Sinai billiards under small external forces II

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Abstract

We study perturbations of Sinai billiards, where a small stationary force acts on the moving particle between its collisions with scatterers. In the previous work [7] we proved that the collision map preserved a unique Sinai-Ruelle-Bowen (SRB) measure that was Bernoulli and had exponential decay of correlations. Here we add several other statistical properties, including bounds on multiple correlations, the almost sure invariance principle (ASIP), the law of iterated logarithms, and a Kawasaki-type formula. We also show that the corresponding flow is Bernoulli and satisfies a central limit theorem.

Keywords: Sinai billiards, SRB measure, central limit theorem, Bernoulli, Kawasaki formula.

1 Introduction

This a continuation of our paper [7], and we use the symbols and notation of the latter for compatibility.

Let $\mathcal{B}_1, \dots, \mathcal{B}_s$ be open convex domains on the unit 2D torus $\mathbb{T}^2$. Assume that $\bar{\mathcal{B}}_i \cap \bar{\mathcal{B}}_j = \emptyset$ for $i \neq j$, and for each i the boundary $\partial\mathcal{B}_i$ is a C^3 smooth closed curve with nonvanishing curvature.

Let a particle of unit mass move in $\mathcal{D} = \mathbb{T}^2 \setminus \cup_i \mathcal{B}_i$ according to equations

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = \mathbf{F} \tag{1.1}$$

where $\mathbf{q} = (x, y)$ is the position vector, $\mathbf{p} = (u, v)$ is the momentum (velocity) vector, and $\mathbf{F}(x, y, u, v) = (F_1, F_2)$ is a stationary force (independent of

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time). Upon reaching the boundary $\partial\mathcal{D} = \cup_i \partial\mathcal{B}_i$, the particle gets reflected elastically, according to the classical rule

$$\mathbf{p}^+ = \mathbf{p}^- - 2\,(\mathbf{n}(\mathbf{q})\cdot\mathbf{p}^-)\,\mathbf{n}(\mathbf{q}); \tag{1.2}$$

here $\mathbf{q} \in \partial\mathcal{D}$ is the point of reflection, $\mathbf{n}(\mathbf{q})$ is the inward unit normal vector to $\partial\mathcal{D}$, and $\mathbf{p}^-$, $\mathbf{p}^+$ are the incoming and outgoing velocity vectors, respectively.

The case $\mathbf{F} = 0$ corresponds to the ordinary billiard dynamics on the table $\mathcal{D}$. It preserves the kinetic energy, so that one can fix it by setting $\|\mathbf{p}\| = 1$. Then the phase space of the system is a compact 3D manifold $\Omega_0 = \mathcal{D} \times S^1$. The billiard flow Φ_0^t on Ω_0 preserves the Liouville measure μ_0 (which is uniform on Ω_0).

In the study of billiards, one uses the collision space

$$\mathcal{M}_0 = \{(\mathbf{q},\mathbf{p}) \in \Omega_0 \colon \mathbf{q} \in \partial\mathcal{D},\ (\mathbf{p}\cdot\mathbf{n}(\mathbf{q})) \geq 0\}, \tag{1.3}$$

which consists of all outgoing velocity vectors at reflection points. The first return map $\mathcal{F}_0 \colon \mathcal{M}_0 \to \mathcal{M}_0$ is called the billiard map (or the collision map). The space $\mathcal{M}_0$ can be parameterized by (r,φ), where r is the arclength along $\partial\mathcal{D}$ and $\varphi \in [-\pi/2, \pi/2]$ is the angle between $\mathbf{p}$ and $\mathbf{n}(\mathbf{q})$. In these coordinates, $\mathcal{M}_0 = \partial\mathcal{D} \times [-\pi/2, \pi/2]$. The map $\mathcal{F}_0$ preserves a finite smooth measure on $\mathcal{M}_0$ with density $d\nu_0 = \text{const}\cdot\cos\varphi\, dr\, d\varphi$. The flow Φ_0^t is a suspension flow over the base map $\mathcal{F}_0$ under the ceiling function $\tau_0(X) = \min\{t > 0 \colon \Phi_0^t X \in \mathcal{M}\}$ (the next collision time).

Billiards on tables $\mathcal{D} = \mathbb{T}^2 \setminus \cup_i \mathcal{B}_i$ as described above are known as dispersing billiards or Sinai billiards. The map $\mathcal{F}_0$ is ergodic, mixing [28], Bernoulli [15], and has other strong statistical properties, such as exponential decay of correlations [32, 6] and the central limit theorem [3]. The billiard flow Φ_0^t is also ergodic, mixing [28], and Bernoulli [15]. Under an additional assumption of finite horizon (see below) the flow Φ_0^t enjoys stretched exponential decay of correlations [9] and satisfies the central limit theorem [3].

Various perturbations of Sinai billiards have been studied in [1, 18, 19, 20, 21, 29, 30], see also a survey [22]. Most notably, when $\mathbf{F}$ is a small constant force with a Gaussian thermostat (see below), then one can rigorously prove a one-particle version of classical Ohm's law and the Einstein relation [4, 5]. More recently, perturbed Sinai billiards were used in the analysis of the Galton board [12] and self-similar Lorentz channels [2, 11].

A general class of Sinai billiards with small external forces $\mathbf{F} \neq 0$ was studied in [7] under the following assumptions:

Assumption A (additional integral). A smooth function $\mathcal{E}(\mathbf{q},\mathbf{p})$ is preserved by the dynamics (1.1)–(1.2). Its level surface, $\Omega = \{\mathcal{E}(\mathbf{q},\mathbf{p}) = \text{const}\}$ is a compact 3-D manifold such that $\|\mathbf{p}\| \neq 0$ on Ω and for each $\mathbf{q} \in \mathcal{D}$ and $\mathbf{p} \in S^1$ the ray $\{(\mathbf{q}, s\mathbf{p}),\ s > 0\}$ intersects the manifold Ω in one point.

Under Assumption A, Ω can be parameterized by (x, y, θ), where $(x, y) = \mathbf{q} \in \mathcal{D}$ and $0 \leq \theta < 2\pi$ is a cyclic coordinate, the angle between $\mathbf{p}$ and the positive x axis. The dynamics (1.1)–(1.2) restricted to Ω is a flow that we denote by Φ^t. In the coordinates (x, y, θ) the equations of motion (1.1) can be rewritten as

$$\dot{x} = p\cos\theta, \qquad \dot{y} = p\sin\theta, \qquad \dot{\theta} = ph, \tag{1.4}$$

where

$$p = \|\mathbf{p}\| > 0 \qquad \text{and} \qquad h = (-F_1 \sin\theta + F_2\cos\theta)/p^2.$$

It is also useful to note that

$$\dot{p} = F_1\cos\theta + F_2\sin\theta. \tag{1.5}$$

Both $h = h(x, y, \theta)$ and $p = p(x, y, \theta)$ are assumed to be C^2 smooth functions on Ω, and note that

$$0 < p_{\min} \leq p \leq p_{\max} < \infty. \tag{1.6}$$

There are two particularly interesting types of forces satisfying Assumption A. One is a potential force $\mathbf{F} = -\nabla U$, where $U = U(\mathbf{q})$ is a potential function; it preserves the total energy $T = \frac{1}{2}\|\mathbf{p}\|^2 + U(\mathbf{q})$. The other type is isokinetic forces satisfying $(\mathbf{F}\cdot\mathbf{p}) = 0$, they preserve the kinetic energy $K = \frac{1}{2}\|\mathbf{p}\|^2$, so one can set $\|p\| = 1$ as in billiards. For example, given any force $\mathbf{F}$, one can construct an isokinetic force by adding Gaussian thermostat:

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = \mathbf{F} - \alpha\mathbf{p} \quad \text{where} \quad \alpha = (\mathbf{F}\cdot\mathbf{p})/(\mathbf{p}\cdot\mathbf{p}). \tag{1.7}$$

For a function f on Ω, let f_x, f_y, f_θ denote its partial derivatives and $\|f\|_{C^2}$ the maximum of f and its first and second partial derivatives over Ω. Put

$$B_0 = \max\{p_{\min}^{-1}, \|p\|_{C^2}, \|h\|_{C^2}\}. \tag{1.8}$$

Assumption B (smallness of the force). We assume that the force $\mathbf{F}$ and its first derivatives are small, i.e.

$$\max\{|h|, |h_x|, |h_y|, |h_\theta|\} \leq \delta_0$$

More precisely, we require that for any given $B_* > 0$ there should be a small $\delta_* = \delta_*(\mathcal{D}, B_*)$ such that all our results will hold whenever $B_0 < B_*$ and $\delta_0 < \delta_*$.

We note that the smallness of δ_* is required in [7] at several crucial steps, some of those requirements are more severe than others. Physical implications of those requirements are discussed in Remark on p. 232 in [7].

Assumption C (finite horizon). There is an $L > 0$ so that every straight line of length L on the torus $\mathbb{T}^2$ crosses at least one obstacle $\mathcal{B}_i$.

Now we introduce the collision space of the system (1.1)–(1.2):

$$\mathcal{M} = \{(\mathbf{q}, \mathbf{p}) \in \Omega \colon \mathbf{q} \in \partial\mathcal{D},\ (\mathbf{p} \cdot \mathbf{n}(\mathbf{q})) \geq 0\} \tag{1.9}$$

and the corresponding collision map $\mathcal{F} \colon \mathcal{M} \to \mathcal{M}$. The space $\mathcal{M}$ can be parameterized by (r, φ) as before, in these coordinates $\mathcal{M}$ is identical to $\mathcal{M}_0$ in (1.3).

Theorem 1.1 ([7])**.** *Under Assumptions A, B, and C, the map $\mathcal{F} \colon \mathcal{M} \to \mathcal{M}$ is a smooth hyperbolic map with singularities that has uniform expansion and contraction rates. It admits a unique SRB measure ν, which is positive on open sets, K-mixing and Bernoulli. It enjoys exponential decay of correlations (with bounds uniform in the force $\mathbf{F}$) and satisfies the central limit theorem.*

We remark that in unperturbed Sinai billiards the hyperbolicity results from a rather obvious geometric fact that divergent families of trajectories remain divergent (under the action of the flow) and grow exponentially in size at time goes on. In our perturbed billiards, the same property holds for the so-called *strongly divergent families* of flow lines (whose orthogonal cross-section has curvature bounded below by a positive constant), see precise definition in [7, p. 208]. Similarly, strongly convergent families are defined (those remain convergent and expand in the time-reversal flow).

2 Advanced statistical properties of $\mathcal{F}$

Here we derive further statistical properties of the collision map $\mathcal{F}$ by employing the coupling method recently introduced by L.-S. Young [33] and modified by D. Dolgopyat [10, Appendix A]. It is based on iterations of probability measures supported on unstable curves.

First we recall a few definitions and facts following [7]. An unstable (or stable) curve $\gamma \subset \mathcal{M}$ is a trace of a strongly divergent (resp., convergent) family of flow lines [7, p. 213]. We may assume [7, p. 215] that the curvature of unstable and stable curves is uniformly bounded. To ensure distortion control we cut $\mathcal{M}$ into countably many homogeneity strips [7, p. 216] that accumulate near $\partial\mathcal{M}$, thus $\mathcal{F}$ becomes discontinuous on the preimages of the boundaries of those strips. A curve is homogeneous if it lies in a single homogeneity strip (i.e. in one connected component of $\mathcal{M}$).

For any $X \in \gamma$ denote by $\mathcal{J}_\gamma \mathcal{F}^n(X)$ the Jacobian of the map $\mathcal{F}^n$ restricted to γ at X. If $\mathcal{F}^i(\gamma)$ is a homogeneous unstable curve for all $0 \le i \le n$, then we have the following *distortion bound*, see [7, Lemma 4.2]:

$$|\ln \mathcal{J}_\gamma \mathcal{F}^n(X) - \ln \mathcal{J}_\gamma \mathcal{F}^n(Y)| \le C|\mathcal{F}^n(\gamma)|^{1/3}, \quad X, Y \in \gamma \tag{2.1}$$

where $|\gamma|$ denotes the length of γ, and by C we will denote various positive constants independent of the force $\mathbf{F}$. Accordingly, if γ^u is a homogeneous unstable manifold (called h-fiber in [7]) and ρ_{γ^u} is the u-SRB density on γ^u, i.e. the unique probability density satisfying

$$\frac{\rho_{\gamma^u}(X)}{\rho_{\gamma^u}(Y)} = \lim_{n\to\infty} \frac{\mathcal{J}_{\gamma^u}\mathcal{F}^{-n}(X)}{\mathcal{J}_{\gamma^u}\mathcal{F}^{-n}(Y)}, \qquad X, Y \in \gamma^u, \tag{2.2}$$

then (2.1) implies $\left|\frac{d}{dX} \ln \rho_{\gamma^u}(X)\right| \le C|\gamma^u|^{-2/3}$; see, e.g. [8, Section 5.6][1].

If γ_1, γ_2 are unstable curves and ξ a stable h-fiber crossing each γ_i in a point X_i, then the Jacobian of the holonomy map $\mathbf{h}\colon \gamma_1 \to \gamma_2$ at X_1 satisfies

$$e^{-C(\beta+\delta^{1/3})} \le \mathcal{J}\mathbf{h}(X_1) \le e^{-C(\beta+\delta^{1/3})} \tag{2.3}$$

where $\delta = |\xi(X_1, X_2)|$ is the length of the segment of ξ between X_1 and X_2, and β is the angle between the tangent vectors to γ_1 and γ_2 at X_1 and X_2, respectively. A little cruder estimate was proved in [7, Lemma 4.3], but a close examination of the proof shows that it in fact implies (2.3). Alternatively, one can prove (2.3) directly, as in [8, Theorem 5.42].

Given $X, Y \in \mathcal{M}$, denote by $\mathbf{s}_+(X, Y) \ge 0$ the future separation time (the first time when the images $\mathcal{F}^n(X)$ and $\mathcal{F}^n(Y)$ for $n \ge 0$ lie in different connected components of the collision space $\mathcal{M}$), and similarly let

[1] The book [8] is devoted to classical (unperturbed) billiards with smooth invariant measures, but many technical facts proven there hold in our case as well.

$\mathbf{s}_-(X,Y) \geq 0$ denote the past separation time (this definition takes into account that $\mathcal{M}$ is cut along the boundaries of the homogeneity strips, cf. [7, p. 223]). Observe that if X and Y lie on one unstable curve $\gamma \subset \mathcal{M}$, then $|\gamma(X,Y)| \leq C\Lambda^{-\mathbf{s}_+(X,Y)}$, where $\Lambda > 1$ is the hyperbolicity constant for $\mathcal{F}$, cf. [8, Eq. (5.32)]. Now (2.3) implies (see, e.g. [8, Proposition 5.48]) that for any $X, Y \in \gamma_1$

$$|\ln \mathcal{J}\mathbf{h}(X) - \ln \mathcal{J}\mathbf{h}(Y)| \leq C\vartheta^{\mathbf{s}_+(X,Y)}, \tag{2.4}$$

where $\vartheta = \Lambda^{-1/6} < 1$. Following Young [32, p. 597], we call the property (2.4) the 'dynamically defined Hölder continuity' of $\mathcal{J}\mathbf{h}$.

Next we define a class of probability measures supported on unstable curves, following [10, 8]. A *standard pair* $\ell = (\gamma, \nu)$ is a homogeneous unstable curve $\gamma \subset \mathcal{M}$ with a probability measure $\mathbb{P}_\ell$ on it, whose density ρ (with respect to the Lebesgue measure on γ) satisfies

$$|\ln \rho(X) - \ln \rho(Y)| \leq C_{\mathrm{r}}\, \vartheta^{\mathbf{s}_+(X,Y)}. \tag{2.5}$$

Here $C_{\mathrm{r}} > 0$ is a sufficiently large constant (independent of $\mathbf{F}$). For any standard pair $\ell = (\gamma, \rho)$ and $n \geq 1$ the image $\mathcal{F}^n(\gamma)$ is a finite or countable union of homogeneous unstable curves (h-components) on which the density of the measure $\mathcal{F}^n(\mathbb{P}_\ell)$ satisfies (2.5); hence the image of a standard pair under $\mathcal{F}^n$ is a family of standard pairs (with a factor measure).

More generally, a *standard family* is an arbitrary (countable or uncountable) collection $\mathcal{G} = \{\ell_\alpha\} = \{(\gamma_\alpha, \rho_\alpha)\}$, $\alpha \in \mathfrak{A}$, of standard pairs with a probability factor measure $\lambda_\mathcal{G}$ on the index set $\mathfrak{A}$. Such a family induces a probability measure $\mathbb{P}_\mathcal{G}$ on the union $\cup_\alpha \gamma_\alpha$ (and thus on $\mathcal{M}$) defined by

$$\mathbb{P}_\mathcal{G}(B) = \int \mathbb{P}_\alpha(B \cap \gamma_\alpha)\, d\lambda_\mathcal{G}(\alpha) \qquad \forall B \subset \mathcal{M}.$$

Any standard family $\mathcal{G}$ is mapped by $\mathcal{F}^n$ into another standard family $\mathcal{G}_n = \mathcal{F}^n(\mathcal{G})$, and $\mathbb{P}_{\mathcal{G}_n} = \mathcal{F}^n(\mathbb{P}_\mathcal{G})$.

For every $\alpha \in \mathfrak{A}$, any point $X \in \gamma_\alpha$ divides the curve γ_α into two pieces, and we denote by $r_\mathcal{G}(X)$ the length of the shorter one. Now the quantity $\mathcal{Z}_\mathcal{G} = \sup_{\varepsilon>0} \varepsilon^{-1}\mathbb{P}_\mathcal{G}(r_\mathcal{G} < \varepsilon)$ reflects the 'average' size of curves γ_α in $\mathcal{G}$; we only consider standard families with $\mathcal{Z}_\mathcal{G} < \infty$. The growth lemma [7, Proposition 5.3] implies that $\mathcal{Z}_{\mathcal{G}_n} \leq C(\theta^n \mathcal{Z}_\mathcal{G} + 1)$ for all $n \geq 0$ and some constant $\theta \in (0,1)$, see a proof in [8, Proposition 7.17]; this estimate effectively asserts that standard families grow under $\mathcal{F}^n$ exponentially fast.

A standard pair (γ, ρ) is *proper* if $|\gamma| \geq \delta_{\rm p}$, where $\delta_{\rm p} > 0$ is a small but fixed constant. A standard family $\mathcal{G}$ is *proper* if $\mathcal{Z}_{\mathcal{G}} \leq C_{\rm p}$, where $C_{\rm p}$ is a large but fixed constant (chosen so that a family consisting of a single proper standard pair is proper, as a family). The image of a proper standard family under $\mathcal{F}^n$ is proper for every $n \geq 1$.

A smooth foliation of $\mathcal{M}$ by (long enough) unstable curves gives us a proper standard family $\mathcal{G}$ such that $\mathbb{P}_{\mathcal{G}} = \nu_0$, the billiard invariant measure, see [8, p. 172]. Also, there is a special standard family $\mathcal{E}$ consisting of (maximal) unstable h-fibers γ^u for the map $\mathcal{F}$ with the SRB densities ρ_{γ^u} on them and the factor measure generated by ν; in that case $\mathbb{P}_{\mathcal{E}} = \nu$, the family $\mathcal{E}$ is proper (due to [7, Proposition 5.6]) and obviously $\mathcal{F}$-invariant.

Next we present the key tool of the Young-Dolgopyat approach – the coupling lemma (for a detailed account see [8, Appendix A] and [8, Section 7.5]). Given a standard pair $\ell = (\gamma, \rho)$, we consider a 'rectangle' $\hat{\gamma} = \gamma \times [0, 1]$ and equip it with a probability measure $\hat{\mathbb{P}}_\ell$ with density

$$\hat{\rho}(X, t) = \rho(X)\, dX\, dt; \tag{2.6}$$

the map $\mathcal{F}^n$ can be naturally defined on $\hat{\gamma}$. Given a standard family $\mathcal{G} = (\gamma_\alpha, \rho_\alpha)$ with a factor measure $\lambda_{\mathcal{G}}$, we denote by $\hat{\mathcal{G}} = (\hat{\gamma}_\alpha, \hat{\rho}_\alpha)$ the family of the corresponding rectangles and equip it with the same factor measure $\lambda_{\mathcal{G}}$; we denote by $\hat{\mathbb{P}}_{\mathcal{G}}$ the induced measure on the union $\cup_\alpha \hat{\gamma}_\alpha$.

Lemma 2.1 (Coupling Lemma)**.** *Let $\mathcal{G} = (\gamma_\alpha, \rho_\alpha)$, $\alpha \in \mathfrak{A}$, and $\mathcal{F} = (\gamma_\beta, \rho_\beta)$, $\beta \in \mathfrak{B}$, be two proper standard families. Then there exist a bijection (called coupling map) $\Theta\colon \cup_\alpha \hat{\gamma}_\alpha \to \cup_\beta \hat{\gamma}_\beta$ that preserves measure; i.e. $\Theta(\hat{\mathbb{P}}_{\mathcal{G}}) = \hat{\mathbb{P}}_{\mathcal{E}}$, and a (coupling time) function $\Upsilon : \cup_\alpha \hat{\gamma}_\alpha \to \mathbb{N}$ such that*

A. *Let $(X, t) \in \hat{\gamma}_\alpha$, $\alpha \in \mathfrak{A}$, and $\Theta(X, t) = (Y, s) \in \hat{\gamma}_\beta$, $\beta \in \mathfrak{B}$. Denote $m = \Upsilon(X, t) \in \mathbb{N}$. Then the points $\mathcal{F}^m(X)$ and $\mathcal{F}^m(Y)$ lie on the same stable h-fiber in $\mathcal{M}$.*

B. *There is a uniform exponential tail bound on the function Υ:*

$$\hat{\mathbb{P}}_{\mathcal{G}_1}\big(\Upsilon > n\big) \leq C_\Upsilon \vartheta_\Upsilon^n, \tag{2.7}$$

for some constants $C_\Upsilon > 0$ and $\vartheta_\Upsilon < 1$ (independent of $\mathbf{F}$, in the sense of Assumption B).

A detailed (and lengthy) proof is given in [8, Chapter 7] for unperturbed Sinai billiards. It applies to our case with one little modification. While for unperturbed billiards the construction of the so called 'magnet' rectangle, see [8, Proposition 7.83], is relatively simple as it deals with one (billiard) map, in our case it requires a more elaborate argument, as we deal with a class of maps and need uniformity in $\mathbf{F}$. In fact, an analogue of the 'magnet rectangle' (called rhombus) is constructed in [7, Lemma 6.5] and its necessary properties are proved in [7, Corollary 6.8].

The coupling lemma has many remarkable implications, some of them we state next. Motivated by (2.4), we say that a function $f\colon \mathcal{M} \to \mathbb{R}$ is *dynamically Hölder continuous* if there are $\vartheta_f \in (0,1)$ and $K_f > 0$ such that for any X and Y lying on one unstable curve

$$|f(X) - f(Y)| \leq K_f \vartheta_f^{\mathbf{s}_+(X,Y)} \tag{2.8}$$

and for any X and Y lying on one stable curve

$$|f(X) - f(Y)| \leq K_f \vartheta_f^{\mathbf{s}_-(X,Y)}. \tag{2.9}$$

We denote the space of such functions by $\mathcal{H}$. It contains every piecewise Hölder continuous function whose discontinuities coincide with those of $\mathcal{F}^{\pm m}$ for some $m > 0$. For example, the return time function $\tau(X) = \min\{t > 0\colon \Phi^t(X) \in \mathcal{M}\}$ belongs in $\mathcal{H}$.

Proposition 2.2 (Equidistribution)**.** *Let $\mathcal{G}$ be a proper standard family. For any dynamically Hölder continuous function $f \in \mathcal{H}$ and $n \geq 0$*

$$\left| \int_{\mathcal{M}} f \circ \mathcal{F}^n \, d\mathbb{P}_{\mathcal{G}} - \int_{\mathcal{M}} f \, d\nu \right| \leq B_f \theta_f^n \tag{2.10}$$

where $B_f = 2C_\Upsilon\big(K_f + \|f\|_\infty\big)$ and $\theta_f = \big[\max\{\vartheta_\Upsilon, \vartheta_f\}\big]^{1/2} < 1$.

In other words, iterations of measures on standard pairs converge to the SRB measure exponentially fast. For the proof, see [8, Theorem 7.31].

To estimate multiple correlations, let $f_0, f_1, \dots, f_r \in \mathcal{H}$ and $g_0, g_1, \dots, g_k \in \mathcal{H}$ be such that f's have identical parameters $\vartheta_f = \vartheta_{f_i}$, $K_f = K_{f_i}$, and $\|f\|_\infty = \|f_i\|_\infty$ for all $0 \leq i \leq r$, and g's have identical parameters $\vartheta_g = \vartheta_{g_i}$, $K_g = K_{g_i}$, and $\|g\|_\infty = \|g_i\|_\infty$ for all $0 \leq i \leq k$. Consider two products

$$\tilde{f} = f_0 \cdot (f_1 \circ \mathcal{F}^{i_{-1}}) \cdot (f_2 \circ \mathcal{F}^{i_{-2}}) \cdots (f_r \circ \mathcal{F}^{i_{-r}})$$

for some $0 > i_{-1} > \cdots > i_{-r}$ and

$$\tilde{g} = g_0 \cdot (g_1 \circ \mathcal{F}^{i_1}) \cdot (g_2 \circ \mathcal{F}^{i_2}) \cdots (g_k \circ \mathcal{F}^{i_k})$$

for some $0 < i_1 < \cdots < i_k$. We use a short-hand notation $\nu(f) = \int_{\mathcal{M}} f \, d\nu$.

Theorem 2.3 (Exponential bound on multiple correlations)**.** *For all* $n > 0$

$$\left|\nu\big(\tilde{f} \cdot (\tilde{g} \circ \mathcal{F}^n)\big) - \nu(\tilde{f})\nu(\tilde{g})\right| \le B_{\tilde{f},\tilde{g}}\, \theta_{f,g}^n \tag{2.11}$$

where

$$\theta_{f,g} = \left[\max\left\{\vartheta_\Upsilon, \vartheta_f, \vartheta_g, \alpha_1\right\}\right]^{1/4} < 1,$$

$\alpha_1 < 1$ *is a constant from [7, Corollary 5.4], and*

$$B_{\tilde{f},\tilde{g}} = C\|f\|_\infty^r \|g\|_\infty^k \left[\frac{K_f\|g\|_\infty}{1-\vartheta_f} + \frac{K_g\|f\|_\infty}{1-\vartheta_g} + \|f\|_\infty\|g\|_\infty\right].$$

For the proof, see [8, Theorem 7.41]. We remark that the theorem remains valid if f_i's only satisfy (2.8) and g_i's only satisfy (2.9). Not only the exponential bound (2.11) is novel and important itself, but the exact formulas for $\theta_{f,g}$ and $B_{\tilde{f},\tilde{g}}$ are essential in the proof of the central limit theorem and the almost sure invariance principle (ASIP), see [8, Sections 7.8–7.9]:

Theorem 2.4 (Almost Sure Invariance Principle)**.** *Let* $f \in \mathcal{H}$ *such that*

$$\nu(f) = 0, \qquad \sigma_f^2 = \sum_{n=-\infty}^{\infty} \nu\big(f \cdot (f \circ \mathcal{F}^n)\big) \neq 0 \tag{2.12}$$

Denote $S_n = f + f \circ \mathcal{F} + \cdots + f \circ \mathcal{F}^{n-1}$ *and define a continuous function* $W_N(s; X)$ *of* $s \in [0, 1]$ *by*

$$W_N\Big(\frac{n}{N}; X\Big) = \frac{S_n(X)}{\sigma_f \sqrt{N}}$$

at rational points $s = n/N$ *and by linear interpolation in between. Then there is a standard Wiener process (a Brownian motion)* $\mathbb{B}(s; X)$ *on* $\mathcal{M}$ *with respect to the measure* ν *so that for some* $\lambda > 0$

$$|W_N(s; X) - \mathbb{B}(s; X)| = \mathcal{O}(N^{-\lambda}) \tag{2.13}$$

for ν*-almost all* $X \in \mathcal{M}$*.*

Corollary 2.5 (Law of Iterated Logarithm)**.** *For* ν*-a.e. point* $X \in \mathcal{M}$

$$\limsup_{n\to\infty} S_n / \sqrt{2n\sigma_f^2 \log\log n} = 1.$$

For proofs, see [8, Sections 7.9]. We emphasize that all our constants are independent of $\mathbf{F}$ and the convergence is always uniform in $\mathbf{F}$.

We should note that our bound on the error term in (2.13) is not optimal; better bounds, e.g. $\mathcal{O}(N^{-1/4+\varepsilon})$, can be obtained by using methods of recent works [14, 26]. A more general (vector) version of the ASIP can be derived based on the results of the manuscript [25] (not yet published).

Since $\tau(X)$ is dynamically Hölder continuous, it satisfies the central limit theorem, i.e. $(t_n - n\nu(\tau))/\sqrt{n}$ converges to a normal distribution $\mathcal{N}(0, \sigma_\tau^2)$; here $t_n = \tau + \tau \circ \mathcal{F} + \cdots + \tau \circ \mathcal{F}^{n-1}$ is the time of the nth collision. Actually, $\sigma_\tau > 0$ (this follows from the mixing property of the flow proved in the next section, as explained in [8, Remark 7.63]). Also, let $n_X(T)$ denote the number of collisions on the trajectory $\Phi^t(X)$, $0 < t < T$. Then $T/n_X(T) \to \nu(\tau)$ for a.e. $X \in \mathcal{M}$ and $(n_X(T) - T/\nu(\tau))/\sqrt{T}$ converges to a normal distribution $\mathcal{N}(0, \sigma^2)$ with $\sigma^2 = \sigma_\tau^2/[\nu(\tau)]^3$; for a proof, see [8, Sections 7.10].

Next we derive a formula specific to the case $\mathbf{F} \neq 0$ (motivated by Kawasaki formulas in nonlinear response theory [31]). For any $f \in \mathcal{H}$

$$\begin{aligned}
\nu(f) &= \lim_{n\to\infty} \nu_0(f \circ \mathcal{F}^k) \\
&= \nu_0(f) + \lim_{n\to\infty} \sum_{k=1}^{n} \nu_0\big[(f \circ \mathcal{F}^k) - (f \circ \mathcal{F}^{k-1})\big] \\
&= \nu_0(f) + \lim_{n\to\infty} \sum_{k=1}^{n} \nu_0\big[(f \circ \mathcal{F}^k)(1 - g)\big],
\end{aligned} \tag{2.14}$$

where $g = d\mathcal{F}^{-1}\nu_0/d\nu_0$ is the Jacobian of the map $\mathcal{F}$ with respect to the billiard invariant measure ν_0. A direct calculation gives

$$g(X) = \frac{p(X)}{p(\mathcal{F}(X))} \exp\left[\int_0^{\tau(X)} \operatorname{div}\Gamma(X)\, dt\right]$$

where $\Gamma = \langle p\cos\theta, p\sin\theta, ph\rangle$ is the vector field in Ω generating the flow Φ^t, cf. (1.4). Since

$$\operatorname{div}\Gamma = p_x \cos\theta + p_y \sin\theta + p_\theta h + ph_\theta = \tfrac{d\ln p}{dt} + ph_\theta,$$

we have $g(X) = \exp\big[\int_0^{\tau(X)} ph_\theta\, dt\big]$. Observe that $g = 1 + \mathcal{O}(\delta_0)$, where δ_0 is a small constant in Assumption B. Also note that $\nu_0(g) = 1$ and $g(X)$ is a piecewise smooth function whose discontinuities coincide with those of the map $\mathcal{F}$. Furthermore, $\ln g$ is dynamically Hölder continuous with $\vartheta_{\ln g} = \Lambda^{-1/2}$ and $K_{\ln g} = C\|p\|_{C^2}\|h\|_{C^2}$ being independent of $\mathbf{F}$.

We now use the special standard family $\mathcal{E}$ consisting of maximal unstable h-fibers $\gamma^u \subset \mathcal{M}$ with SRB densities ρ_{γ^u}. The probability measure $d\tilde{\nu} = g\,d\nu_0$ on $\mathcal{M}$ induces a conditional density $\tilde{\rho}_{\gamma^u}$ on each γ^u, which is proportional to $g\rho_{\gamma^u}$, hence its logarithm is dynamically Hölder continuous:

$$|\ln \tilde{\rho}_{\gamma^u}(X) - \ln \tilde{\rho}_{\gamma^u}(Y)| \leq (C_{\rm r} + K_{\ln g})\,\vartheta^{\mathbf{s}_+(X,Y)}.$$

for $X, Y \in \gamma^u$. Of course, the density $\tilde{\rho}_{\gamma^u}$ may not satisfy (2.5), but its images under $\mathcal{F}^m$ will smooth out (due to distortion bounds [8, p. 203]) and then satisfy (2.5) for all $m \geq m_0$, where $m_0 = m_0(\|p\|_{C^2}, \|h\|_{C^2})$.

Thus, $\mathcal{F}^{m_0}(\tilde{\nu})$ will coincide with $\mathbb{P}_{\mathcal{G}}$ for some proper standard family $\mathcal{G}$. Now Proposition 2.2 implies that both $\nu_0(f \circ \mathcal{F}^k)$ and $\nu_0\big(g \cdot (f \circ \mathcal{F}^k)\big) = \tilde{\nu}(f \circ \mathcal{F}^k)$ converge to $\nu(f)$ exponentially fast (and uniformly in $\mathbf{F}$), thus the sum in (2.14) is bounded by a geometric series. This yields the desired Kawasaki formula:

$$\nu(f) = \nu_0(f) + \sum_{k=1}^{\infty} \nu_0\big[(f \circ \mathcal{F}^k)(1-g)\big], \tag{2.15}$$

where the series converges exponentially fast and uniformly in $\mathbf{F}$.

3 Bernoulli property of the flow Φ^t

In this section we study the flow $\Phi^t\colon \Omega \to \Omega$. It is shown in [7] that Φ^t is a hyperbolic flow with uniform expansion and contraction rates. Its weakly unstable (and stable) manifolds are 2D surfaces in Ω that are made by families of strongly divergent (resp., convergent) flow lines. Strongly unstable and stable manifolds of the flow Φ^t are cross-sections (but not necessarily orthogonal!) of the corresponding families of flow lines.

Clearly, Φ^t is a suspension flow over the base map $\mathcal{F}\colon \mathcal{M} \to \mathcal{M}$ under a ceiling function τ. Strictly speaking, in a suspension flow the velocity must be equal to one, which can be achieved by changing the metric within Ω, but this change does not affect the existence or ergodicity of the SRB measure. Thus Theorem 1.1 easily implies the following:

Corollary 3.1. *The flow $\Phi^t\colon \Omega \to \Omega$ admits a unique SRB measure μ, which is positive on open sets and ergodic.*

However, the mixing of the flow Φ^t requires a more elaborate argument. Loosely speaking, a hyperbolic flow is not mixing if its stable and unstable foliations are (locally) jointly integrable, i.e. for any phase point $X \subset \Omega$ all short chains consisting of alternating stable and unstable manifolds starting at X lie in a submanifold of codimension one transversal to the flow. For the billiard flow Φ_0, the lack of joint integrability was observed by Sinai [28], it is related to the opposite convexity of stable and unstable manifolds in Ω, see a detailed argument in [8, Section 6.11].

In our case, stable and unstable manifolds of the flow may not have opposite convexity, so we use a roundabout way to establish their nonintegrability.

First we sharpen certain facts established in [7] for the map $\mathcal{F}$. For any curve $\gamma \subset \mathcal{M}$ let m_γ denote the Lebesgue measure on γ. For any point $X \in \mathcal{M}$ we denote by $\gamma^u(X)$ and $\gamma^s(X)$ the stable and unstable h-fibers through X. The point X divides the curve $\gamma^\alpha(X)$, $\alpha = u, s$, into two pieces, and we denote by $r^\alpha(X)$ the length of the shorter one.

Lemma 3.2. *For every homogeneous unstable curve $\gamma \subset \mathcal{M}$ and m_γ-almost every point $X \in \gamma$ the stable h-fiber $\gamma^s(X)$ exists, i.e. $r^s(X) > 0$. Moreover*

$$m_\gamma(r^s(X) < \varepsilon) \leq C\varepsilon \tag{3.1}$$

for all $\varepsilon > 0$. The dual statement holds for stable curves.

The existence of $\gamma^s(X)$ follows from two results of [7]: Eq. (6.4) and the Fact stated on p. 227. The estimate (3.1) is a local version of [7, Proposition 5.6], and for its proof see [8, Theorem 5.66].

Lemma 3.3. *For every stable curve γ we have $\nu(\cup_{X\in\gamma}\gamma^u(X)) > 0$. For every unstable curve γ we have $\nu(\cup_{X\in\gamma}\gamma^s(X)) > 0$.*

Proof. The first statement follows from the proof of [7, Lemma 6.12]. Note that the second statement is *not* dual to the first one, since the SRB measure ν is not preserved under the reversal of time. The second statement follows from Lemma 3.2, the absolute continuity ([7, Lemma 4.3]), and the first statement here. □

Corollary 3.4 (Sinai's fundamental theorem)**.** *Let $X \in \mathcal{M}$ and $\mathcal{F}^n$ be continuous at X for all $n > 0$. Then for any $\varepsilon > 0$ and $A > 0$ there exists an open neighborhood $\mathcal{U} \subset \mathcal{M}$ of X such that for any unstable curve $\gamma \subset \mathcal{U}$*

$$m_\gamma\big(Y \in \gamma \colon r^s(Y) > A|\gamma|\big) \geq (1-\varepsilon)\,|\gamma|.$$

Similarly, if $\mathcal{F}^n$ is continuous at X for all $n < 0$, then for any stable curve $\gamma \subset \mathcal{U}$

$$m_\gamma\big(Y \in \gamma \colon r^u(Y) > A|\gamma|\big) \geq (1-\varepsilon)\, m_\gamma|\gamma|.$$

Observe that if $A \gg 1$ is large, then a vast majority of points $Y \in \gamma$ lie on stable (unstable) h-fibers which are much longer than the curve γ itself. This corollary follows from Lemma 3.2, see [8, Section 5.13].

We recall [7, p. 233] that given a stable h-fiber γ^s and $\varepsilon > 0$ we denote by $\Gamma_\varepsilon(\gamma^s)$ the union of all stable h-fibers in $\mathcal{M}$ that are ε-close to γ^s in the Hausdorff metric. We call γ^s a *density h-fiber* if for every $\varepsilon > 0$ the set $\Gamma_\varepsilon(\gamma^s)$ has positive Lebesgue measure in $\mathcal{M}$. (This is an analogue of Lebesgue density points of subsets of $\mathbb{R}^n$.) The union of density h-fibers has full Lebesgue measure [7, Lemma 6.12].

Now we call γ^s a *ν-density h-fiber* if for every $\varepsilon > 0$ the set $\Gamma_\varepsilon(\gamma^s)$ has positive ν-measure. Similarly, we define unstable ν-density h-fibers.

Lemma 3.5. *Every density h-fiber is also a ν-density h-fiber.*

Proof. For stable h-fibers, this follows from Lemmas 3.3 and the absolute continuity. The claim for unstable h-fibers can be proved by a construction similar to the proof of [7, Proposition 6.13]. Precisely, let γ^u be a density h-fiber and $\varepsilon > 0$. Let γ be a stable curve crossing γ^u. By reducing γ, if necessary, we can ensure that $m_\gamma(\gamma \cap \Gamma_\varepsilon(\gamma^u)) > (1-\varepsilon')m_\gamma(\gamma)$, where $\varepsilon' > 0$ is arbitrary small. Now we pull the entire structure back under $\mathcal{F}^{-n}$ until we get $m_\gamma(\gamma(-n)) \geq \tilde{\beta}_2\, m_\gamma(\gamma)$ (here and below we use the notation of [7, Section 6]). The set $\mathcal{F}^{-n}\gamma(-n)$ consists of stable curves that straddle the fixed rhombus R. Then it is not hard to deduce that the set $\mathcal{F}^n(R^u_{\mathbf{F}}) \cap \Gamma_\varepsilon(\gamma^u)$ has a positive ν measure. □

We will only consider density h-fibers without saying that explicitly.

Consider a continuous curve in $\mathcal{M}$ that is a finite union of segments of stable and unstable h-fibers (of course, stable and unstable h-fibers must alternate). Such curves are called *Hopf chains* or *zigzag lines* or *us-paths* ('us' stays for 'unstable-stable'). We require that at every "junction point" where two segments of h-fibers meet, those can be continued beyond the junction point. If a chain is not simple, i.e. has self-intersections, it can be shortened by the removal of extra loops, hence we will only consider simple chain. A chain is called a *loop* (or n-loop) if it is a simple closed curve in $\mathcal{M}$ (consisting of n segments of h-fibers). Chains and loops are instrumental in many proofs of ergodicity that go back to Hopf [16, 17].

Lemma 3.6 ("Zigzag lemma")**.** *For every open connected set $V \subset \mathcal{M}$ and two curves $\gamma_1, \gamma_2 \subset V$ let $\mathrm{dist}_V(\gamma_1, \gamma_2)$ denote the minimal length of smooth curves lying in V and connecting γ_1 with γ_2. Then there is a zigzag line that starts on γ_1 and ends on γ_2, lies entirely in V, and whose total length is $\leq C \cdot \mathrm{dist}_V(\gamma_1, \gamma_2)$.*

This follows from Sinai's fundamental theorem (Corollary 3.4), one just constructs a zigzag line starting at γ_1, moving in a general direction along a curve $\gamma \subset V$ connecting γ_1 with γ_2, whose length is nearly minimal, and eventually crossing γ_2; a detailed construction of such zigzag lines is described in [8, Section 6.5]. Since stable and unstable h-fibers are uniformly transversal [7, Lemma 3.10], one can easily ensure the necessary bound on the length of the chain.

Lemma 3.7. *There is a global constant $d_0 = d_0(\mathcal{D}) > 0$ such that for every small force* $\mathbf{F}$ *there is a simple 4-loop (four is the minimal number of h-fibers in a loop), where all the h-fibers have length $\geq d_0$.*

Proof. This follows from Corollary 3.4 and Lemma 3.5. □

We now turn to the flow $\Phi^t \colon \Omega \to \Omega$. Let $X \in \mathcal{M}$ and $\mathcal{L} \subset \mathcal{M}$ a loop consisting of segments of h-fibers $\gamma_1, \dots, \gamma_n$, so that $\gamma_i \cap \gamma_{i+1} \neq \emptyset$ and $\gamma_1 \cap \gamma_n = \{X\}$. Consider $Y = \Phi^t X$ for some $t \in (0, \tau(X))$. We now 'lift' the loop $\mathcal{L}$ from $\mathcal{M}$ to Ω in the following way: for each stable or unstable h-fiber γ_i let Γ_i be a stable (resp., unstable) manifold of the flow Φ^t that projects down onto γ_i and such that $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ for $i = 1, \dots, n-1$. (We assume that $\mathcal{L}$ is small enough so that the entire construction lies in the interior of Ω and avoids intersections with the boundary $\partial\Omega$.)

Now it is clear that $\cup_{i=1}^n \Gamma_i$ is a continuous curve in Ω starting at Y and terminating at some point Y' lying on the trajectory $\Phi^t(Y)$, i.e. $Y' = \Phi^{\boldsymbol{\tau}(\mathcal{L})}(Y)$ for some small $\boldsymbol{\tau}(\mathcal{L})$. We call $\mathcal{L}$ a *closed loop* if $\boldsymbol{\tau}(\mathcal{L}) = 0$ and an *open loop* otherwise. Clearly, reversing the orientation of $\mathcal{L}$ (i.e., traversing $\mathcal{L}$ in the opposite direction) simply changes the sign of $\boldsymbol{\tau}(\mathcal{L})$. Given an orientation of $\mathcal{L}$, one can easily verify that $\boldsymbol{\tau}(\mathcal{L})$ does not depend on the choice of the initial point $X \in \mathcal{L}$ or t.

In the billiard systems (where $\mathbf{F} = 0$), all the loops are open, and in fact $|\boldsymbol{\tau}(\mathcal{L})|$ equals the ν_0-measure of the domain bounded by $\mathcal{L}$, see [8, Lemma 6.40]. Next we verify the existence of open loops for small forces.

Lemma 3.8. *For every $\varepsilon > 0$ there is a $\delta'_* = \delta'_*(\mathcal{D}, B_0, \varepsilon) > 0$ such that whenever a force* $\mathbf{F}$ *satisfies Assumptions A–C with $\delta_0 < \delta'_*$, then*

$$\left|1 - \max_{x,y,\theta} p(x, y, \theta) / \min_{x,y,\theta} p(x, y, \theta)\right| < \varepsilon$$

i.e. the function $p(x, y, \theta)$ is almost constant on Ω.

Proof. Due to (1.5), p slowly changes along the trajectories of the flow, and due to (1.2) it does not change at collisions. Now the lemma follows from the uniform bound (1.8) on the derivatives of p, and the uniform bounds on correlations in Theorem 1.1. □

Lemma 3.9. *For every $\varepsilon > 0$ there is a $\delta''_* = \delta''_*(\mathcal{D}, B_0, \varepsilon) > 0$ such that whenever a force* $\mathbf{F}$ *satisfies Assumptions A–C with $\delta_0 < \delta''_*$, then all the first order derivatives of the function $p(x, y, \theta)$ are less than ε.*

This lemma follows from Lemma 3.8 and the following elementary fact:

Sublemma 3.10. *Let Ω be a smooth compact manifold with boundary, $f \colon \Omega \to \mathbb{R}$ a C^2 function whose first and second order derivatives are uniformly bounded by a constant B_0. Then for every $\varepsilon > 0$ there is a $\delta = \delta(\Omega, B_0, \varepsilon) > 0$ such that if $|f(X) - f(Y)| < \delta$ for all $X, Y \in \Omega$, then all the first order derivatives of f are less than ε.*

The following lemma sharpens [7, Lemma 3.6]:

Lemma 3.11. *For every $\varepsilon > 0$ there is a $\delta'''_* = \delta'''_*(\mathcal{D}, B_0, \varepsilon) > 0$ such that whenever a force* $\mathbf{F}$ *satisfies Assumptions A–C with $\delta_0 < \delta'''_*$, then for any strongly divergent family of trajectories on an interval (t_0, ∞) we have $|\alpha_t| < \varepsilon$ for all $t > t_0 + c$ for some constant $c > 0$.*

The proof is a modification of that of Lemma 3.6 in [7]. The main difference is that now, in the equation (3.22), all the terms can be made arbitrarily small, except κ, which is still positive and bounded away from zero. Hence, the term $-\kappa(\alpha - p_\theta/p)$ drives α to zero whenever α is not small enough, and the other terms are not strong enough to stop this drive. □

We now recall the meaning of the function α_t, see [7, p. 204]. Let $\Gamma^u \subset \Omega$ be an unstable manifold of the flow, then its trajectories $\{\Phi^s(Y)\}$, $Y \in \Gamma^u$, $s > 0$, make a strongly divergent family. For every $t > 0$ the image $\Phi^t(\Gamma^u)$ is an unstable manifold, too; its projection onto the table $\mathcal{D}$ is a curve that is a cross-section of the above family of flow lines. Now α_t is the cotangent of the angle between that cross-section and the corresponding flow-line. If $\alpha_t = 0$, then the cross-section is orthogonal (and this is the case for unperturbed billiards); if α_t is small, then the cross-section is almost orthogonal. The above lemma now implies that unstable manifolds of the perturbed flow Φ^t are good approximations to those of the billiard flow Φ_0^t. Due to the time reversibility, a similar property holds for stable manifolds. Now it takes a simple geometric argument to conclude the following:

Corollary 3.12. *Let $\mathcal{L} \subset \mathcal{M}$ be a simple 4-loop from Lemma 3.7. If ε in Lemma 3.11 is small enough, we have $\boldsymbol{\tau}(\mathcal{L}) \neq 0$, i.e. $\mathcal{L}$ is an open loop.*

Next we establish a general fact:

Proposition 3.13. *If there exists an open loop $\mathcal{L} \subset \mathcal{M}$, then the flow Φ^t is mixing and Bernoulli.*

Proof. For simplicity we assume that $\mathcal{L}$ is an open 4-loop (and one exists due to Corollary 3.12); it will be clear from our argument that it applies to arbitrary loops, too. The following lemma can be verified by direct inspection:

Lemma 3.14. *Let $\mathcal{L}_1, \dots, \mathcal{L}_k \subset M$ be simple loops, oriented in the same way (say, all – clockwise), and bounding nonoverlapping domains $V_1, \dots, V_k$. Suppose that these domains are adjacent to each other so that $V = V_1 \cup \cdots \cup V_k$ is a simply connected domain in M. Then V is bounded by a loop $\mathcal{L}$ and we have $\boldsymbol{\tau}(\mathcal{L}_1) + \cdots + \boldsymbol{\tau}(\mathcal{L}_k) = \boldsymbol{\tau}(\mathcal{L})$.*

Now, let γ_1^u, γ_2^s, γ_3^u, and γ_4^s denote the sides of the 4-loop $\mathcal{L}$ (which are alternating unstable and stable h-fibers). Let X_1 and X_2 denote the midpoints of γ_1^u and γ_3^u, respectively, and V the open ε-neighborhood of the straight line joining X_1 with X_2, where $\varepsilon \ll \operatorname{dist}(X_1, X_2)$. Due to Lemma 3.6, there is a zigzag line $\mathcal{L}'$ joining $\gamma_1^u \cap V$ with $\gamma_3^u \cap V$, lying entirely in V, and having length $|\mathcal{L}'| \leq C \cdot \operatorname{dist}(X_1, X_2)$.

The zigzag line $\mathcal{L}'$ divides the domain bounded by $\mathcal{L}$ into 2 subdomains, each bounded by a loop. Due to Lemma 3.14 one of these two loops is open, we denote it by $\mathcal{L}_1$. It has four sides: two unstable sides (≈halves of γ_1^u and γ_3^u) and two stable sides: one is γ_2^s or γ_4^s and the other is $\mathcal{L}'$ (the latter is, of course, a zigzag line, but it stretches along a stable curve joining X_1 and X_2, so we call it a stable side). Now we apply the same argument to construct a zigzag line joining the middle part of γ_2^s (or γ_4^s) with the middle part of $\mathcal{L}'$

and stretching along an unstable curve. This gives us two smaller loops, one of them will be open (again by Lemma 3.14), and we denote it by $\mathcal{L}_2$.

Now we repeat our construction inductively and obtain a sequence of open loops $\mathcal{L}_n$, $n \geq 1$, such that $|\mathcal{L}_n| \to 0$, hence $\boldsymbol{\tau}(\mathcal{L}_n) \to 0$, as $n \to \infty$ (but $\boldsymbol{\tau}(\mathcal{L}_n) \neq 0$ for every n). In other words, there are open loops with arbitrarily small values of $\boldsymbol{\tau}(\mathcal{L})$! Recall that all sides of our loops are density h-fibers, hence there are *plenty* of open loops with arbitrarily small $\boldsymbol{\tau}$ values. More precisely, if we remove an arbitrary collection of h-fibers of the total ν-measure zero from $\mathcal{M}$, then there still remain open loops with arbitrarily small (but non-zero) $\boldsymbol{\tau}$ values.

Next, it is known that, under general assumptions, a completely hyperbolic flow (which Φ^t is) with an ergodic SRB measure is either Bernoulli or Bernoulli times rotation. In the latter case a factor of the flow Φ^t is a circle rotation (and, of course, in this case the flow is not even mixing). Hence, the Bernoulli property is equivalent here to the mixing property. A general result of this sort was obtained by Ornstein and Weiss [27]. In the particular setting of certain perturbations of billiard flows, this fact was proved earlier by Kubo and Murata [21].

Assume now that the flow Φ^t is Bernoulli+rotation, i.e. there is a factor $\Xi\colon \Omega \to S^1$ so that $\Xi \circ \Phi^t \circ \Xi^{-1}$ is the rotation of the circle S^1 at constant speed. We call the sets $\Xi^{-1}(p) \subset \Omega$, for $p \in S^1$, layers. It is clear that μ-almost every stable or unstable manifold $\Gamma \subset \Omega$ lies in one layer; we call such stable and unstable manifolds *typical*. We call the projections of typical stable and unstable manifolds on $\mathcal{M}$, along the trajectories of the flow, *typical* h-fibers; then ν-almost every h-fiber is typical.

Let $\mathcal{L} \subset \mathcal{M}$ be any loop consisting of typical h-fibers. Its 'lift' in Ω, as constructed above, will consist of typical stable and unstable manifolds of the flow Φ^t. Since every typical stable and unstable manifold lies in one layer, the entire lift of the loop $\mathcal{L}$ belongs in one layer, too. Hence, $\boldsymbol{\tau}(\mathcal{L})$ is a multiple of the period of the rotation $\Xi \circ \Phi^t \circ \Xi^{-1}$ of S^1. But we have seen that there are plenty of loops consisting of typical h-fibers with arbitrarily small non-zero $\boldsymbol{\tau}$ values, thus the period of rotation must be zero. This proves

Theorem 3.15. *The flow $\Phi^t\colon \Omega \to \Omega$ is mixing and Bernoulli.*

Lastly we present a central limit theorem for the flow Φ^t. Let $F\colon \Omega \to \mathbb{R}$ be a bounded function such that $f(X) = \int_0^{\tau(X)} F(\Phi^t X)\, dt$ is a dynamically Hölder continuous function on $\mathcal{M}$ (this holds, for example, when F is smooth with bounded derivatives). Note that

$$\mu(F) = \int_\Omega F\, d\mu = \nu(f)/\nu(\tau).$$

Denote $S_t(X) = \int_0^t F(\Phi^t X)\, dt$ for all $t > 0$ and $X \in \Omega$.

Theorem 3.16 (Central Limit Theorem)**.** *There is $\sigma_F \geq 0$ such that the function $(S_t - \mu(F)\, t)/\sqrt{t}$ on Ω converges to a normal distribution $\mathcal{N}(0, \sigma_F^2)$.*

This theorem follows from the central limit theorem for the map $\mathcal{F}$ applied to f and τ, see, e.g., [8, Theorem 7.68]. Note that we are not using the mixing property of the flow Φ^t. In fact, probabilistic limit theorems usually hold for suspension flows regardless of their mixing properties, as long as the base map is strongly chaotic, see e.g. [13, 24].

As a corollary, let $\tilde{\mathbf{q}}(t) = \int_0^t \mathbf{p}(s)\, ds$ denote the position of the moving particle on the universal cover of the torus $\mathbb{T}^2$. Then there exists a 2-vector $\mathbf{a} = \int_\Omega \mathbf{p}\, d\mu$ and a 2×2 positive definite matrix $\mathbf{V}$ such that $(\tilde{\mathbf{q}}(t) - \mathbf{a}t)/\sqrt{t}$ converges to a two-dimensional normal distribution $\mathcal{N}(0, \mathbf{V})$. The matrix $\mathbf{V}$ is close to the diffusion matrix of the unperturbed Sinai billiard, which is known to be non-singular [3], thus $\mathbf{V}$ cannot be singular either.

In Theorem 3.16, $\sigma_F^2 = \sigma_f^2/\nu(\tau)$, where σ_f^2 is defined by the infinite series (2.12). If the flow Φ^t has rapidly decaying correlations, one usually can prove that

$$\sigma_F^2 = \int_{-\infty}^{\infty} \mu\big((F \circ \Phi^t) \cdot F\big)\, dt. \tag{3.2}$$

For the unperturbed billiard flow Φ_0^t, correlations decay at least as fast as a 'stretched exponential' function [9], which ensures (3.2) for that special case. But the results of [9] to dot extend to the perturbed flow Φ^t.

By using a different approach, Melbourne [23] obtained fairly strong ('superpolynomial') bounds on correlations for rather general hyperbolic flows under the assumption that four periodic orbits exist whose periods satisfy a Diophantine-type condition, see (2.1) in [23]. This result implies rapid mixing, and therefore (3.2), for *typical* (in certain topological and measure-theoretic senses) flows Φ^t. It would be interesting to obtain bounds on correlations for all flows Φ^t covered in this paper.

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