Entropy Values and Entropy Bounds

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Abstract

We describe rigorous mathematical results on the Kolmogorov-Sinai entropy for Lorentz gases and hard ball systems (both finite and infinite). Exact formulas and asymptotic estimates of the entropy are discussed for various models.

1 Entropy: general formulas

Entropy is an important numerical characteristic of dynamical systems. It, in a sense, measures the amount of chaos, or complexity, in the system.

Two different versions of entropy are widely used in the study of dynamical systems. The *measure-theoretic entropy* (called also *Kolmogorov-Sinai entropy*) is associated with a measurable transformation $T : X \to X$ preserving a probability measure μ. By contrast, the *topological entropy* is associated with a continuous transformation $T : X \to X$ of a topological space X not equipped with any measure. Generally, the topological entropy $h_{\rm top}(T)$ is greater than the Kolmogorov-Sinai entropy $h_\mu(T)$, and any invariant measure μ with the property $h_\mu(T) = h_{\rm top}(T)$ is called the measure of maximal entropy. For continuous time dynamical systems (flows) the entropy is defined as that of the time-one map. More detailed discussions of entropy, its definition, properties, and history of the subject, can be found, for example, in [ME].

We will primarily work with the Kolmogorov-Sinai entropy (also referred to as KS entropy). Throughout, we denote by $h(T)$ the KS entropy of the billiard map $T : \Omega \to \Omega$ with respect to the smooth invariant measure ν and by $h(\Phi^t)$ the KS entropy of the flow Φ^t with respect to the Liouville invariant measure μ.

There is a remarkable relation between the two entropies, $h(T)$ and $h(\Phi^t)$, that follows from a more general Abramov's formula for the entropy of suspension flows [Ab]:

Proposition 1.1 *We have*

$$h(T) = \bar{\tau}\, h(\Phi^t) \tag{1.1}$$

where $\bar{\tau}$ *is the mean value of the free path* $\tau(x)$ *on* Ω*:*

$$\bar{\tau} = \int_\Omega \tau(x)\, d\nu \tag{1.2}$$

The value of the KS entropy is closely related to those of Lyapunov exponents:

Proposition 1.2 *We have*

$$h(T) = \int_\Omega \sum\nolimits^{+} \chi_i(x)\, d\nu(x) \tag{1.3}$$

where the sum $\sum^+$ *runs over all positive Lyapunov exponents* $\chi_i(x) > 0$ *at every* $x \in \Omega$ *(counting multiplicity). We also have*

$$h(\Phi^t) = \int_M \sum\nolimits^{+} \chi_i^*(x)\, d\mu(x) \tag{1.4}$$

where the sum $\sum^+$ *runs over all positive Lyapunov exponents* $\chi_i^*(x) > 0$ *of the flow* Φ^t *at every* $x \in M$ *(counting multiplicity).*

The above formulas (1.3) and (1.4) are known as *Pesin identities*. They were originally found by Ya. Pesin in the context of smooth hyperbolic systems with smooth invariant measures [P]. Later these formulas were proved for smooth systems with singularities (including billiards) [KS], assuming only partial hyperbolicity, and for invariant measures that only have smooth conditional distributions on unstable manifolds. Such measures are now called Sinai-Ruelle-Bowen measures (also, SRB measures). Ergodic SRB measures in hyperbolic systems are the only physically observable measures, in the sense that they characterize space distributions of typical phase trajectories. It is interesting that SRB measures are the *only* measures for which the Pesin identity holds, so for all the other measures the entropy is strictly less than the average sum of positive Lyapunov exponents. For more discussion of this topic see an excellent survey [Y].

SRB measures correspond to nonequilibrium steady states in statistical mechanics. If one perturbs a Hamiltonian system (that has a smooth invariant measure by the Liouville theorem) by an external force or a boundary condition, then generally the perturbed system does not have any smooth invariant measure. Then physically interesting invariant measures are those that describe the evolution of typical phase points, and such measures are, in many cases, SRB measures. More precisely, if the original system is hyperbolic and the perturbation is small, then an SRB measure is very likely to represent a nonequilibrium steady state.

Various perturbations of hard ball gases and Lorentz gases under external fields or boundary conditions have been studied in the literature. In many cases nonequilibrium steady states in the form of an SRB measures have been observed numerically and sometimes investigated mathematically [CELS, C4]. See also the surveys [Bu, CY] in this volume for more details. In all those cases, Pesin's identity for the entropy is very likely to hold as well, but there is no mathematical proof of that fact in such a generality.

We now get back to our hard balls and Lorentz gases.

The relation of the entropy to Lypunov exponents may not be practically very useful, because the Lyapunov exponents are not easy to compute. They characterize the asymptotic rate of expansion of unstable vectors. One can simplify this relation noticing that, due to the Birkhoff ergodic theorem, the average asymptotic rates of expansion are equal to the average one-step rates of expansion. This is stated below.

Proposition 1.3 *We have*

$$h(T) = \int_\Omega \ln |J^u(x)|\, d\nu(x) \tag{1.5}$$

Here $J^u(x)$ is the Jacobian of the differential map DT restricted to the unstable subspace $E^u_x \subset \mathcal{T}_x\Omega$ (the latter is spanned by all the tangent vectors with positive Lyapunov exponents).

Note that $J^u(x)$ is the factor of expansion of volume in the space E^u_x under the map $DT : E^u_x \to E^u_{Tx}$.

The advantage of the last entropy formula (1.5) over the previous one (1.3) is actually quite deceptive. To find the unstable subspace $E^u_x \subset \mathcal{T}_x\Omega$, one essentially needs an asymptotic procedure practically equivalent to the computation of all positive Lyapunov exponents.

There is, fortunately, an explicit characterization of the unstable subspace E^u_x and an explicit formula for the entropy $h(T)$ in terms of the so called curvature operator B_x. That operator was introduced by Ya. Sinai in the seventies [S1, S3], and it has been the main tool in Sinai's pioneering works on Lorentz gases and hard ball systems. The operator B_x is given in terms of an infinite continued fraction defined below.

For any point $x = (q, v) \in M$ we denote by $dx = (dq, dv)$ tangent vectors in $\mathcal{T}_xM$, so that $dq \in \mathcal{T}_qQ$ and $dv \in \mathcal{T}_vS^{d-1}$. Note that $dv \perp v$, because $||v|| =$const. Denote by J_x the hyperplane in $\mathcal{T}_qQ$ orthogonal to the velocity vector v. It can be naturally identified with $\mathcal{T}_vS^{d-1}$, since both are perpendicular to the vector v. We will define a linear operator $B_x : J_x \to J_x = \mathcal{T}_vS^{d-1}$, with the help of a few auxiliary linear operators.

Let $x_t = (q_t, v_t) = \Phi^t x$. If there is no reflections at ∂Q between x and x_t, then the velocity vectors v and v_t are parallel, hence the spaces J_x and J_{x_t} are parallel and can be naturally identified by parallel translation.

Let t be a moment of reflection at ∂Q, i.e. assume $q_t \in \partial Q$. We have an instantaneous transformation of the velocity vector at time t given by

$$v_t^+ = v_t^- - 2(n(q_t) \cdot v_t^-)n(q_t)$$

Here v_t^- and v_t^+ are the velocity vectors before and after the reflection, respectively, and $n(q_t)$ is the unit normal vector to ∂Q at the point q_t pointing inward Q. We have two hyperplanes in the tangent space $\mathcal{T}_{q_t}Q$, perpendicular to v_t^- and v_t^+, we call them $J_{x_t}^-$ and $J_{x_t}^+$, respectively.

Denote by $U : \mathcal{T}_{q_t}Q \to \mathcal{T}_{q_t}Q$ the reflector across the hyperplane $\mathcal{T}_{q_t}(\partial Q)$ tangent to ∂Q at the reflection point q_t. The reflector U is obviously given by

$$U(w) = w - 2\,(n(q_t)\cdot w)\,n(q_t)$$

for all $w \in \mathcal{T}_{q_t}Q$. It is easy to see that $U(v_t^-) = v_t^+$ and $U(J_{x_t}^-) = J_{x_t}^+$, and U is an isometry. The operator U may be used to identify $J_{x_t}^-$ with $J_{x_t}^+$, and thus we can identify the hyperplanes J_{x_t} for all t, but we will not pursue this goal.

Denote by $\Theta : \mathcal{T}_{q_t}Q \to \mathcal{T}_{q_t}Q$ the unique linear operator specified by two conditions:
(i) $\Theta(v_t^-) = v_t^+$;
(ii) for any vector $w \in J_{x_t}^-$ we have

$$\Theta(w) = 2\,(v_t^+ \cdot n(q_t))\,V_+ K_{q_t} V_-(w) \,\in J_{x_t}^+$$

Here V_- is the projection of $J_{x_t}^-$ onto $\mathcal{T}_{q_t}(\partial Q)$ parallel to the incoming velocity vector v_t^-, and V_+ is the projection of $\mathcal{T}_{q_t}(\partial Q)$ onto $J_{x_t}^+$ parallel to the normal vector $n(q_t)$. Also, K_{q_t} is the curvature operator of the boundary hypersurface ∂Q at the point q_t defined, as usual, by

$$n(q_t + dq) = n(q_t) + K_{q_t}(dq) + o(||dq||)$$

for vectors $dq \in \mathcal{T}_{q_t}(\partial Q)$. Note: since K_{q_t} is a self-adjoint positive-semidefinite operator, then so is ΘU^{-1}.

Assume now that the past trajectory of x is completely defined. Let $0 > t_1 > t_2 > \cdots$ be all the past moments of reflection (note that $t_i \to -\infty$ as $i \to \infty$). At each reflection moment t_i we denote by U_i and Θ_i the two linear operators introduced above. Let $\tau_0 = -t_1$ and $\tau_i = t_i - t_{i+1} > 0$, $i \geq 1$, be the intercollision times. Then

$$B_x = \cfrac{I}{\tau_0 I + \cfrac{I}{\Theta_1 U_1^{-1} + U_1 \cfrac{I}{\tau_1 I + \cfrac{I}{\cdots}} U_1^{-1}}} \tag{1.6}$$

where $\frac{I}{A}$ means A^{-1}. Note that the terms $\Theta_i U_i$ and $\tau_i I$ alternate as the fraction continues downward. In a sense, these two alternating terms describe the contribution of reflections and free paths as they appear on the trajectory $\Phi^t x$, $t < 0$.

Note that

$$B_{x_t} = \cfrac{I}{tI + \cfrac{I}{B_x}}$$

if there is no reflections between x and x_t. At each moment of reflection t_i, the operator B_{x_t} changes discontinuously, and we have

$$B_{x_{t_i+}} = \Theta_i U_i^{-1} + U_i B_{x_{t_i-}} U_i^{-1} \tag{1.7}$$

Hence, the operators B_{x_t} are naturally related to each other along the trajectory $\Phi^t x$.

If $x = (q, v) \in \Omega$, i.e. x is a reflection point, we define

$$B_x^+ = \lim_{t \downarrow 0} B_{x_t}$$

Then it follows from (1.6) and (1.7) that

$$B_x^+ = \Theta_1 U_1^{-1} + U_1 \cfrac{I}{\tau_1 I + \cfrac{I}{\Theta_2 U_2^{-1} + U_2 \cfrac{I}{\tau_2 I + \cfrac{I}{\cdots}} U_2^{-1}}} U_1^{-1} \tag{1.8}$$

Here $0 = t_1 > t_2 > \cdots$ are the past moments of reflections.

One can easily check that B_x maps J_x into itself. In all that follows we restrict B_x onto the hyperplane J_x.

Proposition 1.4 *The operator-valued continued fraction (1.6) converges at every point $x \in \Omega$ with an infinite past trajectory. Moreover, if $B_{x,n}$ is a finite continued fraction obtained from (1.6) by truncation at the n-th reflection, then*

$$||B_x - B_{x,n}|| \leq \frac{1}{|t_n|}$$

The operator B_x is self-adjoint positive semi-definite.

The proof of the convergence is based on the fact that all the operators in (1.6) are self-adjoint positive semi-definite, i.e. $\tau_i > 0$ and $\Theta_i \geq 0$. The first proof was published in [SC], see also [LW]. In a weaker form the statement was given without proof earlier in [S4]. For 2-D Lorentz gases the convergence was proved earlier in [S1].

Remark. The past trajectory $\Phi^t x$, and hence the operator B_x, is well defined unless the following anomalies occur:
(i) The trajectory Φ^t, $t < 0$, hits a "corner point" in the configuration space. No such points exist in the Lorentz gas model where all the scatterers are smooth. In the hard ball model, corner points in the configuration space correspond to multiple collisions of balls (where three or more balls collide simultaneously). The dynamics is discontinuous at such points.
(ii) The trajectory is tangent to the boundary in the configuration space (this situation is called a grazing collision, it is possible in both Lorentz gases and hard ball gases). At such points the dynamics is continuous but not differentiable, i.e. these are singular points for the dynamics.
(iii) The trajectory experiences infinitely many collisions within a finite interval of time. This sort of disaster is possible for some billiard systems. However, as G. Galperin [Gal] and L. Vaserstein [V] showed, this never happens in gases of hard balls or Lorentz gases (more generally, this is impossible in any semidispersing billiards).

As a result, the operator B_x is defined at all regular (nonsingular) phase points. Moreover, it depends on x continuously.

The operator B_x explicitly describes the unstable subspace E^u_x at every point $x \in M$:

Proposition 1.5 *If Lyapunov exponents exist at a point $x \in M$, then the unstable subspace $E^u_x \subset \mathcal{T}_x M$ (the subspace spanned by all the tangent vectors with positive Lyapunov exponents) satisfies*

$$E^u_x = \{(dq, dv) : \, dq \in J^u_x, \; dv = B_x(dq)\}$$

Here $J^u_x \subset J_x$ is the subspace spanned by the eigenvectors of B_x with positive eigenvalues.

We note that since B_x is self-adjoint and positive semi-definite, the space J_x is the orthogonal sum $J^u_x \oplus J^0_x$ of two B_x-invariant subspaces so that B_x is positive on J^u_x and zero on J^0_x. Note also that $\dim E^u_x = \dim J^u_x$.

The entropy can also be explicitly given in terms of the operator B_x:

Theorem 1.6 *We have*

$$h(T) = \int_\Omega \ln \det(I + \tau(x) B^+_x) \, d\nu(x) \tag{1.9}$$

and

$$h(\Phi^t) = \int_M \operatorname{tr} B_x \, d\mu(x) \tag{1.10}$$

The formula (1.10) has a long history. It was first established for 2-D dispersing billiards by Sinai [S1] in 1970. Its multidimensional version for semi-dispersing billiards appeared in 1979 in a preprint by Sinai [S3], with an outline of a proof. A complete proof of both (1.9) and (1.10) for semi-dispersing billiards was provided by Chernov [C2] in 1991. He also extended both formulas to more general classes of billiard tables in [C2] and later in [C3]. In fact, Chernov proved [C3] that (1.9) holds for every billiard table, in any dimension, as long as unstable bundles of trajectories do not focus right on the boundary. He also found a necessary and sufficient condition on a billiard table for the formula (1.10) to hold. The condition is that unstable bundles of trajectories do not focus between collisions [C3] (we note that if they do, the integral in (1.10) diverges).

The proof of the above theorem is based on the following ideas. The formula (1.9) follows from (1.5) provided we can establish

$$J^u(x) = \det(I + \tau(x) B^+_x) \tag{1.11}$$

This is not true in the Euclidean metric $(dq)^2 + (dv)^2$ on Ω, but there is the so called pseudo-metric on Ω (also called the p-metric) in which $J^u(x)$ is indeed given by (1.11). In the p-metric, the distance on unstable manifolds in Ω is induced by the Euclidean metric on the orthogonal cross-section of the corresponding outgoing bundles of trajectories in the configuration space. This construction of a pseudo-metric goes back to Sinai [S1] and is commonly used in other papers on billiard dynamics. The verification of (1.11) is then quite elementary, see, e.g., [C2]. One should also note that by changing metric in Ω one changes the function $J^u(x)$ but its integral entering (1.5) remains unchanged, as it follows from the invariance of the measure ν. Lastly, the formula (1.10) follows from (1.9) and (1.1) by rather standard and simple calculation, see [C2, C3].

2 Entropy of Lorentz gases: asymptotic estimates

Estimation of the entropy and Lyapunov exponents of Lorentz gases have been done by physicists since early eighties. One motivation was to explore the quantitative characteristics of chaotic dynamics and observe transition from a regular motion (on a torus without scatterers) to chaos (starting when a small scatter is placed on the torus).

For a 2-D periodic Lorentz gas with a single circular scatterer of radius $r > 0$ on a unit torus the entropy was estimated [FOK] by

$$h(T) \approx -2 \ln r \tag{2.1}$$

as $r \to 0$. Since the mean free path was long estimated to be $\bar{\tau} \approx (2r)^{-1}$, we have by (1.1)

$$h(\Phi^t) \approx -r \ln r \tag{2.2}$$

Since $h(\Phi^t) \to 0$ as $r \to 0$, one obtains an asymptotic behavior of the entropy near the transition point (between the "regular motion" at $r = 0$ and "chaos" at $r > 0$). The above estimates have been proved, see below.

It was also conjectured in [FOK] that for any d-dimensional periodic Lorentz gas with a spherical scatterer of radius $r > 0$ one should have $h(T) \approx -d \ln r$, which turned out to be wrong, see below. In the analysis of $h(T)$, the following important quantity is involved:

$$\ln \int_\Omega \tau(x)\, d\nu(x) \; - \; \int_\Omega \ln \tau(x)\, d\nu(x) \tag{2.3}$$

It was numerically estimated [FOK] that this quantity remains bounded and has a positive limit ($\approx 0.44 \pm 0.01$) as $r \to 0$. The first part of this conjecture (boundedness) was later rigorously proved, see below. The convergence to a limit is still an open problem.

In the 2-D case, the only positive Lyapunov exponent coincides with the entropy. For multi-dimensional periodic Lorentz gases with a single spherical scatterer of radius r, individual positive Lyapunov exponents for the billiard ball map T have been studied in [BD]. It was estimated that every positive Lyapunov exponent $\chi_i > 0$, as a function of r, increases like const$\cdot|\ln r|$, as $r \to 0$. Moreover, every positive exponent but the maximal one was conjectured to be $\approx -1/4 \ln(r/2)$. The maximal Lyapunov exponent was conjectured to be $\approx -(3d+2)/4 \ln r$. The last two conjectures turn out to be wrong, see (2.9) and (2.10) below. The first one is correct, see (2.10) below.

P. Baldwin [B] gave a theoretical argument supporting the following sharpening of the formula (2.1):

$$h(T) = -2 \ln r + \text{const} + O(r) \tag{2.4}$$

His argument is not a mathematical proof, and so his prediction still remains an open problem.

The following theorem was rigorously proved by Chernov.

Theorem 2.1 ([C2]) *The entropy of the d-dimensional periodic Lorentz gas ($d \geq 2$) with a single spherical scatterer of radius $r > 0$ in a unit torus is given by*

$$h(T) = -d(d-1)\,\ln r + O(1) \tag{2.5}$$

and

$$h(\Phi^t) = -d(d-1)\,|B^{d-1}|\,r^{d-1}\,\ln r + O(r^{d-1}) \tag{2.6}$$

as $r \to 0$. The mean free path is given by

$$\bar{\tau} = \frac{1 - |B^d|\,r^d}{|B^{d-1}|\,r^{d-1}} = \frac{1}{|B^{d-1}\,|r^{d-1}} + O(r) \tag{2.7}$$

Here $|B^k|$ is the volume of the k-dimensional unit ball, see (3.4) below. The difference (2.3) is always positive and uniformly bounded in r for every d.

The proof in [C2] is based on the approximation of the operator B_x^+ in (1.9) by $\Theta_1 U_1^{-1}$, see (1.8). The norm of the error is bounded

$$||B_x^+ - \Theta_1 U_1^{-1}|| \leq 1/\tau_1 \leq \text{const}$$

cf. Proposition 1.4. Therefore, the substitution of $\Theta_1 U_1^{-1}$ for B_x^+ in (1.9) can only change the integral in (1.9) by a uniformly bounded amount.

Next, for small r the operator Θ_1 has eigenvalues of order r^{-1}, which can be found by an elementary calculation for spherical scatterers, details may be found in [C2]. As a result, the integration in (1.9) gives

$$h(T) = (d-1)\left(-\ln r + \int_\Omega \ln \tau(x)\,d\nu(x)\right) + H(d) + o(1) \tag{2.8}$$

The term $H(d)$ here comes from the substitution of $\Theta_1 U_1^{-1}$ for B_x^+ in (1.9). Its value was computed explicitly in [C2]: $H(2) = 2$, $H(3) = \ln 4$, and for $d \geq 4$ we have

$$H(d) = (d-1)\ln 2 - (d-3)\,|S^{d-2}|\int_0^1 t^{d-2}\ln\sqrt{1-t^2}\,dt$$

Here $|S^k|$ is the k-dimensional volume of the unit sphere S^k in $\mathbb{R}^{k+1}$, see (3.3) below.

Lastly, the boundedness of (2.3) that was proved in [C2] gives (2.5). The estimate (2.6) then follows from (1.1) and (2.7). The formula (2.7) is quite elementary, see ??? below.

It also follows from (2.8) that the existence of the limit of the quantity (2.3) is equivalent to the following asymptotic formula:

$$h(T) = -d(d-1)\ln r + \text{const} + o(1)$$

Both remain, however, open questions, as well as the more refined prediction (2.4).

All the open questions involving the entropy $h(T)$ can be equivalently restated for the entropy $h(\Phi^t)$, in view of (1.1) and (2.7).

As for the Lyapunov exponents for T, it follows directly from (2.5) that the maximal one is bounded by

$$-d \ln r + O(1) \le \chi_{\max} \le -d(d-1) \ln r + O(1) \tag{2.9}$$

By using again the approximation of B_x^+ by $\Theta_1 U_1^{-1}$, and the asymptotic eigenvalues of the latter, see [C2] for details, it is easy to estimate *every* positive Lyapunov exponent from below: $\chi_i \ge -d \ln r + O(1)$. Together with (2.5) this gives an asymptotic formula

$$\chi_i = -d \ln r + O(1) \tag{2.10}$$

for every positive Lyapunov exponent $\chi_i > 0$.

Therefore, all positive Lyapunov exponents have the same asymptotics as $r \to 0$. It was also conjectured in [C2] that all the positive Lyapunov exponents should be actually equal. This conjecture is still open. However, it was shown recently [DP, LBD], both analytically and numerically, that in a 3-D *random* Lorentz gas (with a random configuration of scatterers) the two positive Lyapunov exponents are distinct!

Two more general results were proved in [C2].

Consider a periodic Lorentz gas with m disjoint spherical scatterers with radii $r_1, \dots, r_m$ in a unit torus. Put

$$Z_0 = r_1^{d-1} + \cdots + r_m^{d-1}$$

and

$$Z_1 = r_1^{d-1} \ln r_1 + \cdots + r_m^{d-1} \ln r_m$$

The entropy of such a Lorentz gas was proved [C2] to be

$$h(T) = -(d-1)[\ln Z_0 + Z_1/Z_0] + O(1) \tag{2.11}$$

and

$$h(\Phi^t) = -(d-1)\, |B^{d-1}|\, [Z_0 \ln Z_0 + Z_1] + O(Z_0)$$

as $r_1, \dots, r_m \to 0$, while the distances between the scatterers remain bounded away from 0. The mean free path is

$$\bar{\tau} = \frac{1}{|B^{d-1}|\, Z_0} + O(\max r_i)$$

Lastly, consider a periodic Lorentz gas with m disjoint convex scatterers in a unit torus, which are homotetically shrinking with a common scaling factor $\varepsilon > 0$. Let S_1 be the total surface area and V_1 the total volume of the scatterers when $\varepsilon = 1$. Then we have [C2]

$$h(T) = -d(d-1)\, \ln \varepsilon + O(1)$$

and

$$h(\Phi^t) = -d(d-1)\, |B^{d-1}|\, |S^{d-1}|^{-1} S_1 \varepsilon^{d-1} \ln \varepsilon + O(\varepsilon^{d-1})$$

as $\varepsilon \to 0$.

3 Mean free path

Recall that the mean free path

$$\bar{\tau} = \int_\Omega \tau(x)\, d\nu \tag{3.1}$$

relates the entropies of the map T and the flow Φ^t by (1.1). It is interesting that the mean free path can be exactly computed in terms of the geometric characteristics of the Lorentz gas:

$$\bar{\tau} = \frac{|Q| \cdot |S^{d-1}|}{|\partial Q| \cdot |B^{d-1}|} \tag{3.2}$$

Here $|Q|$ is the d-dimensional volume of the domain Q available to the moving particle, $|\partial Q|$ is the $(d-1)$-dimensional area of the boundary of Q. Also,

$$|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{3.3}$$

is the $(d-1)$-dimensional volume of the unit sphere in $\mathbb{R}^d$. Here $\Gamma(x)$ is the gamma function, $\Gamma(n+1) = n!$, $\Gamma(x+1) = x\Gamma(x)$, and $\Gamma(1/2) = \sqrt{\pi}$. Lastly,

$$|B^{d-1}| = |S^{d-2}|/(d-1) \tag{3.4}$$

is the volume of the unit ball in $\mathbb{R}^{d-1}$.

It is also interesting that the expression (3.2) holds for *any* billiard system, in *any* dimension. In particular, for planar billiard tables, $d = 2$, we have

$$\bar{\tau} = \frac{\pi\,|Q|}{|\partial Q|} \tag{3.5}$$

and for 3-D billiard tables we have

$$\bar{\tau} = \frac{4\,|Q|}{|\partial Q|} \tag{3.6}$$

The formula (3.2) follows by a simple calculation involving the invariant measure μ of the flow Φ^t and the invariant measure ν of the map T, see [C3].

The formulas (3.2)-(3.6) are known in integral geometry and geometric probability, see, e.g., Eq. (4-3-4) in [M]. Eq. (3.5) is often referred to as Santalo's formula, since it is given in Santalo's book [Sa].

Next we discuss the free path in the system of hard balls. It is related to the mean intercollision time, one of the basic characteristics of the gases of hard balls.

Consider a gas of hard balls in $\mathbb{R}^k$ of diameter σ and unit mass, mean number density n (the average number of balls per unit volume) and the mean temperature T. The temperature is related to the mean kinetic energy by the classical formulas

$$E = k_B T \quad (k=2) \qquad \text{and} \qquad E = \frac{3}{2} k_B T \quad (k=3)$$

Instead of the mean number density, one can use the mean "volume density" (the fraction of volume occupied by the balls):

$$\rho = |B^k| \cdot \sigma^k n$$

In physics, the classical Boltzmann mean free time formulas [CC, EW]:

$$\bar{t}_{\rm Boltz}(k=2) = \frac{\pi^{1/2}\,\sigma}{8\,E^{1/2}\,\rho} = \frac{1}{2\sigma\,n\,\sqrt{\pi k_B T}} \tag{3.7}$$

and

$$\bar{t}_{\rm Boltz}(k=3) = \frac{\pi^{1/2}\,\sigma}{8\,(6E)^{1/2}\,\rho} = \frac{1}{(2\sigma)^2\,n\,\sqrt{\pi k_B T}} \tag{3.8}$$

give the mean free time $\bar{t}$ between successive collisions for each ball, on the average. The Boltzmann formulas hold in the so called *dilute* mode (or gas mode) when $n \to 0$. For larger densities (dense mode, or fluid mode), there is classical Enskog's correction to the Boltzmann formula, which we give only in the 2-D case:

$$\bar{t}_{\rm Enskog}(k=2) = \frac{\pi^{1/2}\,\sigma}{8\,E^{1/2}\,\rho\,\chi} = \frac{1}{2\sigma\,n\,\chi\,\sqrt{\pi k_B T}} \tag{3.9}$$

see, e.g., [Gas], where χ is the Enskog scaling factor

$$\chi \approx 1 + 0.782 \cdot 2\rho + 0.5327 \cdot (2\rho)^2$$

see, e.g., [CL].

It is remarkable that the Boltzmann equation can be derived mathematically from the billiard free path formula (3.2). This was done in [C3] in the following setup.

Consider a system of N hard balls of diameter σ and unit mass in the k-dimensional torus $\mathbf{T}_L^k$ whose linear dimension is $L > 0$. The k-dimensional volume of the torus $\mathbf{T}_L^k$ is L^k. The number density is $n = N/L^k$, and the "volume density" is $\rho = |B^k|\sigma^k n$.

The balls move freely and collide with each other elastically. Let $q_{i,1}, \dots, q_{i,k}$ and $p_{i,1}, \dots, p_{i,k}$ be the coordinates of the position and velocity vector, respectively, of the ith ball. The configuration space Q of the system is a subset of the kN-dimensional torus $\mathbf{T}_L^{kN}$, which correspond to all feasible (nonoverlapping) positions of the balls. The total kinetic energy of the system is preserved in time, and we fix it: $p_{1,1}^2 + \cdots + p_{N,k}^2 = 2EN$, where the constant $E > 0$ is the mean kinetic energy per particle. The phase space is then $M = Q \times S_1^{kN-1}$ where S_1^{kN-1} is the $(kN-1)$-dimensional sphere of radius $(2EN)^{1/2}$.

The dynamics of the hard balls with elastic collisions correspond to the billiard dynamics in the configuration space Q with specular reflections at the boundary ∂Q. The billiard particle in Q will move at the speed $(2EN)^{1/2}$ rather than the conventional unit speed. The boundary ∂Q consists of $N(N-1)/2$ cylindrical surfaces corresponding to the pairwise collisions of the balls. Denote by $C_{i,j}$ the open solid cylinder corresponding to overlapping positions of the balls $i \neq j$. It is given by the inequality

$$\sum_{r=1}^{k} (q_{i,r} - q_{j,r})^2 < \sigma^2 \quad (\text{mod } L)$$

The configuration space is then $Q = \mathbf{T}_L^{kN} \setminus \cup_{i\neq j} C_{i,j}$, and its boundary is $\partial Q = Q \cap (\cup_{i\neq j} \partial C_{i,j})$.

In order to estimate the mean free path by using Eq. (3.2) one needs to compute the volume $|Q|$ of the space Q and the surface area $|\partial Q|$ of its boundary ∂Q. This is a difficult problem, very hard to solve exactly, since the cylinders $C_{i,j}$ have plenty of pairwise and multiple intersections. However, one can find the asymptotic values of both $|Q|$ and $|\partial Q|$ at very low densities, as $n \to 0$.

Some simple calculations [C3] yield

$$|Q| = L^{kN}(1 - o(1))$$

$$|\partial Q| = \frac{N(N-1)}{2} \cdot |\partial C_{1,2}| \cdot (1 - o(1))$$

A little trickier is the estimation of $|\partial C_{1.2}|$. Certain geometric considerations [C3] yield

$$|\partial C_{1,2}| = \sqrt{2}\, \sigma^{k-1} \cdot |S^{k-1}| \cdot L^{kN-k}\, (1 + o(1))$$

This gives the following:

$$|\partial Q| = \frac{N-1}{\sqrt{2}} \cdot \frac{2^k\, k\rho}{\sigma} \cdot L^{kN}\, (1 + o(1))$$

Now, according to (3.2), the mean free path of the billiard particle in the domain Q is

$$\begin{aligned} \bar{\tau} &= \frac{|Q| \cdot |S^{kN-1}| \cdot (kN-1)}{|\partial Q| \cdot |S^{kN-2}|} \\ &= \frac{\sqrt{2}\, \sigma(kN-1) \cdot |S^{kN-1}|}{2^k\, k\rho(N-1) \cdot |S^{kN-2}|} \cdot (1 + o(1)) \qquad (3.10) \end{aligned}$$

Now comes a somewhat surprizing observation. First, the billiard system in Q is not ergodic. Indeed, the total momentum $\mathbf{P} = (P_1, \dots, P_k)$, where $P_r = \sum_i p_{i,r}$, is invariant under the dynamics. Those phase trajectories whose total momentum $\mathbf{P}$ is large will display slow relative motion of the balls, and thus the mean free path between reflections in ∂Q along such trajectories will be larger than $\bar{\tau}$ in (3.10). On the contrary, the mean free path along trajectories with zero or small $\mathbf{P}$ will be below $\bar{\tau}$. The value of $\bar{\tau}$ in (3.10) only gives the phase space average of the mean free paths taken over individual trajectories.

Physically interesting regime is the one at equilibrium, where the total momentum is zero, $\mathbf{P} = \mathbf{0}$. Let $\bar{\tau}_0$ denote the mean free path on the surface $\mathbf{P} = \mathbf{0}$ in the phase space. A little more computation [C3] gives

$$\begin{aligned} \bar{\tau}_0 &= \frac{\sqrt{2}\,\sigma\,(kN-k-1)\cdot|S^{kN-k-1}|}{2^k\,k\,\rho\,(N-1)\cdot|S^{kN-k-2}|}\cdot(1+o(1)) \\ &= \frac{\sqrt{2\pi}\,\sigma\cdot\Gamma\left(\frac{kN-k+1}{2}\right)}{2^k\,\rho\cdot\Gamma\left(\frac{kN-k+2}{2}\right)}\cdot(1+o(1)) \end{aligned}$$

One can 'translate' this result into the physically sensible mean free time $\bar{t}$ as follows. The speed of the billiard particle in Q is $(2EN)^{1/2}$, and so the mean intercollision time (in the whole system) is $\bar{t}_{\rm sys} = \bar{\tau}_0\,(2EN)^{-1/2}$. The mean intercollision time for every individual particle is simply $\bar{t}_{\rm par} = \bar{t}_{\rm sys}\cdot N/2$, since every collision involves two particles. This gives

$$\bar{t}_{\rm par} = \frac{\pi^{1/2}\cdot\Gamma\left(\frac{kN-k+1}{2}\right)\cdot N\sigma}{2^{k+1}\cdot\Gamma\left(\frac{kN-k+2}{2}\right)\cdot(EN)^{1/2}\,\rho}\cdot(1+o(1)) \qquad (3.11)$$

Now taking the limit in (3.11) as $N\to\infty$ and using a handy formula $\Gamma(N)/\Gamma(N-1/2) = \sqrt{N}(1+o(1))$ yields

$$\bar{t}_{\rm par}(N\to\infty) = \frac{\pi^{1/2}\,\sigma}{2^{k+1}\,(Ek/2)^{1/2}\,\rho}\cdot(1+o(1))$$

In particular, for $k = 2$ and $k = 3$ we recover the Boltzmann mean free time for hard disks and hard balls (3.7) and (3.8).

4 Entropy of infinite gases

Estimation of the entropy and Lyapunov exponents for systems of hard balls has been always difficult, on both numerical and theoretical levels. Relatively little is proved rigorously, and the issue is still pretty much open. For recent estimates based on kinetic theory and numerical experiments we refer the reader to the survey [BZD].

One interesting theoretical estimate of the entropy for a system of two hard disks on a torus was proved by Wojtkowski in 1988 [W]. He showed that as the disks are so large that they always nearly contact each other the entropy of the flow approaches infinity.

Here we concentrate on the entropy of infinite systems of particles. We describe three rare mathematically proven theorems in this direction. First we need to describe basic facts about infinite systems. We avoid some technicalities here, a complete account of the issue may be found in [SC].

Infinite particle systems. We will consider infinitely many particles in $\mathbb{R}^d$ interacting via a pair potential $U(||q-q'||)$ where $q, q' \in \mathbb{R}^d$ are the centers of the interacting particles. The potential U has hard core, i.e. $U(r) = \infty$ for $0 < r \leq r_0$ and finite range, i.e. $U(r) \equiv 0$ for $r \geq r_1$. Here $0 < r_0 \leq r_1$ are some parameters. If $r_0 < r_1$, then for $r_0 < r < r_1$ the potential $U(r)$ must satisfy certain conditions of regularity and smoothness [SC]. This model somewhat generalizes the model of hard balls, which corresponds to the case $r_0 = r_1$.

The configuration space of an infinite system consists of countable subsets $Q_\infty \subset \mathbb{R}^d$ such that $||q - q'|| \geq 2r_0$ for every $q \neq q' \in Q_\infty$. The phase space M_∞ consists of pairs $X = (Q_\infty, P_\infty)$ where Q_∞ is a configuration and P_∞ is a $\mathbb{R}^d$-valued function on Q_∞. The value $p = p(q)$ for $q \in Q_\infty$ is the momentum of the particle at q.

The definition of dynamics on M_∞ is not a trivial task. For systems with potential, one might run into unsolvable problems of integrating infinitely many coupled differential equations. Even for hard balls, some weird developments may occur. For example, the system may "collapse" when infinitely many balls with arbitrary large velocities are coming down "from infinity" into a finite domain of $\mathbb{R}^d$, where they experience infinitely many collisions on a finite interval of time.

The formal definition of dynamics requires a special construction. Let V_n be a sequence of increasing cubes in $\mathbb{R}^d$ with a common center and parallel faces such that $\cup_n V_n = \mathbb{R}^d$. For every $X = (P_\infty, Q_\infty) \in M_\infty$ and each V_n we define a special dynamics $\Phi^t_{V_n}(X)$ as follows. We freeze the particles outside V_n and those whose hard core intersects ∂V_n. Hence, only the particles $x = (q, v)$ with

$$q \in V_n \setminus B_{r_0}(\partial V_n)$$

can move. We regard the boundary ∂V_n as consisting of rigid walls at which the moving particles inside V_n bounce off elastically, as hard balls of radius r_0.

Observe that since the potential U has a finite range, the moving particles inside V_n only feel the influence of finitely many frozen particles outside V_n. Therefore, the dynamics $\Phi^t_{V_n}(X)$ is well defined for every $X \in M_\infty$. The flow $\Phi^t_{V_n}$ is called a *partial flow* (or partial dynamics) in the cube V_n. For each $x = (q, v) \in X$ denote by $x_n(t)$ the trajectory of x in the partial dynamics $\Phi^t_{V_n}$.

Key assumption. Assume that for each $x \in X$ and $s > 0$ there is an $n_0 = n_0(x, s)$ such that the trajectory $x_n(t)$ for $|t| < s$ is the same for all $n > n_0$.

If the Key Assumption holds, then the trajectory $x(t)$ of every particle $x \in X$ for all $t \in \mathbb{R}$ is well defined by simply taking the limit of $x_n(t)$ as $n \to \infty$. We denote by Φ^t the resulting dynamics on the part of M_∞ where the Key Assumption is satisfied.

Gibbs measures. Next, we define a family of the so called Gibbs measures $\mu_{\lambda,e}$ on the phase space M_∞. Consider again the sequence of cubes $V_n \to \mathbb{R}^d$, and in each V_n a finite system of N_n particles with the total energy E_n. Assume that the walls of the container V_n are rigid again, so that the particles in V_n bounce off ∂V_n elastically. The dynamics in V_n preserves the total energy E_n and the Liouville measure μ_n on the surface of constant energy (this measure is called the microcanonical distribution).

Consider a sequence of such finite systems so that $N_n/V_n \to \lambda > 0$ and $E_n/N_n \to e > 0$ as $n \to \infty$. The parameters λ and e characterize the mean number density and the mean kinetic energy per particle, respectively. The limit as $n \to \infty$ is called a thermodynamic limit. The weak limit of the sequence of measures μ_n (if one exists) is a measure $\mu_{\lambda,e}$ on M_∞ called the Gibbs measure.

Theorem 4.1 (Sinai [S2]) *If the potential U satisfies certain regularity and smoothness assumptions and the mean number density λ is low enough, then*
(a) *the Gibbs measure $\mu_{\lambda,e}$ exists;*
(b) *the set of phase points $X \in M_\infty$ satisfying the Key Assumption has full $\mu_{\lambda,e}$-measure, i.e. the dynamics is $\mu_{\lambda,e}$-almost everywhere defined;*
(c) *the measure $\mu_{\lambda,e}$ is preserved under the dynamics Φ^t. It is also preserved under the d-dimensional group of space translations.*

In addition, the Gibbs measures $\mu_{\lambda,e}$ are invariant under the partial dynamics $\Phi^t_{V_n}$ for each cube V_n.

The proof of the theorem is based on the construction of the so called *cluster dynamics*. Let $r > r_1$. For any configuration Q_∞ consider the union of balls of radius r centered at all the points $q \in Q_\infty$. A connected component of that union is called an r-cluster. Sinai proved that with $\mu_{\lambda,e}$-probability one, each particle $x = (q,p)$ belongs in a finite r-cluster that does not interact with any other cluster during a certain interval of time. Of course, within a finite cluster the dynamics is well defined. This observation allows the construction of the dynamics in the entire system.

For infinite systems of hard balls the above existence theorem was proved by Alexander [Al], without restrictions on the density λ. In this case the Gibbs measure $\mu_{\lambda,e}$ can be characterized more explicitly:

1. For $\mu_{\lambda,e}$-almost every configuration Q_∞ the conditional distribution on the space of momenta $p \in P_\infty$ is a direct product of Gaussian distributions with density

$$\left(\frac{\beta}{2\pi}\right)^{d/2} e^{-\frac{1}{2}\beta||v_q||^2}$$

where $\beta = d/(2e)$.

2. The marginal distribution on the space of configurations is a d-dimensional Poisson measure with density λ. This means that for any bounded set $B \subset \mathbb{R}^d$ the number of points $q \in Q_\infty \cap B$ is a Poisson random variable with parameter $\lambda \cdot \mathrm{Vol}(B)$.

The parameter β is related to the temperature T by $\beta = (k_B T)^{-1}$, where k_B is Boltzmann's constant. The temperature is then related to the mean kinetic energy by $e = (d/2)k_B T$.

Space-time translation group. Consider the d-dimensional group S^u, $u \in \mathbb{R}^d$, of space translations on M_∞. The translation S_u shifts all the particles by the vector u and leaves their momenta unchanged. Space translations commute with the dynamics

Φ^t and together they generate a $(d+1)$-dimensional abelian group $\Gamma^{t,u} = \Phi^t \circ S^u$ on M_∞ of space-time translations.

The Gibbs measure $\mu_{\lambda,e}$ is invariant under the group $\Gamma^{t,u}$. Denote by $h_{\lambda,e}(\Gamma^{t,u})$ the measure-theoretic entropy of the group $\Gamma^{t,u}$ with respect to the measure $\mu_{\lambda,e}$. For the definition and basic properties of the entropy of multidimensional groups of measure-preserving transformations see Conze [Co]. One can consider $h_{\lambda,e}(\Gamma^{t,u})$ as the natural entropy characteristic of the Gibbs measure $\mu_{\lambda,e}$, it is called the *space-time entropy*.

The following estimate for the space-time entropy was proved by Chernov.

Theorem 4.2 ([C1]) *Assume that the density λ is low enough, i.e. $\lambda < \lambda_0(e)$ for some $\lambda_0(e) > 0$ (the system is in a dilute mode). Then the space-time entropy $h_{\lambda,e}(\Gamma^{t,u})$ is finite and satisfies the following estimate:*

$$h_{\lambda,e}(\Gamma^{t,u}) < \lambda \cdot \text{const}(e)$$

The proof is based on Sinai's construction of cluster dynamics.

We note that this theorem does not ensure that $h_{\lambda,e}(\Gamma^{t,u}) > 0$, even though this seems very likely. For now, this remains an open problem.

Lyapunov spectrum. The second theorem due to Sinai deals with the Lyapunov spectrum of infinite systems of particles. In order to state the result we need to describe an algorithm for computation of Lyapunov exponents for finite systems.

Let $M = M_{V,N}$ be the phase space of a system of N particles in a cube V with rigid walls (with no restrictions on the energy so far). Let $\mathcal{T}_X(M)$ denote the tangent space to M at a point $X \in M$. Denote the dynamics on M by Ψ_V^t. It generates the family of Jacobi maps (derivatives of Ψ_V^t)

$$J_X^t : \mathcal{T}_X^{(1)}(M) \to \mathcal{T}_{\Psi_V^t X}^{(1)}(M)$$

Denote by $\mathcal{T}_X^{(k)}(M)$ the kth exterior power of $\mathcal{T}_X^{(1)}(M)$. It is the space of all exterior products $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ where $e_i \in \mathcal{T}_X^{(1)}(M)$. The Jacobi maps J_X^t genetare the maps

$$J_X^t(k) : \mathcal{T}_X^{(k)}(M) \to \mathcal{T}_{\Psi_V^t X}^{(k)}(M)$$

Now let us fix the total energy E. Then the dynamics Ψ_V^t restricted to the energy surface $M_{V,N,E} \subset M_{V,N}$ preserves the Liouville measure $\mu_{V,N,E}$ (the microcanonical distribution). This measure has $m = 2dN - 1$ Lyapunov exponents, which we write down in the decreasing order

$$\chi_1^{(N)} \geq \chi_2^{(N)} \geq \cdots \geq \chi_m^{(N)}$$

A version of an idea of Benettin et al. [BGGS] implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_{M_{V,N,E}} \ln \text{tr}\, J_X^t(k)\, [J_X^t(k)]^*\, d\mu_{V,N,E}(X) = 2 \sum_{i=1}^{k} \chi_i^{(N)} \tag{4.1}$$

Here $*$ denotes the adjoint transformation.

Now, consider the thermodynamic limit as $N \to \infty$, $N/V \to \lambda > 0$ and $E/N \to e > 0$. The measure $\mu_{V,N,E}$ weakly converges to the Gibbs measure $\mu_{\lambda,e}$ on M_∞. We would like to characterize the Lyapunov spectrum of the Gibbs measure $\mu_{\lambda,e}$ by a function $\varphi(p) = \varphi_{\lambda,e}(p)$ for $0 < p < 2d$ such that

$$\lim_{N\to\infty} \chi^{(N)}_{[pN]} = \varphi(p) \tag{4.2}$$

(here $[pN]$ is the integral part of pN). The function φ would describe the distribution of Lyapunov exponents in many particle systems. Obviously, $\varphi(p)$ must be a decreasing function. Of course, the above formula (4.2) is just a conjecture at present.

In terms of the function $\varphi(p)$, we can state another conjecture:

$$\frac{1}{N}\sum_{i=1}^{[pN]} \chi_i^{(N)} \longrightarrow h(p) := \int_0^p \varphi(s)\,ds \tag{4.3}$$

as $N \to \infty$. We note that since $\varphi(p)$ is a decreasing function, $h(p)$ must be a concave function.

Substituting (4.1), we can rewrite (4.3) as

$$\lim_{N\to\infty} \frac{1}{N} \lim_{t\to\infty} \frac{1}{t} \int_{M_{V,N,E}} \ln \operatorname{tr} J_X^t([pN])\,(J_X^t([pN]))^*\, d\mu_{V,N,E}(X) = 2h(p) \tag{4.4}$$

Instead of proving (4.4) as such, Sinai argues as follows. It is not really finite systems that are physically interesting, but rather an infinite system of particles. So, the thermodynamic limit $N \to \infty$ should be taken first, and then the time limit $t \to \infty$. This would better fit the concept of a Lyapunov spectrum of the Gibbs measure $\mu_{\lambda,e}$. So, Sinai changes the order in which the limits are taken and conjectures that

$$\lim_{t\to\infty} \frac{1}{t} \lim_{N\to\infty} \frac{1}{N} \int_{M_{V,N,E}} \ln \operatorname{tr} J_X^t([pN])\,(J_X^t([pN]))^*\, d\mu_{V,N,E}(X) = 2h(p) \tag{4.5}$$

Furthermore, since we now take the limit $N \to \infty$ first, we can as well replace the finite dynamics Ψ_V^t in V by the partial dynamics Φ_V^t in the same cube V as defined earlier. Accordingly, the maps $J_X^t(k)$ must be defined in terms of Φ_V^t, and X be a point in M_∞. This is yet another step closer to working directly with an infinite system.

Theorem 4.3 (Sinai [S5]) *Let $\mu_{\lambda,e}$ be a Gibbs measure on M_∞. Assume that the density λ is low enough and the temperature (i.e. the mean energy e) is high enough. Then for every $t > 0$ and $\mu_{\lambda,e}$-almost every point $X \in M_\infty$ there exists*

$$\lim_{V\to\mathbb{R}^d} \frac{1}{\lambda\cdot \operatorname{Vol} V} \ln \operatorname{tr} J_X^t([pN])\,(J_X^t([pN]))^* = 2h_t(p)$$

where $h_t(p)$ is independent of X. Furthermore, there exists

$$\lim_{t\to\infty} \frac{1}{t} h_t(p) = h(p)$$

The function $h(p)$ is continuous and concave.

The proof of the theorem is based on the cluster dynamics constructed in the earlier paper by Sinai [S2].

The entire function $h(p)$ can be regarded as an entropy-like characteristic of the Gibbs measure $\mu_{\lambda,e}$. But particularly important is its maximum

$$h_{\max} = \max_p h(p)$$

Note that $h_{\max} = h(p_0)$ where p_0 is selected so that $\varphi(p_0) = 0$. Hence, $h_{\max}$ corresponds to “the sum of all positive Lyapunov exponents”. The following generalization of Pesin’s identity (1.4) was also proposed by Sinai [S5]:

Sinai’s conjecture. The value $h_{\max}$ coincides with the space-time entropy of the Gibbs measure $\mu_{\lambda,e}$:

$$h_{\max} = h_{\lambda,e}(\Gamma^{t,u})$$

Another intriguing question is the asymptotic behavior of the largest Lyapunov exponent χ_1 as $V \to \mathbb{R}^d$, either for the finite dynamics Ψ_V^t or the partial dynamics Φ_V^t. Note that the value of the function $\varphi(p)$ at $p = 0$ only gives a lower bound for χ_1. Sinai remarks in [S5] that the largest exponent χ_1 either remains bounded or grows slowly (e.g., logarithmically) with the volume of V. Numerical estimates of χ_1 indicate a very slow growth, but do not rule out the boundedness of χ_1.

Entropy per particle. Now we turn to the third, earlier theorem by Sinai and Chernov [SC]. It deals with another entropy-like characteristic of an infinite gas of hard balls.

Let $\mu_{\lambda,e}$ be a Gibbs measure on M_∞. Pick a sequence of cubes V_n with a common center and parallel faces and once again consider the partial dynamics $\Phi_{V_n}^t$. Note that under $\Phi_{V_n}^t$ the balls inside V_n move freely, collide with each other and bounce off the walls of the cube V_n and the frozen balls sticking out of the walls (those balls act like bumps).

For each n the partial dynamics $\Phi_{V_n}^t$ preserve the Gibbs measure $\mu_{\lambda,e}$. It is not ergodic, though, for the number of moving balls N, their total energy E, and the positions of the frozen balls are all the obvious integrals of motion. Fixing N, E and the positions of the frozen balls intersecting the walls (the balls outside V_n can be ignored altogether) yields a finite hard ball system in a container, though with somewhat peculiar boundary. The boundary is composed of the flat faces of V_n and the fragments of spherical surfaces of the frozen balls sticking out.

Let $h_{\lambda,e}(\Phi_{V_n}^t)$ be the entropy of the flow $\Phi_{V_n}^t$ with respect to the measure $\mu_{\lambda,e}$. It can be computed with the help of (1.10) as follows. For every cube V_n and phase point $X \in M_\infty$ let B_{X,V_n} be the operator defined in Section 1 for the trajectory of the phase point X under the dynamics $\Phi_{V_n}^t$. Of course, only the coordinates and momenta of the moving balls in V_n are included in the construction of B_{X,V_n}, the frozen balls are either a part of the boundary or ignored completely (if outside of V_n). Integrating the equation (1.10) over the phase space M_∞ gives

$$h_{\lambda,e}(\Phi_{V_n}^t) = \int_{M_\infty} \operatorname{tr} B_{X,V_n} \, d\mu_{\lambda,e} \tag{4.6}$$

Clearly, the entropy $h_{\lambda,e}$ increases as the cube V_n grows, because more and more moving balls are captured in the cube V_n. We are interested in the *entropy per unit volume*

$$\frac{1}{\operatorname{Vol} V_n} h_{\lambda,e}(\Phi^t_{V_n})$$

The related quantity $(\lambda \operatorname{Vol} V_n)^{-1} h_{\lambda,e}(V_n)$ can be called the *entropy per particle*.

Theorem 4.4 ([SC]) *Let the cubes V_n have sides $L_n = 2^n L_0$, where $L_0 > 0$ is a constant. Assume that the density is low enough, i.e. $\lambda < \lambda_0(e)$ for some $\lambda_0(e) > 0$ (the system is in a dilute mode). Then there is an $h = h(\lambda, e) > 0$ such that*
(a) *We have*

$$\lim_{n\to\infty} \frac{1}{\operatorname{Vol} V_n} h_{\lambda,e}(\Phi^t_{V_n}) = h$$

(b) *For $\mu_{\lambda,e}$-almost every phase point $X \in M_\infty$*

$$\lim_{n\to\infty} \frac{1}{\operatorname{Vol} V_n} \operatorname{tr} B_{X,V} = h$$

A weaker version of this theorem was obtained by Sinai in 1978 [S3], where he proved that $\liminf (\operatorname{Vol} V_n)^{-1} h_{\lambda,e}(\Phi^t_{V_n}) > 0$.

Sinai and Chernov conjectured that the quantity $h = h(\lambda, e)$ actually coincides with the space-time entropy:

$$h(\lambda, e) = h_{\lambda,e}(\Gamma^{t,u})$$

If this is true, it would imply that $h_{\lambda,e}(\Gamma^{t,u}) > 0$ solving the open problem stated after Theorem 4.2. If this is not true, then $h_{\lambda,e}$ can be regarded as yet another entropy-like characteristic of the Gibbs measure $\mu_{\lambda,e}$. It would be interesting to further investigate its properties, in particular its asymptotics as $\lambda \to 0$.

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