{W}| \gg L'_H$. But this is impossible because $t_m \ge t_i$ and $\Phi^{-t_m}\mathcal{W} \in \mathbf{H}_0$, hence $|\Phi^{-t_m}\mathcal{W}| \approx L'_H$.

Next we verify that the constructed collection $\Upsilon_n^{(m)}$ of solid rectangles has all the properties claimed in Proposition 4.1. The clauses (a), (c), (d) and the bound on the diameter in the clause (b) obviously hold. To ensure an exponential upper bound on the number $J^{(m)}$ of rectangles $\Upsilon_n^{(m)}$ we use the large deviation lemma proved in [CD, Proposition A.5]:

Lemma 4.3 (Large deviations)**.** *There is a constant $A = A(\mathcal{D}) > 1$ such* *that* $\nu\big(x \in \Omega\colon |\mathcal{J}_{W^s(x)} T^m(x)| < A^{-m}\big) \le c_6\theta_6^m$ *for all* $m \ge 1$

Let $m = gn/\tau_{\min}$. We define $\mathbf{W}_{n,0}$ to be the collection of stable fibers $\mathcal{W}_\gamma^s \in \mathbf{H}_0$ whose projections onto Ω contract by less than A^{-m} during the first m iterations of T (i.e. $|\mathcal{J}_{W^s} T^m| > A^{-m}$ on such fibers). Then (4.1) will hold due to the above lemma. On the other hand, the images of our fibers $\Phi^{gn}(\mathcal{W}_\gamma^s)$ will have length $\ge$ const $\cdot\, A^{-gn/\tau_{\min}}$, hence the s-sides of the refined solid rectangles $Q \in \Upsilon_n^{(m)}$ will be longer than const $\cdot\, A^{-gn/\tau_{\min}}$. Therefore

$$\nu(R_i) \ge c_7\theta_7^n \qquad \text{and} \qquad J^{(m)} \le c_7^{-1}\theta_7^{-n} \tag{4.3}$$

with $\theta_7 = \theta_4^2 A^{-g/\tau_{\min}}$. Note that the u-sides and s-sides of $Q \in \Upsilon_n^{(m)}$ are necessarily longer than const$\cdot\theta_7^n$.

There are a few 'final touches' we should make. First, observe that for any fiber $\mathcal{W} = \mathcal{W}_\gamma^s \in \mathbf{W}_{n,0}$ its image $\Phi^{t_i}(\mathcal{W})$ has length $\ge$ const $\cdot\, A^{-t_i/\tau_{\min}}$, hence it will exceed θ_3^n for all $t_i \le c'n$, where $c' = \tau_{\min}|\ln\theta_3|/\ln A > 0$ is a constant. Thus the actual refinement of Υ_n starts at time $t_i \ge c'n$, i.e. for $i \ge c'n/d$. It may happen that some stable fibers $\Phi^{t_i}(\mathcal{W}_\gamma^s)$ cross Ω (as their points may experience a collision at time t_i), which would complicate our refinement procedure. But the size of such fibers is exponentially small for every $i \ge c'n/d$, hence their union lies in an exponentially small neighborhood of Ω. Therefore the measure of their union is exponentially small, and we can simply remove all of them from $\mathbf{H}_{n,0}$ to avoid possible complications.

Second, the projection of some stable fiber $\Phi^{t_i}(\mathcal{W}_\gamma^s)$ onto Ω may intersect solid rectangles $Q \in \Upsilon_n^{(i)}$ only partially, as a significant portion of that projection may land in $\Omega \setminus \cup Q$. But we can simply require that portion be exponentially small (relative to the length of the whole projection). Indeed, the union of fibers violating this requirement will have an exponentially small measure, so they can be just removed from $\mathbf{H}_{n,0}$.

Third, it is essential that the projections of the fibers $\Phi^{t_m}(\mathcal{W}_\gamma^s)$ onto Ω are not cut by u-sides of the solid rectangles $Q \in \Upsilon_n^{(m)}$ in their middle parts corresponding to the images of the 'bars' in the H-structures. Recall that the middle part makes a small fraction of each stable fiber (as $L_H \gg R_H$, cf. Section 2), so there is a plenty of room for the u-sides of Q's to cut our fibers. On the other hand, the construction of Υ_n is quite flexible, as was remarked in the proof of Proposition 4.1. In fact the boundaries $\partial^u\xi$ and $\partial^s\xi$ of the pre-Markov partition ξ were constructed in [BSC1, Section 3] by first positioning some initial unstable and stable curves fairly arbitrarily in

Ω and then adjusting them iteratively to ensure the Markov property. This freedom can be used to place the curves in ξ^u so that they avoid undesirable intersections with the projections of our fibers in their middle parts. This calls for certain modifications in the constructions of [BSC1, Section 3] that are rather technical, so we leave them out.

From now on we discard the original collection $\Upsilon_n^{(0)}$ and denote by $\Upsilon_n = \{Q_1, \dots, Q_J\}$ the refined collection $\Upsilon_n^{(m)}$. We will assume that Υ_n satisfies Proposition 4.1 (where the value of θ_4 must be reset according to (4.3)).

5 Markov approximations

Here we recall and modify the construction of Markov approximations for the billiard map $T\colon \Omega \to \Omega$ developed in [BSC2, C1]. For every pair of integers $N > n > 0$ we will construct a finite partition $\mathfrak{R}_{n,N} = \{R_0, R_1, \dots, R_I\}$ of Ω. Its atoms $R_1, \dots, R_I$ will be small (Cantor) rectangles, and the remaining atom R_0 will be a 'large' open dense subset of Ω, but its measure will be small.

Let $\Upsilon_n = \{Q_1, \dots, Q_J\}$ be the collection of solid rectangles constructed in the previous section. For every Q_i consider the (Cantor) rectangle

$$R_i = R(Q_i) \cap \big(\cap_{k=-N}^{N} T^k(\cup_j Q_j)\big). \tag{5.1}$$

Proposition 5.1. *If $\nu\big(T^k(R_i) \cap R_j\big) \neq 0$ for some $1 \le i, j \le I$ and $1 \le k \le N$, then $T^k(R_i)$ intersects R_j properly. Moreover,*

$$\nu\big(\Omega \setminus \cup_i R_i\big) \le c_8 N \theta_8^n. \tag{5.2}$$

Proof. The properness of the intersections is verified by a direct inspection, which is fairly standard [BSC2, Section 4]. Then, $\nu\big(\cup_{j=1}^J Q_j \setminus \cup_{j=1}^J R(Q_j)\big) \le \text{const} \cdot \theta_3^{an}$ due to Proposition 4.1 (c) and (1.2). Now (5.2) follows from Corollary 3.3 and Proposition 4.1 (d). $\square$

In our further calculations, N does not exceed n^3, so that $N\theta_i^n$, $N^2\theta_i^n$, etc., are always small numbers.

Given $\theta \in (0,1)$, we say that a rectangle R_i in (5.1) is θ-dense if

$$\nu(R_i)/\nu(Q(R_i)) \ge 1 - \theta^n.$$

Lemma 5.2. *For some $\theta_9 > 0$ the union of θ_9-dense rectangles R_i has measure $\ge 1 - c_9 N \theta_9^n$.*

Proof. Let $\theta_9 = \theta_3^{1/2}$. If R_i does is not θ_9-dense, then $\nu\big(Q_i \setminus R_i\big) \geq \theta_9^n \nu(Q_i)$. Summing up over all such rectangles and using (5.2) proves the lemma. $\square$

We will only keep θ_9-dense rectangles. Then, according to (3.2), every such rectangle will satisfy

$$\rho(R_i) \geq 1 - \text{ const} \cdot \theta_9^n, \tag{5.3}$$

i.e. our rectangles have 'high density' on their stable and unstable fibers. Furthermore, we will only keep rectangles whose measure is $\geq \theta_4^{2n}$; the union of the abandoned rectangles will have measure $\leq \theta_4^n$ according to Proposition 4.1 (b). Let $R_1, \dots, R_I$ denote the remaining rectangles and $\mathfrak{R}_{n,N} = \{R_0, R_1, \dots, R_I\}$ the (mod 0) partition of Ω with $R_0 = \Omega \setminus \cup_{i=1}^I R_i$. We emphasize that

$$\mu(R_0) \leq c_{10} N \theta_{10}^n \qquad \text{and} \qquad \mu(R_i) \geq \theta_4^{2n} \quad \forall i \geq 1. \tag{5.4}$$

Proposition 5.3. *The following 'short-memory' approximation holds*[1]*:*

$$\begin{aligned} &\nu(T^{i_1} R_{j_1} \cap T^{i_2} R_{j_2} \cap \cdots \cap T^{i_{l-1}} R_{j_{l-1}} / T^{i_l} R_{j_l} \cap \ldots \cap T^{i_k} R_{j_k}) \\ &\qquad = \nu(T^{i_1} R_{j_1} \cap \cdots \cap T^{i_{l-1}} R_{j_{l-1}} / T^{i_l} R_{j_l}) \cdot (1 + \Delta) \end{aligned} \tag{5.5}$$

where the 'remainder term' Δ satisfies $|\Delta| \leq c_9 \theta_9^n$ for all rectangles $R_{j_1}, \dots, R_{j_k} \in \mathfrak{R}_{n,N}$ and all $1 \leq i_1 < i_2 < \cdots i_k \leq N$ (note that R_0 is not a rectangle, so it is not allowed here).

Proof. This follows from the properness of intersection (Proposition 5.1) and the approximation of ν by a product measure in every rectangle (Proposition 3.1); see a detailed proof in [BSC2, Section 4]. $\square$

In [C1, C3], a Markov approximation for the map $T\colon \Omega \to \Omega$ based on any partition of M into subsets $\{R_0, R_1, \dots, R_I\}$ was defined to be a probabilistic stationary Markov chain with states $\{0, 1, \dots, I\}$, transition probabilities

$$\pi_{ij} = \nu(R_j / T R_i) = \nu(R_j \cap T R_i) / \nu(R_i) \tag{5.6}$$

and the stationary distribution

$$p_i = \nu(R_i). \tag{5.7}$$

[1] Here and further on $\nu(A/B)$ means the conditional measure, $= \nu(A \cap B)/\nu(B)$, and we always set it to zero whenever $\nu(B) = 0$.

This is, perhaps, one of the simplest realizations of the popular physical concept of 'coarse-graining' of phase space, see, e.g., discussions in [NMT, CELS].

The following quantity was introduced in [C1, C3] to characterize the discrepancy between the above Markov approximation and the actual N iterations of T:

$$\chi_N\colon = \sup_{L\le N} \sum_{i_0,i_{-1},\dots,i_{-L}} |\nu(R_{i_0}/TR_{i_{-1}}\cap\cdots\cap T^L R_{i_{-L}}) - \nu(R_{i_0}/TR_{i_{-1}})| \tag{5.8}$$
$$\times\, \nu(TR_{i_{-1}}\cap\cdots\cap T^L R_{i_{-L}}).$$

The properties of our Markov approximation $\mathfrak{R}_{n,N} = \{R_0, R_1, \dots, R_I\}$ ensure that

$$\chi_N \le c_9\theta_9^n + c_8 N\theta_8^n \le c_{10}N\theta_{10}^n.$$

It was proved in [C1, Section 5] that

$$\sum_{i_0,i_{-1},\dots,i_{-N}} |\nu(R_{i_0}\cap TR_{i_{-1}}\cap\cdots\cap T^N R_{i_{-N}}) - p_{i_{-N}}\pi_{i_{-N}i_{-N+1}}\cdots\pi_{i_{-1}i_0}| \tag{5.9}$$
$$\le (N-1)\chi_N \le c_{10}N^2\theta_{10}^n.$$

The meaning of this is that the ν-measure is close to the Markov measure on 'cylindrical sets' of length N.

Another quantity characterizing a partition $\{R_0,\dots,R_I\}$ of Ω, see [C3, Section 3], is $D = \sum_{i=0}^I \nu(R_i)\,\mathrm{diam}(R_i)$. In our case

$$D \le \theta_3^n + \mathrm{diam}\,\Omega\cdot c_8 N\theta_8^n \le c_{11}N\theta_{11}^n. \tag{5.10}$$

Next we recall necessary constructions of [C3] related to the flow $\Phi^t\colon \mathcal{M}\to\mathcal{M}$, of which we always think as a suspension flow with the base automorphism $T\colon \Omega\to\Omega$ and the ceiling function $\tau(x)$.

Let $\mathfrak{R}_{n,N}$ be the above partition of Ω. For any $x\in M$ let $R(x)$ denote the atom $R\in\mathfrak{R}_{n,N}$ containing x. Now

$$\bar\tau(x) = [\nu(R(x))]^{-1}\cdot\int_{R(x)}\tau(y)\,d\nu(y)$$

is the return time function $\tau(x)$ conditioned on the partition $\mathfrak{R}_{n,N}$.

Lemma 5.4. *We have* $|\tau(x)-\bar\tau(x)|\le\theta_3^{n/2}$ *for all* $x\in\Omega\setminus R_0$.

Proof. Observe that $\tau(x)$ is Hölder continuous with Hölder exponent $= 1/2$ on every connected component of $\Omega \setminus \mathcal{S}_1$, then use Proposition 4.1 (b). $\square$

Let $\delta > 0$ be a small number, a 'quantum of time'. Put $\hat{\tau}(x) = \bigl([\bar{\tau}(x)/\delta] + 2\bigr)\delta$. The function $\hat{\tau}(x)$ on Ω approximates $\tau(x)$, but it is constant on every atom of $\mathfrak{R}_{n,N}$ and its values are integral multiples of δ. Denote by $\hat{\Phi}^t$ the suspension flow with the base automorphism $T\colon \Omega \to \Omega$ and under the ceiling function $\hat{\tau}(x)$. Let

$$\hat{\mathcal{M}} = \{(x, s)\colon x \in \Omega, \quad 0 \le s < \hat{\tau}(x)\}$$

denote the phase space of this flow. It preserves the measure $d\hat{\mu} = c_{\hat{\mu}}\, d\nu \times ds$, which is proportional to μ on $\mathcal{M} \cap \hat{\mathcal{M}}$, and $1 \le c_\mu / c_{\hat{\mu}} \le 1 + \mathcal{O}(\delta)$. The flow $\hat{\Phi}^t$ was called a discrete version of Φ^t in [C3], we will call it a *box flow* here.

The map $\hat{F} = \hat{\Phi}^\delta$ acts on $\hat{\mathcal{M}}$ and preserves the measure $\hat{\mu}$. Let $\mathcal{R}_{n,N,\delta}$ denote the partition of $\hat{\mathcal{M}}$ into atoms $R_j \times [s\delta, (s+1)\delta)$, where $R_j \in \mathfrak{R}_{n,N}$ and $s = 0, 1, \dots, \hat{\tau}(x)/\delta - 1$ for $x \in R_j$. We denote the atoms of this partition by X_i, $1 \le i \le \hat{I} = \hat{I}_{n,N,\delta}$, numbered in an arbitrary order.

For every atom $X_i = R_j \times [s\delta, (s+1)\delta) \in \mathcal{R}_{n,N,\delta}$ we put $R(X_i) = R_j$ (the atom's base) and $s(X_i) = s$ (the atom's level). Over every 'base' atom $R_j \in \mathfrak{R}_{n,N}$ there is a column of atoms X_i with $R(X_i) = R_j$ and $s(X_i) = 0, 1, \dots, \hat{\tau}(x)/\delta - 1$ for $x \in R_i$. The first atom in every column is called its *bottom*, and the last one its *top*. The space $\hat{\mathcal{M}}$ consists of columns of atoms, each of height δ built over the atoms of the partition $\mathfrak{R}_{n,N}$. The map $\hat{F}$ shifts (elevates) every atom X_i, except the top ones, one level up, so that $\hat{F}X_i$ is another atom in the same column. Every top atom breaks, under $\hat{F}$, into pieces which fall into some bottom atoms according to the action of the map T on Ω.

The atoms $X_i \in \mathcal{R}_{n,N,\delta}$ constructed over rectangles R_j have a 3D direct product structure, we call them *boxes*. (Recall that $R_j \in \mathfrak{R}_{n,N}$ is a rectangle if $j \ge 1$.) 'Bad' atoms $R_0 \times [s\delta, (s+1)\delta)$ constructed over the 'leftover set' R_0 are called 'nonboxes', they will not be of much use. The measure $\hat{\mu}$ restricted to any box X_i is approximately a 3D product measure. Every box $X_i = R_j \times [s\delta, (s+1)\delta)$ is a Cantor-like set enclosed in a 'solid box' $\mathcal{Q}(X_i) = Q_j \times [s\delta, (s+1)\delta)$, where Q_j is the solid rectangle corresponding to R_j according to (5.1). The solid boxes are closed domains bounded by six smooth hypersurfaces (faces), which include: two u-faces and two s-faces (which project down to the u-sides and s-sides of Q_j, respectively), the top face $Q_j \times \{(s+1)\delta\}$ and the bottom face $Q_j \times \{s\delta\}$. Solid boxes $\mathcal{Q}(X_i)$'s have disjoint interiors and line up in columns that enclose the columns of (Cantor) boxes X_i's.

For any $t > 0$ the map $\hat{F}^{[t/\delta]}$ on $\hat{\mathcal{M}}$ approximates the time t map Φ^t on $\mathcal{M}$. The two flows Φ^t and $\hat{\Phi}^t$ have (slightly) different ceiling functions but the same base map $T\colon \Omega \to \Omega$. Therefore, for any point $y = (x, s) \in \mathcal{M} \cap \hat{\mathcal{M}}$ its images $\Phi^t y$ and $\hat{\Phi}^t y$ follow the same orbit, but with a time delay. We call this effect *asynchronism*. Precisely, there is a $\Delta_t(y)$ such that $\Phi^{t+\Delta_t(y)} = \hat{\Phi}^t y$ (in the case $\hat{\Phi}^t y \notin \mathcal{M}$ the flow Φ^t can be obviously extended to the point $\hat{\Phi}^t y$). The value $\Delta_t(y)$ is, generally, small, unless the orbit crosses the 'bad' set $R_0 \in \mathfrak{R}_{n,N}$. More precisely, Lemma 5.4 implies

Lemma 5.5. *For any $t > 0$, either the trajectory $\{\Phi^s y\}$ crosses the 'bad' set R_0 (i.e. enters a 'bad' atom of $\mathcal{R}_{n,N,\delta}$) for some $0 < s < t$, or we have*

$$|\Delta_t(y)| \le c_{12}(\theta_{12}^n + \delta)\,|t|. \tag{5.11}$$

Next, for any $t_1, t_2 > 0$ such that $t_1 + t_2 \le t$ we will define a Markov chain approximating the map $\hat{\Phi}^{t_1+t_2}$. Its states $\{1, \dots, \hat{I}\}$ will correspond to the atoms $X_i \subset \hat{\mathcal{M}}$ and its stationary vector will be

$$\hat{P} = \|\hat{p}_i\|, \qquad \hat{p}_i = \hat{\mu}(X_i). \tag{5.12}$$

For $r = 1, 2$ consider a Markov chain with transition probabilities

$$\hat{\pi}_{ij}^{(r)} = \hat{\mu}(X_j / \hat{F}^{[t_r/\delta]} X_i). \tag{5.13}$$

Both stochastic matrices $\hat{\Pi}^{(r)} = \|\hat{\pi}_{ij}^{(r)}\|$ with $r = 1, 2$ preserve the probability vector (5.12). It follows from (5.4) that

$$\hat{p}_{\min} = \min_i \{\hat{p}_i\} \ge c_{\hat{\mu}} \delta c_4 \theta_4^{2n} \tag{5.14}$$

and the total measure of 'nonboxes' is bounded by

$$\hat{\mu}\big(\cup_{R(X_i)=R_0} X_i\big) \le 2\,\tau_{\max}\, c_{10} N \theta_{10}^n. \tag{5.15}$$

Also, for any integer η such that $2 < \eta < \delta^{-1}$ let $\Pi^{(\eta)} = \|\pi_{ij}^{(\eta)}\|$ be a stochastic matrix defined by

$$\pi_{ij}^{(\eta)} = \begin{cases} 1/(2\eta+1) & \text{if } R(X_i) = R(X_j) \text{ and } |s(X_i) - s(X_j)| \le \eta \\ 0 & \text{otherwise} \end{cases} \tag{5.16}$$

for all $i \neq j$, and $\pi_{ii}^{(\eta)} = 1 - \sum_{j \neq i} \pi_{ij}^{(\eta)}$ for all i. Roughly speaking, under the action of $\Pi^{(\eta)}$ the mass of every atom $X_i \in \mathcal{R}_{n,N,\delta}$ is 'smothered' uniformly into $(2\eta + 1)$ neighboring atoms around X_i in the same column. We need this random perturbation to compensate for the asynchronism between the flows Φ^t and $\hat{\Phi}^t$ described in Lemma 5.5.

Since the distribution (5.12) is uniform within every column of atoms of $\mathcal{R}_{n,N,\delta}$, it is also invariant under $\Pi^{(\eta)}$.

Now, the Markov chain defined by the stochastic matrix $\hat{\Pi} = \hat{\Pi}^{(1)}\Pi^{(\eta)}\hat{\Pi}^{(2)}$ is the one that will approximate the map $\hat{\Phi}^{t_1+t_2}$. It preserves the probability vector (5.12), and its transition probabilities are

$$\hat{\pi}_{ij} = \sum_{i_1,i_2} \hat{\pi}_{ii_1}^{(1)} \cdot \pi_{i_1 i_2}^{(\eta)} \cdot \hat{\pi}_{i_2 j}^{(2)}. \tag{5.17}$$

6 Final estimates

Let $K_1 = [t_1/\delta]$ and $K_2 = [t_2/\delta]$. For any i, j we put

$$\begin{aligned} \hat{\gamma}_{i,j} &= \sum_k \frac{\hat{\pi}_{ik}\hat{\pi}_{jk}}{\hat{p}_k} \\ &\geq \sum_k \sum_{l_1 l_2 l_3 l_4}^{(c)} \frac{\hat{\mu}(\hat{T}^{K_1}X_i \cap X_{l_1})\hat{\mu}(X_{l_3} \cap \hat{T}^{-K_2}X_k)\hat{\mu}(\hat{T}^{K_1}X_j \cap X_{l_2})\hat{\mu}(X_{l_4} \cap \hat{T}^{-K_2}X_k)}{(2\eta+1)^2\hat{\mu}(X_k)\hat{\mu}(X_i)\hat{\mu}(X_{l_3})\hat{\mu}(X_j)\hat{\mu}(X_{l_4})} \end{aligned} \tag{6.1}$$

where the summation in $\sum^{(c)}$ is taken over the quadruples (l_1, l_2, l_3, l_4) satisfying the following 'coupling' condition: the atoms X_{l_1} and X_{l_3} must be in the same column separated by no more than $\eta - 1$ other atoms, and the same must hold for the atoms X_{l_2} and X_{l_4}.

We will only need a lower bound on $\hat{\gamma}_{i,j}$, and we will reduce its value by excluding 'bad' atoms ('nonboxes') from the consideration as follows. First, we set $\hat{\gamma}_{i,j} = 0$ if either X_i or X_j is not a box (i.e. if one of them is constructed over R_0). Second, we assume that the summation $\sum_k$ is taken over boxes X_k only. Third, we restrict the summation $\sum_{l_1 l_2 l_3 l_4}^c$ to quadruples $\{X_{l_1}, X_{l_2}, X_{l_3}, X_{l_4}\}$ of boxes only. Forth, we reduce the sets $\hat{T}^{K_1}X_i \cap X_l$ by excluding all the points $x \in \hat{T}^{K_1}X_i \cap X_l$ for which $\hat{T}^{-r}x$ belongs to a 'nonbox' for any $r = 1, \ldots, K_1$ (we do the same for X_j). Similarly, we reduce the sets

$X_l \cap \hat{T}^{-K_2} X_k$ by excluding all the points $x \in X_l \cap \hat{T}^{-K_2} X_k$ for which $\hat{T}^r x$ belongs to a 'nonbox' for any $r = 1, \dots, K_2$.

We will only use (6.1) when $K_1 \ll N/\delta$ and $K_2 \ll N/\delta$. Now, after all the above reductions, we can describe the intersections $\hat{T}^{K_1} X_i \cap X_l$ as follows. First of all, given a box $X = R \times [s\delta, (s+1)\delta)$ and a u-subrectangle $R' \subset R$, we call the set $X' = R' \times [s\delta, (s+1)\delta)$ a u-subbox of X. Similarly we define s-subboxes. Now due to Proposition 5.1 every intersection $\hat{T}^{K_1} X_i \cap X_l$ that has a positive measure is a u-subbox in X_l. Similarly, every intersection $\hat{T}^{-K_2} X_k \cap X_l$ that has a positive measure is an s-subbox in X_l.

Next, for any $\boldsymbol{\gamma} > 0$ let

$$\hat{Q}(\boldsymbol{\gamma}) = \sum_{(i,j):\, \hat{\boldsymbol{\gamma}}_{i,j} < \boldsymbol{\gamma}} \hat{\mu}(X_i) \hat{\mu}(X_j) \tag{6.2}$$

We note that all the above reductions of $\hat{\boldsymbol{\gamma}}_{i,j}$ only increased the value of $\hat{Q}(\boldsymbol{\gamma})$.

We now fix a $t > 1$. Throughout, $\alpha_i > 0$ and $\kappa > 0$ are constants depending on the billiard table $\mathcal{D}$ alone. The following proposition is proved in [C3, Propositions 5.4 and 7.1]:

Proposition 6.1. *Let F, G be two generalized Hölder continuous functions on $\mathcal{M}$. For any $t_1, t_2 > 1$ such that $t_1 + t_2 < t$, any $\delta, \eta, \boldsymbol{\gamma} > 0$, and any partition $\mathcal{R}_{n,N}$ such that $N > \kappa t$, we have*

$$\begin{aligned} |\mathbf{C}_{F,G}(t)| \leq\ & \mathrm{const} \cdot \mathrm{var}_\alpha(F)\, \mathrm{var}_\alpha(G)\, t^{\alpha_1} \boldsymbol{\gamma}^{-1} \\ & \times \left[(\eta\delta)^{\alpha_2} + D_{n,N}^{\alpha_3} + \chi_N + \hat{p}_{\min} + \hat{Q}(\boldsymbol{\gamma}) + \hat{p}_{\min}^{-2} (1 - \boldsymbol{\gamma}/80)^{\frac{t}{t_1 + t_2}} \right] \end{aligned}$$

Now the results of the previous sections gives

Corollary 6.2. *Under the same conditions,*

$$\begin{aligned} |\mathbf{C}_{F,G}(t)| \leq\ & \mathrm{const} \cdot \mathrm{var}_\alpha(F)\, \mathrm{var}_\alpha(G)\, N^{\alpha_4} \boldsymbol{\gamma}^{-1} \\ & \times \left[(\eta\delta)^{\alpha_5} + \theta_{11}^n + \delta\theta_4^{2n} + \hat{Q}(\boldsymbol{\gamma}) + \delta^{-2} \theta_4^{-4n} (1 - \boldsymbol{\gamma}/80)^{\frac{t}{\kappa(t_1 + t_2)}} \right] \end{aligned}$$

More specifically, the following theorem proved in [C3, Theorem 7.2] based on Proposition 6.1 gives exact sufficient conditions for a stretched exponential bound on correlations.

Theorem 6.3. *Assume that there are constants $\boldsymbol{\gamma}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 > 0$, and $0 < \theta_{13} < \theta_{14} < 1$ depending on the billiard table $\mathcal{D}$ alone, such that for all large n with the choice of $N = n^3$, $\delta = \theta_{13}^n$, $\eta = [\theta_{14}^{-n}]$, $t_1 = \boldsymbol{\beta}_1 n$, $t_2 = \boldsymbol{\beta}_2 n$ we have*

$$\hat{Q}(\boldsymbol{\gamma}) \leq c_{15}\theta_{15}^n. \tag{6.3}$$

Then there is a constant $a = a(\mathcal{D}, \alpha) > 0$ such that for any generalized Hölder continuous functions F and G

$$|\mathbf{C}_{F,G}(t)| \leq \text{ const} \cdot \text{var}_\alpha(F)\, \text{var}_\alpha(G)\, e^{-a\sqrt{t}} \tag{6.4}$$

for all $t > 0$.

Theorem 6.3 follows from Corollary 6.2 if we set $n = z\sqrt{t}$ with a sufficiently small $z > 0$, see [C3, Section 7].

It remains to verify the conditions of Theorem 6.3. Our arguments follow the scheme developed in [C3, Section 16] for Anosov flows, but in addition we have to deal with irregularities caused by the billiard dynamics. The key element in our scheme will be the estimation of (6.1) from below.

Let $X_i = R_j \times [s\delta, (s+1)\delta)$ be a box in $\hat{\mathcal{M}}$. Recall that each solid rectangle $Q_j \in \Upsilon_n$ has sides shorter than θ_3^n, see Proposition 4.1 (b). We pick $\theta_{13} > \theta_3$ so that $\delta = \theta_{13}^n \gg \theta_3^n \geq \operatorname{diam} Q_j$. Then each solid box $\mathcal{Q}(X_i)$ will look like a 'julienne': a relatively toll prism with a narrow base. Thus there will be an unstable fiber $\mathcal{W}_i^u \subset \mathcal{Q}(X_i)$ stretching from one s-face of $\mathcal{Q}(X_i)$ to the other. We fix such a fiber $\mathcal{W}_i^u$ in every box.

Now let $t > 0$ and $K = [t/\delta]$. As the map $\hat{T}^K$ approximates Φ^t (see Section 5), the image $\hat{T}^K X_i$ of the box X_i will, roughly speaking, stretch along the smooth components $\Phi^t(\mathcal{W}_i^u)$. Let $\mathcal{W}_{t,i}^u(\mathbf{H}_\tau^k)$ denote the union of the smooth components of $\Phi^t(\mathcal{W}_i^u)$ satisfying the conditions (W1) and (W2) of Section 2 for some $k = 1, 2$ and $|\tau| < r_H$ (note though that the index i has now a different meaning than in (W1)–(W2)).

First we need to guarantee the existence and abundance of the components in $\mathcal{W}_{i,t}^u(\mathbf{H}_\tau^k)$. Recall that the curve $\mathcal{W}_i$ has length $\geq$ const $\cdot\, \theta_7^n$. Now we set $\boldsymbol{\beta}_1 = 2a_H|\ln\theta_7|$. Then we can apply Corollary 2.6 to any pair of boxes X_i, X_j and respective unstable fibers $\mathcal{W}_i^u \subset \mathcal{Q}(X_i)$, $\mathcal{W}_j^u \subset \mathcal{Q}(X_j)$ and the time moment $t = t_1 = \boldsymbol{\beta}_1 n$. Thus, for every pair of boxes X_i, X_j there is an $s \in [0, s_H]$ such that

(a) the components of $\Phi^{t_1}(\mathcal{W}_i^u)$ stretching along some unstable fibers $W_\alpha^u \in \mathbf{H}_{-s}^1$ will make a subset in $\Phi^t(\mathcal{W}_i^u)$ of relative measure $\geq d_H$;

(b) the components of $\Phi^{t_1}(\mathcal{W}_j^u)$ stretching along some unstable fibers $W_\beta^u \in \mathbf{H}_{-s}^2$ will make a subset in $\Phi^t(\mathcal{W}_j^u)$ of relative measure $\geq d_H$.

Now let $K_1 = [t_1/\delta]$. For every component $\mathcal{W}' \subset \Phi^{t_1}(\mathcal{W}_i^u)$ its preimage $\Phi^{-t_1}(\mathcal{W}')$ is a subcurve of $\mathcal{W}_i^u \subset \mathcal{Q}(X_i)$ which delimits an s-subbox $X' \subset X_i$. Let $X'' \subset X'$ consist of points $x \in X'$ whose trajectories $\{\Phi^t x\}$ do not cross the bad set R_0 at any time $t \in (0, t_1)$. Then the set $\hat{T}^{K_1}(X'')$ consists of u-subboxes in some boxes X_l. These subboxes lie in the columns of solid boxes constructed over solid rectangles $Q \in \Upsilon_n$ that are crossed by the curve $\pi_\Omega(\mathcal{W}') \subset \Omega$.

More precisely, for every solid rectangle $Q \in \Upsilon_n$ that is crossed by the curve $\pi_\Omega(\mathcal{W}')$ the set $\hat{T}^{K_1}(X'')$ intersects at most one solid box $\mathcal{Q}(X_l)$ in the column over Q. And if the intersection $X_l \cap \hat{T}^{K_1}(X'')$ has positive measure, then it is a u-subbox in X_l. Due to the asynchronism (Section 5), each u-subbox $X_l \cap \hat{T}^{K_1}(X'')$ may be shifted from the curve $\mathcal{W}'$ up or down the respective column; more precisely it is located the distance $\leq \Delta_{t_1} = c_{12}\big(\theta_{12}^n + \delta\big)|t_1|$ from the curve $\mathcal{W}'$, due to (5.11). We can choose $\theta_{13} > \theta_{12}$, then we simply have $\Delta_{t_1} = \mathcal{O}(\delta|t_1|)$.

Thus there is a chain of u-subboxes $\{X_l \cap \hat{T}^{K_1}(X'')\}$ lining up along the curve $\mathcal{W}'$. The set $\cup_{X''}\pi_\Omega\big(\hat{T}^{K_1}(X'')\big)$ is a (long and narrow) rectangle in Ω stretching along the curve $\pi_\Omega(\mathcal{W}')$. We will only use the chains stretching along the components $\mathcal{W}'$ fitting the description (a) above, i.e. along $\mathcal{W}' \subset \mathcal{W}_{t_1,i}^u(\mathbf{H}_\tau^1)$. As there may be many such components, there are just as many chains. Note, however, that some chains may be rather 'holey', arbitrarily small (in measure), or even empty. To ensure the abundance of 'sufficiently dense' chains we will utilize the 'high density' properties of rectangles $R \in \mathfrak{R}_{n,N}$, see (5.3).

Summarizing, the set $\hat{T}(X_i)$ contains chains of u-subboxes stretching along some unstable fibers $W_\alpha^u \in \mathbf{H}_{-s}^1$. Similarly, the set $\hat{T}(X_j)$ contains chains of u-subboxes stretching along some unstable fibers $W_\beta^u \in \mathbf{H}_{-s}^2$. Due to Corollary 2.6, there are stable fibers $\mathcal{W}_{\alpha\beta}^s$ linking the former chains with the latter chains, and the distance from $\mathcal{W}_{\alpha\beta}^s$ to the corresponding chains is $\mathcal{O}(\Delta_{t_1})$. Furthermore, due to property (H5) of the H-structures there are plenty of stable fibers $\mathcal{W}_\gamma^s \in \mathbf{H}_{-s}$ around the linking fiber $\mathcal{W}_{\alpha\beta}^s$.

Now we set $\boldsymbol{\beta}_2 = g$ and $t_2 = \boldsymbol{\beta}_2 n$, where g was defined in the previous section. Then for 'almost' every fiber $\mathcal{W}_\gamma^s$ (in the sense of (4.1)) its image $\Phi^{t_2}\mathcal{W}_\gamma^s$ ends up in a solid box $\mathcal{Q}(X_k)$ and its length is comparable to the length of the maximal stable fiber in that box. Let $K_2 = [t_2/\delta]$. Then the set $\hat{T}^{-K_2}X_k$ will contain a chain of s-subboxes stretching along the curve $\mathcal{W}^s_\gamma$ and staggering from it (up and down the corresponding columns of boxes) by less than $\Delta_{t_2} = c_{12}(\theta_{12}^n + \delta)|t_2| = \mathcal{O}(\delta|t_2|)$, again due to (5.11) and the assumption $\theta_{13} > \theta_{12}$. Note also that the projection of that chain down to Ω is a (long and narrow) rectangle stretching along the curve $\pi_\Omega(\mathcal{W}^s_\gamma)$.

We choose $\theta_{14} > \theta_{13}$ and set $\eta = [\theta_{14}^{-n}]$. Then, on the one hand, $\eta\delta < (\theta_{13}/\theta_{14})^n$ is exponentially small and, on the other hand, $\eta\delta \gg \max\{\Delta_{t_1}, \Delta_{t_2}\}$.

The rest of the argument closely follows [C3, Section 16]. Let $\tilde{X}_i^{\zeta_1}$, $\zeta_1 = 1, \ldots, Z_i$, denote all the chains of u-subboxes in $\hat{T}^{K_1}X_i$ described above, i.e. stretching along some curves $\mathcal{W}^u_\alpha \in \mathbf{H}^1_{-s}$. Similarly, let $\tilde{X}_j^{\zeta_2}$, $\zeta_2 = 1, \ldots, Z_j$, be all the chains of u-subboxes in $\hat{T}^{K_1}X_j$ stretching along some curves $\mathcal{W}^u_\beta \in \mathbf{H}^2_{-s}$. For any pair ζ_1, ζ_2 consider the connecting curve $\mathcal{W}^s_{\alpha\beta}$ and all nearby stable fibers $\mathcal{W}^s_\gamma$; then for any k denote by $\tilde{X}_k^1$ the chain of s-subboxes in $\hat{T}^{-K_2}X_k$ stretching along some W^s_γ (due to the results of the previous section, there can be at most one such chain for any k).

Consider any pair of chains $\tilde{X}_i^{\zeta_1}$, $\tilde{X}_j^{\zeta_2}$ and any chain $\tilde{X}_k^1$ described above. The projections $\pi_\Omega(\tilde{X}_i^{\zeta_1})$ and $\pi_\Omega(\tilde{X}_k^1)$ may intersect each other inside at most one solid rectangle $Q \in \Upsilon_n$. Hence, there is at most one column of solid boxes in $\hat{\mathcal{M}}$ in which both chains have 'representatives', i.e. there is at most one u-subbox $X_{l_1} \cap \tilde{X}_i^{\zeta_1} \subset X_{l_1}$ and at most one s-subbox $X_{l_3} \cap \tilde{X}_k^1 \subset X_{l_3}$ so that the boxes X_{l_1} and X_{l_3} belong to the same column (over Q). We put $\Gamma_{i,k}^{\zeta_1} = 1$ if such boxes X_{l_1} and X_{l_3} exist and $|s(X_{l_1}) - s(X_{l_3})| < \eta$ and $\Gamma_{i,k}^{\zeta_1} = 0$ otherwise[2]. Thus, every pair of chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_k^1$ has at most one representative in (6.1); in fact it does have one if and only if $\Gamma_{i,k}^{\zeta_1} = 1$, according to the "coupling" condition on l_1, l_3 in the setup of equation (6.1). Similar conclusions, of course, hold for every pair of chains $\tilde{X}_j^{\zeta_2}$ and $\tilde{X}_k^1$.

The following estimate easily results from the approximation of the measure $\hat{\mu}$ by a product measure within boxes, cf. [C3, Lemma 16.1]:

$$\hat{\mu}(\hat{T}^{K_1}X_i \cap X_{l_1}) \cdot \hat{\mu}(\hat{T}^{-K_2}X_k \cap X_{l_3}) \geq \text{const} \cdot \delta^{-1}\hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_k^1)\hat{\mu}(X_{l_3})$$

A similar estimate holds for any pair of chains $\tilde{X}_j^{\zeta_2}$ and $\tilde{X}_k^1$. This allows us

[2] The difference $|s(X_{l_1}) - s(X_{l_3})| - 1$ is the number of boxes between X_{l_1} and X_{l_3} in the corresponding column over Q.

to 'decouple' the indices i, j, k from l_1, l_2, l_3, l_4 in (6.1):

$$\begin{aligned}\hat{\gamma}_{i,j} &\geq \hat{c}_1 \sum_k \sum_{\zeta_1,\zeta_2} \Gamma^{\zeta_1}_{i,k}\Gamma^{\zeta_2}_{j,k} \frac{\hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_j^{\zeta_2})\hat{\mu}(\tilde{X}_k^1)}{[(2\eta+1)\delta]^2\hat{\mu}(X_i)\hat{\mu}(X_j)} \\ &\geq \hat{c}_1 \sum_{\zeta_1,\zeta_2} \left(\sum_k \Gamma^{\zeta_1}_{i,k}\Gamma^{\zeta_2}_{j,k} \frac{\hat{\mu}(\tilde{X}_k^1)}{[(2\eta+1)\delta]^2} \right) \frac{\hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_j^{\zeta_2})}{\hat{\mu}(X_i)\hat{\mu}(X_j)} \qquad (6.5)\end{aligned}$$

where $\hat{c}_1 > 0$ is a constant (cf. [C3, Equation (16.2)]; also note that if the chain $\tilde{X}_k^1$ exists, then $\hat{\mu}(\tilde{X}_k^1) \geq$ const $\cdot\, \hat{\mu}(X_k)$ due to (4.2)).

Next, we fix a sufficiently small constant $\hat{c}_2 > 0$ and say that a pair of chains $\tilde{X}_i^{\zeta_1}$, $\tilde{X}_j^{\zeta_2}$ makes a 'good couple' if

$$\sum_k \Gamma^{\zeta_1}_{i,k}\Gamma^{\zeta_2}_{j,k}\, \hat{\mu}(\tilde{X}_k^1) \geq \hat{c}_2[(2\eta+1)\delta]^2. \qquad (6.6)$$

Recall that $\Gamma^{\zeta_1}_{i,k} = \Gamma^{\zeta_2}_{j,k} = 1$ whenever the 'stable' chain $\tilde{X}_k^1$ is $\eta\delta$-close to both 'unstable' chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_j^{\zeta_2}$. Those 'unstable' chains stretch along some fibers $\mathcal{W}^u_\alpha \in \mathbf{H}^1_{-s}$ and $\mathcal{W}^u_\beta \in \mathbf{H}^2_{-s}$, respectively, and there is a 'connecting' stable fiber $\mathcal{W}^s_{\alpha\beta}$. Due to Lemma 2.2 every stable fiber $\mathcal{W}^s_\gamma$ passing through the $\eta\delta$-neighborhood of the point $\mathcal{W}^u_\alpha \cap \mathcal{W}^s_{\alpha\beta}$ will pass through the $C\eta\delta$-neighborhood of the point $\mathcal{W}^u_\beta \cap \mathcal{W}^s_{\alpha\beta}$, where $C > 0$ is a constant. And the property (H5) of the H-structures guarantees the abundance of such stable fibers, thus their union has volume $\geq$ const $\cdot\, (\eta\delta)^2$. For this reason one might expect that the union of 'stable' chains $\tilde{X}_k^1$ that are $\delta\eta$-close to both 'unstable' chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_j^{\zeta_2}$ has volume $\geq$ const $\cdot\, (\eta\delta)^2$ as well, which would obviously imply (6.6). We will argue below that this is indeed true for 'typical' (if not all) pairs of chains.

For any 'good couple' of chains $\tilde{X}_i^{\zeta_1}$, $\tilde{X}_j^{\zeta_2}$ the interior sum in (6.5) is bounded below by $\hat{c}_2$. We say that a pair of boxes X_i, X_j makes a 'good couple' if at least 50% (in terms of measure) of their chains $\tilde{X}_i^{\zeta_1}$ and $\tilde{X}_j^{\zeta_2}$ make good couples. Now by Corollary 2.6 and the 'high density' property (5.3) it follows that

$$\sum_{\zeta_1} \hat{\mu}(\tilde{X}_i^{\zeta_1}) \geq \tfrac{1}{2}\, d_H \hat{\mu}(X_i) \quad \text{and} \quad \sum_{\zeta_2} \hat{\mu}(\tilde{X}_j^{\zeta_1}) \geq \tfrac{1}{2}\, d_H \hat{\mu}(X_j).$$

Thus, for every 'good couple' of boxes X_i, X_j we have

$$\sum_{\zeta_1,\zeta_2 :\, (6.6) \text{ holds}} \hat{\mu}(\tilde{X}_i^{\zeta_1})\hat{\mu}(\tilde{X}_j^{\zeta_2}) \geq \tfrac{1}{8}\, d_H^2 \hat{\mu}(X_i)\hat{\mu}(X_j),$$

and so $\hat{\boldsymbol{\gamma}}_{i,j}$ will be bounded below by $\boldsymbol{\gamma} := \hat{c}_1\hat{c}_2 d_H^2/8 > 0$.

It remains to verify that pairs of boxes that do not make 'good couples' are 'rare', in the sense that they satisfy (6.3). For Anosov systems [C3, Section 16] *all* pairs of boxes make good couples. But the billiard dynamics is far less regular and boxes may fail to make 'good couples' for several reasons. First, some fibers $\mathcal{W}_\gamma^s$ may not belong to the set $\mathbf{W}_{n,-s}$, cf. (4.1), then there may not be enough 'stable' chains $\tilde{X}_k^1$ around to ensure(6.6). Second, recall that our chains $\tilde{X}_i^{\zeta_1}$, $\tilde{X}_j^{\zeta_2}$, and $\tilde{X}_k^1$ are Cantor-like 'holey' structures. If the 'holes' are too wide, then $\pi_\Omega(\tilde{X}_k^1)$ may not intersect either $\pi_\Omega(\tilde{X}_i^{\zeta_1})$ or $\pi_\Omega(\tilde{X}_j^{\zeta_2})$, in which case either $\Gamma_{i,k}^{\zeta_1} = 0$ or $\Gamma_{j,k}^{\zeta_2} = 0$ (or both).

To assess losses caused by 'holes' in the chains and 'bad' fibers $\mathcal{W}_\gamma^s$ we can use the overall exponential bound on 'bad' atoms ('nonboxes') in $\hat{\mathcal{M}}$, cf. (5.15), the bound on the relative measure of the union of 'holes' in every box, cf. (5.3), and the bound on the measure of the union of 'bad' stable fibers $\mathcal{W}_\gamma^s \notin \mathbf{W}_{n,-s}$, cf. (4.1). Loosely speaking, all these bad phenomena occur with exponentially small probability. Thus for 'typical' pairs of boxes X_i, X_j the losses are exponentially small and we get our lower bound $\hat{\boldsymbol{\gamma}}_{i,j} \geq \boldsymbol{\gamma}$; the total measure of 'nontypical' pairs X_i, X_j satisfies an overall exponential bound (6.3) with some $\theta_{15} > 0$. The verification amounts to a straightforward (but tedious) estimation, which we leave out.

Thus all the conditions of Theorem 6.3 are met, hence Theorem 1.1 is proved.

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Appendix

Here we prove Proposition 2.4. Our argument is based on the mixing property of the flow Φ^t; it closely follows that of [BSC2, Theorems 3.12 and 3.13], see a more detailed presentation in [CM, Chapters 5 and 7], so we will only sketch it here.

First, there is a $d_0 > 0$ and are a finite number of boxes of positive measure, $X_1, \dots, X_K \subset \mathcal{M}$ such that any fiber $\mathcal{W}^u$ of length $\geq d_0$ fully crosses the solid box $\mathcal{Q}(X_j)$ (i.e. crosses both s-faces of it) for some $j = 1, \dots, K$; in fact we can guarantee that $\mathcal{W}^u$ crosses $\mathcal{Q}(X_j)$ somewhere in its middle half (with respect to the μ measure). The boxes X_j are build over certain rectangles R_j constructed in [BSC2], see also [CM, Lemma 7.87], which we can assume to have high density, say

$$\rho^s(R_j) > 0.99 \tag{6.7}$$

for every j. We also assume that the height of every solid box $\mathcal{Q}(X_j)$ is less than $s_H/200$.

Now consider the set $K_1 = \pi_\Omega(V_1) \subset \Omega$. Fix a subset $\tilde{K}_1 \subset K_1$ such that $\nu(\tilde{K}_1) > 0$ and there is a $\delta_0 > 0$ such that for any stable fiber W^s of the billiard map T with length $< \delta_0$ and the s-SRB measure ν^s on it we have $\nu^s(W^s \cap K_1) > 0.99$ whenever $W^s \cap \tilde{K}_1 \neq \emptyset$ (the existence of $\tilde{K}_1$ is proved in [BSC2, page 68]). Then fix a subset $\tilde{V}_1 \subset V_1$ such that $\pi_\Omega(\tilde{V}_1) = \tilde{K}_1$ and on any surface $\Phi^{[s_\alpha - s_H, s_\alpha]}(\mathcal{W}^u_\alpha \cap B_1)$, see (H4), the set $\tilde{V}_1$ intersects only the bottom 1% of it, i.e. the subsurface $\Phi^{[s_\alpha - s_H, s_\alpha - 0.99 s_H]}(\mathcal{W}^u_\alpha \cap B_1)$.

The mixing property of Φ^t ensures that there are $t_H > 0$ and $\tilde{d}_H > 0$ such that for all $t > t_H$ we have $\mu(\Phi^t X_j \cap \tilde{V}_1) > \tilde{d}_H$ for every $j = 1, \dots, K$. An obvious lifting of the arguments in the proof of [BSC2, Theorem 3.13] from Ω to $\mathcal{M}$ shows that there are components $\mathcal{W}^u_{t,i}$ of $\Phi^t \mathcal{W}^u$ such that

(i) $\operatorname{dist}\big(W^u_{t,i}, \Phi^{[-s_H/100, s_H/100]} \tilde{V}_1\big) \le C_\Phi \lambda^t_\Phi$;

(ii) the curve $W^u_{t,i}$ intersects the ball B_1 and sticks out of it by at least L_H in both directions;

(iii) the ν^u_t-measure of the union of those components is greater than some $d_H > 0$.

In fact the property (ii) follows from (6.7) and our construction of $\tilde{V}_1$. Now, Proposition 2.4 readily follows from (H4).

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