Statistical properties of piecewise smooth hyperbolic systems in high dimensions

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Abstract

We study smooth hyperbolic systems with singularities and their SRB measures. Here we assume that the singularities are submanifolds, the hyperbolicity is uniform aside from the singularities, and one-sided derivatives exist on the singularities. We prove that the ergodic SRB measures exist, are finitely many, and mixing SRB measures enjoy exponential decay of correlations and a central limit theorem. These properties have been proved previously only for two-dimensional systems.

Keywords: Decay of correlations, Sinai-Ruelle-Bowen measures, hyperbolic dynamics.

1 Introduction

Let M be an open connected domain in a d-dimensional C^∞ Riemannian manifold, such that $\bar{M}$ is compact, and let $\Gamma \subset \bar{M}$ be a closed subset. Assume that $\mathcal{S} := \Gamma \cup \partial M$ is a finite union of smooth compact submanifolds of codimension one, possibly with boundary. We denote by $\mathcal{S}_1, \mathcal{S}_2, \dots \mathcal{S}_r$ the smooth components of $\mathcal{S}$. We consider a map $T: M \setminus \mathcal{S} \to M$ such that

(H1) T is a C^2 diffeomorphism of $M \setminus \mathcal{S}$ onto its image. We also assume that T and T^{-1} are twice differentiable up to the boundaries of their domains (only one-sided derivatives are required at the boundary).

The set $\mathcal{S}$ will be referred to as the singularity set for T. For $n \geq 1$ denote by

$$\mathcal{S}^{(n)} = \mathcal{S} \cup T^{-1}\mathcal{S} \cup \cdots \cup T^{-n+1}\mathcal{S}$$

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the singularity set for T^n. Define

$$M^+ = \{x \in M : T^n x \notin \mathcal{S},\, n \geq 0\}, \qquad M^- = \cap_{n\geq 0} T^n(M \setminus \mathcal{S}^{(n)})$$

and

$$M^0 = \cap_{n\geq 0} T^n(M^+) = M^+ \cap M^-$$

The sets M^+ and M^- consist, respectively, of points were all the future and past iterations of T are defined, and M^0 is the set of points where all the iterations of T are defined. We denote by ρ the Riemannian metric in M and by $\mathrm{Vol}(\cdot)$ the Lebesgue measure (volume) in M.

Three most interesting classes of maps are

1. **Conservative case:** volume-preserving maps, or, more generally, maps with an absolutely continuous invariant measure (a.c.i.m.).

2. **Dissipative case:** no a.c.i.m. exist, yet $T(M \setminus \mathcal{S})$ is dense in M. In this case, like in the previous one, M^0 has full Lebesgue measure.

3. **Attractor case:** when the closure of $T(M \setminus \mathcal{S})$ is a proper subset of M. In this case $\mathrm{Vol}(M^0)$ $<$Vol(M), and often $\mathrm{Vol}(M^0) = 0$. The set $\overline{M^0}$ is then called an attractor.

Below we list our additional assumptions on T.

(H2) T is uniformly hyperbolic, i.e. there exist two families of cones C^u_x and C^s_x in the tangent spaces $\mathcal{T}_x M$, $x \in \bar{M}$, such that $DT(C^u_x) \subset C^u_{Tx}$ and $DT(C^s_x) \supset C^s_{Tx}$ whenever DT exists, and

$$|DT(v)| \geq \Lambda_{\min}|v| \quad \forall v \in C^u_x$$
$$|DT^{-1}(v)| \geq \Lambda_{\min}|v| \quad \forall v \in C^s_x$$

with some constant $\Lambda_{\min} > 1$. These families of cones are continuous on $\bar{M}$ and the angle between C^u_x and C^s_x has a positive lower bound.

Technically, the families of cones $C^{u,s}_x$ are specified by two continuous families of linear subspaces $P^{u,s}_x \subset \mathcal{T}_x M$ such that $P^u_x \oplus P^s_x = \mathcal{T}_x M$, and two continuous functions $\alpha^{u,s}(x) > 0$. The cones $C^{u,s}_x$ are defined by

$$\angle(v, P^{u,s}_x) := \min_{w \in P^{u,s}_x} \angle(v, w) \leq \alpha^{u,s}(x) \qquad \forall v \in C^{u,s}_x$$

The angle between the cones C^u_x and C^s_x is set to $\min\{\angle(v, w) : v \in C^u_x, w \in C^s_x\}$. We denote $d_{u,s} = \dim P^{u,s}_x$ (these are independent of x, since $P^{u,s}_x$ are continuous and M is connected, and $d_u + d_s = d = \dim M$).

Denote $\Lambda_{\max} = \max\{\sup_x ||DT(x)||, \sup_x ||DT^{-1}||\}$. In plain words, $\Lambda_{\min}$ and $\Lambda_{\max}$ are lower and upper bounds on the expansion factor of unstable vectors and contraction factor of stable vectors.

For any submanifold $W \subset M$ we denote by ρ_W the metric on W induced by the Riemannian metric in M, and by ν_W the Lebesgue measure on W generated by ρ_W. We call U a u-manifold if it is a smooth d_u-dimensional submanifold in M of finite diameter (in the inner metric ρ_U) and at every $x \in U$ the tangent space $\mathcal{T}_x U$ lies in C^u_x. Any u-manifold is expanded (locally) by T in every direction by a factor between $\Lambda_{\min}$ and $\Lambda_{\max}$. Similarly, s-manifolds are defined.

(H3) The angle between $\mathcal{S}$ and C^u has a positive lower bound.

Technically, the angle between $\mathcal{S}$ and C^u_x at $x \in \mathcal{S}$ is defined to be $\max\{0, \angle(P^u_x, \mathcal{T}_x\mathcal{S}) - \alpha^u(x)\}$. Here $\angle(P^u_x, \mathcal{T}_x\mathcal{S}) = \max_{v \in P^u_x} \min_{w \in \mathcal{T}_x\mathcal{S}} \angle(v, w)$.

As a consequence of (H3), any u-manifold intersects $\mathcal{S}$ transversally, and the angle between them has a positive lower bound.

It is convenient to assume that for every $\mathcal{S}_i \subset \Gamma$ we have $\partial\mathcal{S}_i \subset \cup_{j \neq i} \mathrm{int}\mathcal{S}_j \cup \partial M$, i.e. every interior singularity manifold with boundary terminates on some other singularity manifolds or on ∂M. This is not a restrictive assumption, since if this is not the case for some $\mathcal{S}_i \subset \Gamma$, we can extend $\mathcal{S}_i$ until it terminates on other hypersurfaces of $\mathcal{S}$ or on the boundary of M.

A point $x \in \mathcal{S}^{(m)}$ of the singularity set $\mathcal{S}^{(m)}$ of T^m is said to be multiple if it belongs to $l \geq 2$ smooth components of $\mathcal{S}^{(m)}$, and then l is called the multiplicity of x in $\mathcal{S}^{(m)}$.

(H4) There are $K_0 \geq 1$ and $m \geq 1$ such that the multiplicity of any point $x \in \mathcal{S}^{(m)}$ does not exceed K_0, and $K_0 < \Lambda^m_{\min} - 1$.

This is a standard assumption which ensures that the singularity manifolds of $\mathcal{S}^{(m)}$ do not pile up too fast anywhere as m grows. The expansion of any u-manifold U under T^m is hereby guaranteed to be stronger than the cutting (shredding) of U inflicted by $\mathcal{S}^{(m)}$. We make this claim precise below in Section 2. The necessity of an assumption of this kind is explained in [13].

It is also standard to assume that $m = 1$ here, which we do, since we can simply consider T^m instead of T. (The assumptions (H1)-(H3) obviously hold for all T^m, $m \geq 1$.)

For any $x \in M^+$ and $y \in M^-$ we set

$$E^s_x = \cap_{n \geq 0} DT^{-n}(C^s_{T^n x}), \qquad E^u_y = \cap_{n \geq 0} DT^n(C^u_{T^{-n} y})$$

respectively. It is standard, see, e.g., [9], that

(a) E^s_x, E^u_x are linear subspaces in $\mathcal{T}_x M$, $\dim E^{u,s}_x = d_{u,s}$, and $E^s_x \oplus E^u_x = \mathcal{T}_x M$ for $x \in M^0$;

(b) $DT(E^{u,s}_x) = E^{u,s}_{Tx}$, and DT expands vectors in E^u_x and contracts vectors in E^s_x;

(c) the subspaces E^u_x and E^s_x are continuous in x (on M^- and M^+, respectively), and the angle between them on M^0 has a positive lower bound.

As a consequence, there can be no zero Lyapunov exponents on M^0. The space E^u_x is spanned by all vectors with positive Lyapunov exponents, and E^s_x by those with negative Lyapunov exponents.

We call a submanifold $W^u \subset M$ a local unstable manifold (LUM), if T^{-n} is defined and smooth on W^u for all $n \geq 0$, and $\forall x, y \in W^u$ we have $\rho(T^{-n}x, T^{-n}y) \to 0$ as $n \to \infty$

exponentially fast. Similarly, local stable manifolds (LSM), W^s, are defined. Obviously, $\dim W^{u,s} = d_{u,s}$, and at any point $x \in W^{u,s}$ the tangent space $\mathcal{T}_x W^{u,s}$ coincides with $E_x^{u,s}$. We denote by $W^u(x)$, $W^s(x)$ local unstable and stable manifolds containing x, respectively. The existence and abundance of LUM's and LSM's in M is proved in Sect. 3.

We state our main result, with necessary definitions following it.

Theorem 1.1 *Let T satisfy (H1)-(H4). Then*

(a) *Existence: T admits a Sinai-Ruelle-Bowen (SRB) measure μ;*

(b) *Ergodic properties: any SRB measure μ has a finite number of ergodic components, on each of which it is, up to a finite cycle, mixing and Bernoulli;*

(c) *Statistical properties: if (T^n, μ) is ergodic $\forall n \geq 1$, then (T, μ) has exponential decay of correlations and satisfies the central limit theorem for Hölder continuous functions on M.*

Definition 1. A T-invariant measure μ concentrated on M^0 is called a Sinai-Ruelle-Bowen (SRB) measure if the conditional measures of μ on local unstable manifolds are absolutely continuous with respect to the Lebesgue measures on those manifolds.

The part (b) of the theorem means that μ has a finite number of ergodic components $M_1^0, \ldots, M_s^0$, and on each M_i^0 the map $(T, \mu|M_i^0)$ either is mixing and Bernoulli, or else M_i^0 is further decomposed into a finite number of subcomponents $M_i^0 = M_{i,1}^0 \cup \cdots \cup M_{i,s_i}^0$ which are permuted cyclicly by T. In the latter case the map $(T^{s_i}, \mu|M_{i,j}^0)$ is mixing and Bernoulli for every $M_{i,j}^0$. The part (c) of the theorem applies to the dynamical system $(T^{s_i}, \mu|M_{i,j}^0)$ for each $M_{i,j}^0$. It is also standard that *any* SRB measure on M^0 is a weighted sum of (unique) ergodic SRB measures concentrated on the components $M_1^0, \ldots, M_s^0$. Thus, all the SRB measures for T make an s-dimensional simplex, whose vertices are ergodic SRB measures.

SRB measures are the only physically observable invariant measures for smooth or piecewise smooth hyperbolic dynamical systems. In the conservative case (Case 1 above), any a.c.i.m. is an SRB measure automatically, and vice versa. In the dissipative and attractor cases 2 and 3, SRB measures are weak Cesaro limits of iterations of smooth measures on M. Furthermore, for any ergodic SRB measure μ there is a positive volume set consisting of μ-generic points, i.e. points $x \in M$ such that $\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i(x) \to \int f\, d\mu$ for all continuous functions $f : M \to \mathbb{R}$, see, e.g., [13]. (This property is sometimes taken as the definition of SRB measures.) In the dissipative case 2, μ is typically singular, i.e. $\mathrm{Vol}(M^0) = 0$, but the support of μ can coincide with $\bar{M}$, i.e. the μ-measure of every open set may be positive. That happens to typical transitive Anosov diffeomorphisms [12] and some nonequilibrium stationary distributions studied in modern statistical physics [7]. In the attractor case (3), the support of μ normally has zero volume, the best studied examples here being Lorenz, Lozi and Belykh attractors [1].

Now, let $\mathcal{H}_\eta$ be the class of Hölder continuous functions on M with exponent $\eta > 0$:

$$\mathcal{H}_\eta = \{f : M \to \mathbb{R} \,|\, \exists C > 0 : \, |f(x) - f(y)| \leq C\rho(x,y)^\eta, \ \forall x, y \in M\}$$

Definition 2. We say that (T, μ) has exponential decay of correlations for Hölder continuous functions if $\forall \eta > 0$ $\exists \gamma = \gamma(\eta) \in (0, 1)$ such that $\forall f, g \in \mathcal{H}_\eta$ $\exists C = C(f, g) > 0$ such that

$$\left| \int_M (f \circ T^n) g \, d\mu - \int_M f \, d\mu \int_M g \, d\mu \right| \le C \gamma^{|n|} \qquad \forall n \in \mathbb{Z}$$

Definition 3. We say that (T, μ) satisfies a central limit theorem (CLT) for Hölder continuous functions if $\forall \eta > 0, f \in \mathcal{H}_\eta$, with $\int f \, d\mu = 0$, $\exists \sigma_f \ge 0$ such that

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f \circ T^i \xrightarrow{\text{distr}} \mathcal{N}(0, \sigma_f^2)$$

Furthermore, $\sigma_f = 0$ iff $f = g \circ T - g$ for some $g \in L^2(\mu)$

History. The study of SRB invariant measures for smooth hyperbolic dynamical systems with singularities was initiated by Pesin [9] who proved the existence of SRB measures under some general assumptions, see (H5)-(H6) in our Section 3. In general, SRB measures may have countably many ergodic components. Under some more restrictive assumptions, see (H5) and (H7), Sataev [10] showed that SRB measures have finitely many ergodic components (the so called finitude). We prove in Sect. 3 that our assumptions (H1)-(H4) imply (H5)-(H7), hence the existence and finitude of SRB measures. Such a verification was previously done only in the 2-D case (first in ref. [9] under a more restrictive assumption than our (H4) and then in ref. [1] under the one equivalent to (H4)). We do that verification in any dimensions.

The main results of this paper, however, are statistical prrperties – the exponential bound on correlations and the central limit theorem. Recently, Young [13] developed new techniques for proving exponential decay of correlations and CLT for hyperbolic systems. We apply her results and some techniques developed earlier in the context of billiards in ref. [5] to obtain exponential bounds on correlations under our assumptions (H1)-(H4). This has been previously done only in the 2-D case, first by Liverani [8] and then by Young [13]. A weaker bound (the so called "stretched exponential") on correlations was obtained earlier in two cases: 2-D hyperbolic attractors [1] and multidimensional Lorentz gases with finite horizon [5].

Applications. The interest to hyperbolic systems with singularities is due to certain popular physical models, such as billiards [3, 5] and attractors [1]. In particular, the realistic 3-D Lorentz gas and multiparticle systems, such as gases of hard balls, are multidimensional hyperbolic systems with singularities. While three most popular hyperbolic attractors – Lorentz, Lozi and Belykh attractors [1] – are two-dimensional, one can study arbitrary many coupled attractors, i.e. small perturbations of their direct product. In this way one gets multidimensional hyperbolic systems satisfying (H1)-(H4). One can also consider linear hyperbolic toral maps defined by matrices with determinant one and noninteger entries (a popular 2-D example is $A = \begin{pmatrix} 1+a & a \\ 1 & 1 \end{pmatrix}$ with a positive $a \notin \mathbb{Z}$), such maps have singularities. We will not elaborate examples like this. Our main purpose is to find a working approach to statistical properties of multidimensional systems in general, aiming Lorentz gases and hard balls. While billiards do not satisfy our assumption (H1), because the derivatives of billiard maps are always unbounded, we plan to extend our results to billiards in a separate paper.

2 Expansion and filtration of u-manifolds

Here we study u-manifolds. The general theme will be showing that the expansion of u-manifolds by T is "stronger" (in many senses) than cutting by singularity manifolds $\mathcal{S}$. More precisely, we will show that the images of small u-manifolds under T^n, $n \geq 0$, grow in size exponentially in n 'on the average', until they reach a certain 'fixed' size (δ_1 below).

Notations. Let U be a u-manifold. We denote by $\mathrm{diam}U$ the diameter of U in the ρ_U metric. For any point $x \in U \setminus \mathcal{S}$ denote by $J^u(x) = |\det(DT|\mathcal{T}_x U)|$ the jacobian of the map T restricted to U at x, i.e. the factor of the volume expansion on U at the point x. For $n \geq 1$ the connected components of $T^n(U \setminus \mathcal{S}^{(n)})$ are called *components* of $T^n U$.

The following is standard, and we omit the proofs:

(a) *Curvature.* $\exists B' > B'' > 0$ such that if the sectional curvature of a u-manifold U is $\leq B''$, then all the components of $T^n U$, $n \geq 1$, have sectional curvature $\leq B'$. As a result, sectional curvature of any LUM W^u is bounded above by B'. We will always assume that sectional curvature of our u-manifolds is bounded above by B'.

(b) *Distorsions.* Let $x, y \in U \setminus \mathcal{S}^{(n-1)}$ and $T^n x, T^n y$ belong in one component of $T^n U$, denote it by V. Then

$$\log \prod_{i=0}^{n-1} \frac{J^u(T^i x)}{J^u(T^i y)} \leq C' \rho_V(T^n x, T^n y) \tag{2.1}$$

with some $C' = C'(T) > 0$.

(c) *Absolute continuity.* Let U_1, U_2 be two sufficiently small u-manifolds, so that any local stable manifold W^s intersects each of U_1 and U_2 in at most one point. Let $U_1' = \{x \in U_1 : W^s(x) \cap U_2 \neq \emptyset\}$. Then we define a map $h : U_1' \to U_2$ by sliding along stable manifolds. This map is often called a holonomy map. It is absolutely continuous with respect to the Lebesgue measures ν_{U_1} and ν_{U_2}, and its jacobian (at any point of density of U_1') is bounded, i.e.

$$1/C'' \leq \frac{\nu_{U_2}(h(U_1'))}{\nu_{U_1}(U_1')} \leq C'' \tag{2.2}$$

with some $C'' = C''(T) > 0$.

Our assumption (H4) implies that $\exists \bar{\delta} > 0$ such that any $\bar{\delta}$-ball in M intersects at most K_0 smooth components of $\mathcal{S}$.

We now fix a $\delta_0 \ll \bar{\delta}$ and will assume that it is small enough for all our future needs.

Definition. We say that a connected u-manifold U is *admissible* if
(a) its sectional curvature is $\leq B'$ everywhere;
(b) $\text{diam}U \leq \delta_0$;
(c) its boundary ∂U is piecewise smooth, i.e. it is a finite union of smooth compact submanifolds of dimension dimension $d_u - 1$, possibly with boundary.

Key Remark. Let U be an admissible u-manifold. Since δ_0 is very small, the tangent spaces $\mathcal{T}_x U$ are almost parallel at all points $x \in U$. If $n \geq 1$ and $U' \subset T^n U$ is another admissible u-manifold, then $T_*^n \nu_U | U'$ (the nth iterate of the Lebesgue measure on U conditioned on U') has an almost constant density with respect to $\nu_{U'}$, due to (2.1). These important observations will allow us to approximate any admissible u-manifold by a d_u-dimensional flat domain in $\mathbb{R}^d$, i.e. a domain on a d_u-dimensional linear subspace of $\mathbb{R}^d$ with piecewise smooth boundary. In addition, we assume that $\delta_0 \ll$ the minimum radius of curvature of singularity manifolds $\mathcal{S}^{(i)} \subset \mathcal{S}$. Thus, if a u-manifold U intersects a singularity manifold $\mathcal{S}^{(i)}$, all the tangent spaces to $\mathcal{S}^{(i)}$ at the points of $\mathcal{S}^{(i)} \cap U$ are almost parallel. Hence, the manifold $\mathcal{S}^{(i)}$ is almost flat on the 'microscopic' scale of $\text{diam}U \leq \delta_0$.

Let U be an admissible u-manifold. The components of its iterates, $T^n U$, $n \geq 1$, may not be admissible, since they grow in size. We will partition them into smaller, admissible u-manifolds.

Definition. Let U be an admissible u-manifold, and $V \subset U$ an open subset with piecewise smooth boundary (i.e., ∂V consist of a finite number of smooth compact $(d_u - 1)$-dimensional submanifolds in $\bar{M}$). $\forall x \in V$ we denote by $V(x)$ the connected component of V that contains x. We say that V is *n-admissible*, $n \geq 0$, if T^n is smooth on V and $\forall x \in V$ the u-manifold $T^n V(x)$ is admissible.

Observe that V need not be connected, in fact, it almost never is in our arguments. Observe also that for an n-admissible open set V we have $V \subset U \setminus \mathcal{S}^{(n)}$.

Let U be an admissible u-manifold, and $V \subset U$ an n-admissible open subset. Let

$$r_{V,n}(x) = \rho_{T^n V(x)}(T^n x, \partial T^n V(x)) \tag{2.3}$$

be the distance from $T^n x$ to the boundary of the connected component of $T^n V$ where this point belongs. (The distance is measured in the induced Riemannian metric on that component.) We put

$$Z[U, V, n] = \sup_{\varepsilon > 0} \frac{\nu_U(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot \nu_U(U)} \tag{2.4}$$

This supremum is finite because $\partial T^n V(x)$ in (2.3) is piecewise smooth $\forall x \in V$. In the case $\nu_U(U \setminus V) = 0$, the value of $Z[U, V, n]$ characterizes, in a certain way, the 'average size' of the components of $T^n V$ – the larger they are the smaller $Z[U, V, n]$.

In particular, the value of $Z[U, U, 0]$ characterizes the size of U in a way illustrated by the following examples:

Examples. Let U be a ball of radius r, then $Z[U,U,0] \sim r^{-1}$. Let U be a cylinder whose base is a ball of radius r and height $h \gg r$, then again $Z[U,U,0] \sim r^{-1}$. Let U be a rectangular box with dimensions $l_1 \times l_2 \times \cdots \times l_{d_u}$, then $Z[U,U,0] \sim 1/\min\{l_1,\ldots,l_{d_u}\}$.

Definition Let U be an admissible u-manifold. A decreasing sequence of open subsets $U = U_0 \supset U_1 \supset U_2 \supset \cdots$ is called a *u-filtration* of U if
(a) $\forall n \geq 0$ the set U_n is n-admissible;
(b) $\forall n \geq 0$ the set U_n is dense in U, i.e. $\bar{U}_n = \bar{U}$.
We also put $U_\infty = \cap_{n\geq 0} U_n$

Observe that all U_n and U_∞ have full ν_U-measure. On the other hand, U_∞ has to be totally disconnected.

Let $\{U_n\}$ be a u-filtration of an admissible u-manifold U. We then put for brevity $r_n = r_{U_n,n}$ a function on U_n defined by (2.3) and $Z_n = Z[U,U_n,n]$ for all $n \geq 0$. The value of Z_n characterizes the 'average size' of the connected components of $T^n U_n$.

Theorem 2.1 *There are $\alpha = \alpha(T) \in (0,1)$ and $\beta = \beta(T) > 0$ such that for any admissible u-manifold U there is a u-filtration $\{U_n\}$ such that*
(i) *we have*

$$Z_1 \leq \alpha Z_0 + \beta\delta_0^{-1} \tag{2.5}$$

and $\forall n \geq 2$

$$Z_n \leq \alpha^n Z_0 + \beta\delta_0^{-1}(1+\alpha+\cdots+\alpha^{n-1}) \tag{2.6}$$

(ii) *let $\bar{\beta} = 2\beta/(1-\alpha)$: then $Z_n \leq \max\{Z_0, \bar{\beta}/\delta_0\}$ for all $n \geq 0$;*
(iii) *$Z_n \leq \bar{\beta}/\delta_0$ for all $n \geq a\ln Z_0 + b$.*
Here $a = -(\ln\alpha)^{-1}$ and $b = \max\{0, -\ln(\delta_0(1-\alpha)/\beta)/\ln\alpha\}$ are independent of U.

Remark. Effectively, the theorem asserts that if a u-manifold U is small or thin, so that Z_0 is very large, then the connected components of $T^n U$ grow larger, on the average, so that Z_n decreases exponentially in n until it becomes small enough, $\leq \bar{\beta}/\delta_0$. This is our exact version of the well-known concept 'small unstable manifolds grow exponentially in size' in the context of high dimensions.

Proof of Theorem 2.1. We start with a construction of an open dense 1-admissible subset $U_1 \subset U$ that satisfies (2.5). For brevity, we will write ν instead of ν_U and ρ instead of ρ_U.

Step 1. Assume first that $U \cap \mathcal{S} = \emptyset$ and $\mathrm{diam} U \leq \delta_0 \Lambda_{\max}^{-1}$. Then TU is an admissible u-manifold, and we set $U_1 = U$. Then $r_1(x) \geq \Lambda_{\min} r_0(x)$ for any $x \in U$, and so

$$\nu(r_1 < \varepsilon) \leq \nu(r_0 < \varepsilon/\Lambda_{\min}) \leq Z_0 \Lambda_{\min}^{-1} \nu(U) \cdot \varepsilon \tag{2.7}$$

Step 2. Assume that $\mathrm{diam} U > \delta_0 \Lambda_{\max}^{-1}$. Then we will define an open dense subset $U_1' \subset U$ whose connected components will have diameter $< \delta_0 \Lambda_{\max}^{-1}$.

According to our Key Remark, the manifold U is almost flat. We first assume that U is exactly a flat d_u-dimensional surface in $\mathbb{R}^d$ with piecewise smooth boundary. We choose a coordinate system in $\mathbb{R}^d$ so that U is parallel to the first d_u coordinate axes, i.e. $x_{d_u+1} = \cdots = x_d = 0$ on U. Also, we assume that ν is the d_u-dimensional volume on U.

For each $i = 1, \ldots, d_u$ we take an array of parallel hyperplanes $\{x_i = a_i + m\delta'\}$, $m \in \mathbb{Z}$, where $\delta' = \delta_0 \Lambda_{\max}^{-1}/\sqrt{2d_u}$, and with some fixed $a_i \in [0, \delta')$. All these hyperplanes together 'shred' (or 'dice') the domain U into cubic pieces of diameter $\delta_0 \Lambda_{\max}^{-1}/\sqrt{2} < \delta_0 \Lambda_{\max}^{-1}$. Then the set

$$U_1' := U \setminus (\cup_{i,m}\{x_i = a_i + m\delta'\}) \tag{2.8}$$

is open, dense in U, and completely determined by the vector $(a_1, \ldots, a_{d_u})$, which will be fixed shortly. For each $i = 1, \ldots, d_u$ and $m \in \mathbb{Z}$ put $D_{m,a_i} = U \cap \{x_i = a_i + m\delta'\}$. Observe that $\partial U_1' = \partial U \cup (\cup_{i,m} D_{m,a_i})$. For $\varepsilon > 0$ put $\mathcal{U}_\varepsilon^0 = \{x \in U : \rho(x, \partial U) < \varepsilon\}$ and $\mathcal{U}_\varepsilon' = \{x \in U : \rho(x, \partial U_1') < \varepsilon\}$.

Now we will optimize the parameters $a_1, \ldots, a_{d_u}$ so that $\nu(\mathcal{U}_\varepsilon')$ will be small enough, $\forall \varepsilon > 0$. For every D_{m,a_i} denote by

$$\mathcal{C}_{m,a_i}(\varepsilon) = D_{m,a_i} \times [a_i + m\delta' - \varepsilon \le x_i \le a_i + m\delta' + \varepsilon]$$

the solid cylinder in $\mathbb{R}^{d_u}$ of height 2ε whose middle cross-section is D_{m,a_i}. Observe that for any point $x \in \mathcal{U}_\varepsilon' \setminus \mathcal{U}_\varepsilon^0$ the d_u-dimensional ball in U of radius $\rho(x, \partial U_1')$ centered at x is touching one of the $(d_u - 1)$-dimensional domains D_{m,a_i}. Therefore, the region $\mathcal{U}_\varepsilon' \setminus \mathcal{U}_\varepsilon^0$ is covered by the union of the cylinders $\mathcal{C}_{m,a_i}(\varepsilon)$. Therefore

$$\nu(\mathcal{U}_\varepsilon' \setminus \mathcal{U}_\varepsilon^0) \le 2\varepsilon \sum_{i=1}^{d_u} S_{a_i} \tag{2.9}$$

where S_{a_i} is the total $(d_u - 1)$ dimensional volume of the domains D_{m,a_i}, $m \in \mathbb{Z}$.

We now fix $a_i \in [0, \delta')$ so that S_{a_i} takes its minimum value. In particular, this fixes our subset U_1' defined by (2.8)! Obviously, for each $i = 1, \ldots, d_u$ we have

$$\nu(U) = \int_0^{\delta'} S_{a_i}\, da_i$$

so that $\min_{a_i} S_{a_i} \le \nu(U)/\delta'$. Therefore

$$\nu(\mathcal{U}_\varepsilon' \setminus \mathcal{U}_\varepsilon^0) \le 2\varepsilon d_u \nu(U)/\delta' < 4 d_u^{3/2} \delta_0^{-1} \Lambda_{\max} \nu(U) \cdot \varepsilon \tag{2.10}$$

Step 3. Assume now that $U \cap \mathcal{S} \ne \emptyset$. Since $\delta_0 \ll \bar{\delta}$, then, according to (H4), U intersects at most K_0 singularity manifolds $\mathcal{S}_j \subset \mathcal{S}$. We again assume that U is a flat d_u-dimensional surface in $\mathbb{R}^d$ with piecewise smooth boundary. Besides, we assume that each singularity manifold $\mathcal{S}_j$ intersecting U is a hyperplane in $\mathbb{R}^d$, cf. Key Remark. It may happen that some $\mathcal{S}_j$ terminates inside U, then it must terminate on some other $\mathcal{S}_{j'}$, see Introduction. In that case we treat $\mathcal{S}_j$ as a hyperplane cutting one part of U after U was previously cut into two parts by the hyperplane $\mathcal{S}_{j'}$.

The set $U_1'' = U \setminus \mathcal{S}$ is open and dense in U. It is obtained by cutting the domain U by $k \le K_0$ hyperplanes in $\mathbb{R}^d$. Unlike Step 2, however, we no longer can control the position of the new cutting hyperplanes. So, we need the following lemma:

Lemma 2.2 *Let Σ be an arbitrary hyperplane cutting U. For any $\varepsilon > 0$ put $\mathcal{U}_\varepsilon^0 = \{x \in U : \rho(x, \partial U) < \varepsilon\}$ and $\mathcal{U}_{\varepsilon,1}'' = \{x \in U : \rho(x, \Sigma) < \varepsilon\}$. Then $\nu(\mathcal{U}_{\varepsilon,1}'' \setminus \mathcal{U}_\varepsilon^0) \le \nu(\mathcal{U}_\varepsilon^0)$.*

Proof. If $x \in \mathcal{U}_{\varepsilon,1}'' \setminus \mathcal{U}_\varepsilon^0$, then the d_u-dimensional ball in U of radius $\rho(x, \Sigma \cap U)$ centered at x is touching the $(d_u - 1)$-dimensional region $\Sigma \cap U$. Therefore, the set $\mathcal{U}_{\varepsilon,1}'' \setminus \mathcal{U}_\varepsilon^0$ is foliated by segments in U orthogonal to $\Sigma \cap U$ in such a way that each segment crosses $\Sigma \cap U$ and sticks out by $\le \varepsilon$ on each side of $\Sigma \cap U$. On the other hand, the line containing any of those segments intersects $\mathcal{U}_\varepsilon^0$ by two segments of length $\ge \varepsilon$ each. Hence the lemma. $\square$

Remark. We will later need the following modification of Lemma 2.2. Let $B \subset U$ be some d_u-dimensional ball, and $\mathcal{U}_\varepsilon''' = \{x \in U \setminus B : \rho(x, B) < \varepsilon\}$. Then $\nu(\mathcal{U}_\varepsilon''' \setminus \mathcal{U}_\varepsilon^0) \le \nu(\mathcal{U}_\varepsilon^0)$, $\forall \varepsilon > 0$. The proof of this is similar to that of Lemma 2.2 if one uses the foliation of $\mathcal{U}_\varepsilon'''$ by segments of rays emanating from the center of B.

Lemma 2.2 asserts that cutting U by any hyperplane effectively adds at most as much volume to the ε-neighborhood of the boundary as there was originally.

Corollary 2.3 *Let $\Sigma_1, \ldots, \Sigma_k$ be arbitrary hyperplanes crossing U. For any $\varepsilon > 0$ put $\mathcal{U}_\varepsilon'' = \{x \in U : \rho(x, \cup_i \Sigma_i) < \varepsilon\}$. Then $\nu(\mathcal{U}_\varepsilon'' \setminus \mathcal{U}_\varepsilon^0) \le k \cdot \nu(\mathcal{U}_\varepsilon^0)$.*

Applying Corollary 2.3 to the $k \le K_0$ singularity hyperplanes $\mathcal{S}_j$ that cut U gives

$$\nu(\mathcal{U}_\varepsilon'' \setminus \mathcal{U}_\varepsilon^0) \le K_0 \cdot \nu(\mathcal{U}_\varepsilon^0) \tag{2.11}$$

Step 4. We put $U_1 = U_1' \cap U_1''$. Observe that $\{x \in U : \rho(x, \partial U_1) < \varepsilon\} = \mathcal{U}_\varepsilon^0 \cup \mathcal{U}_\varepsilon' \cup \mathcal{U}_\varepsilon''$. Combining (2.10) and (2.11) gives

$$\nu(x \in U : \rho(x, \partial U_1) < \varepsilon) \le (K_0 + 1) \cdot \nu(\mathcal{U}_\varepsilon^0) + 4 d_u^{3/2} \delta_0^{-1} \Lambda_{\max} \nu(U) \cdot \varepsilon$$

This last estimate combined with (2.7) gives (formally) the bound (2.5) with $\alpha = (K_0 + 1)/\Lambda_{\min}$ and $\beta = 4 d_u^{3/2} \Lambda_{\max}/\Lambda_{\min} > 0$. Note that $\alpha < 1$ due to (H4).

Due to the actual nonflatness of both U and $\mathcal{S}$ we have to slightly (depending on δ_0) increase the above values of α and β, and we can keep α below 1 assuming δ_0 be small enough.

Step 5. Next, (2.6) follows from (2.5) by induction on n. To define U_n inductively, assume that U_{n-1} is defined. Every connected component V of $T^{n-1} U_{n-1}$ is an admissible u-manifold. Applying the proof of (2.5) to V defines an open dense subset $V_1 \subset V$. Then U_n is defined to be the union of $T^{-n+1} V_1$ over all $V \subset T^{n-1} U_{n-1}$. Lastly, the measure $T_*^n \nu_U$ conditioned on any admissible u-manifold $V \subset T^n U_n$ is almost uniform (depending on δ_0) with respect to ν_V, cf. Key Remark. Its actual nonuniformity, however, requires an additional slight increase of α and β in the above calculations, which we can afford assuming that δ_0 is small enough.

The clauses (ii) and (iii) trivially follow from (i). Theorem 2.1 is proved. $\square$

Remark. The choice of the vector $(a_1, \dots, a_{d_u})$ made in Step 2 defines the subset U_1. Applying this choice to every connected component of $T^{n-1}U_{n-1}$ defines the subset U_n. Thus, the entire u-filtration $\{U_n\}$ is defined. We say that the u-filtration so defined is *admissible*. Admissible u-filtrations always satisfy (i)-(iii) of the above theorem.

Remark. Our estimates (2.7), (2.10) and (2.11) yield a little more than the part (i) of the theorem. In fact, for any admissible u-manifold U, an admissible u-filtration $\{U_n\}$ of U, and $\forall \varepsilon > 0$ we have

$$\nu_U(r_1 < \varepsilon) \le \alpha \Lambda_{\min} \cdot \nu_U(r_0 < \varepsilon / \Lambda_{\min}) + \varepsilon \beta \delta_0^{-1} \cdot \nu_U(U) \tag{2.12}$$

and hence $\forall n \ge 2$

$$\nu_U(r_n < \varepsilon) \le (\alpha \Lambda_{\min})^n \cdot \nu_U(r_0 < \varepsilon / \Lambda_{\min}^n) + \varepsilon \beta \delta_0^{-1}(1 + \alpha + \cdots + \alpha^{n-1}) \cdot \nu_U(U) \tag{2.13}$$

Let $\delta_1 = \delta_0/(2\bar{\beta})$. According to the part (iii) of Theorem 2.1, $Z_n \le (2\delta_1)^{-1}$ for all $n \ge a \ln Z_0 + b$. Hence,

$$\nu_U(r_{U_n,n}(x) > \delta_1) > \nu_U(U)/2 \tag{2.14}$$

In other words, at least 50% of the points in T^nU (with respect to the measure induced by ν_U) lie a distance $\ge \delta_1$ away from the boundaries of T^nU.

3 Existence and ergodic properties of SRB measures

Here we prove the parts (a) and (b) of our main theorem 1.1. In [9], Pesin proved the existence and ergodic properties of SRB measures for a wide class of hyperbolic maps with singularities (he called them generalized hyperbolic attractors), covering the class we study here, under two extra assumptions, which in our notation are:

(H5) $\exists C > 0, q > 0$ such that $\forall \varepsilon > 0, n \ge 1$

$$\mathrm{Vol}(T^{-n}\mathcal{U}_\varepsilon(\mathcal{S})) \le C\varepsilon^q$$

(H6) $\exists z \in M^0$ with a local unstable manifold[1] $W^u(z)$ and $C > 0, q > 0$ such that $\forall \varepsilon > 0, n \ge 1$

$$\nu_{W^u(z)}(W^u(z) \cap T^{-n}\mathcal{U}_\varepsilon(\mathcal{S})) \le C\varepsilon^q$$

Here $\mathcal{U}_\varepsilon(\cdot)$ stands for ε-neighborhood in the ρ metric.

[1] $W^u(z)$ is defined [9] via a function $\phi^u : B_z^u \to E_z^s$, where B_z^u is a ball in E_z^u centered at z, whose graph is then mapped onto M by the exponential map. Observe that such $W^u(z)$ has smooth boundary, so it is admissible.

One should note that, under these assumptions, Pesin [9] proved that any SRB measure has at most countable number of ergodic components – a weaker statement than we claim in Theorem 1.1. Sataev [10] showed that the number of ergodic components is finite under the above (H5) and the following:

(H7) $\exists C > 0, q > 0$ such that for any ball-like u-manifold U (i.e., a u-manifold U that is a ball in the ρ_U metric) there are $n_U > 0$ and $B_U > 0$ such that $\forall \varepsilon > 0$
(a) $\nu_U(U \cap T^{-n}\mathcal{U}_\varepsilon(\mathcal{S})) \le \nu_U(U) \cdot C\varepsilon^q \quad \forall n > n_U$
(b) $\nu_U(U \cap T^{-n}\mathcal{U}_\varepsilon(\mathcal{S})) \le \nu_U(U) \cdot B_U\varepsilon^q \quad \forall n > 0$

Proposition 3.1 *If the map T satisfies (H1)-(H4), then it satisfies (H5)-(H7).*

Proof. The property (H5) with $q = 1$ is obvious for $n = 0$, cf. [6]. To prove it for $n \ge 1$, we foliate M by admissible u-manifolds $\mathcal{U}^f = \{U\}$ in a smooth way. Let ν_U^f be the conditional measures on $U \in \mathcal{U}^f$ induced by the Lebesgue volume in M, and $d\mu^f(U)$ the factor measure on $\mathcal{U}^f$. If the foliation is smooth enough and δ_0 small enough, every ν_U^f will have almost uniform density with respect to the Lebesgue measure ν_U on U. For every $U \in \mathcal{U}^f$ let $\{U_n\}$ be an admissible u-filtration of U, and $r_{U_n,n}(x)$ be defined by (2.3). Observe that

$$r_{U_{n+1},n+1}(x) < C'\varepsilon\Lambda_{\max} \qquad \forall x \in T^{-n}\mathcal{U}_\varepsilon(\mathcal{S}) \cap U \tag{3.1}$$

where C' is determined by the minimum angle between the unstable cone family and $\mathcal{S}$. Now we have

$$\begin{aligned}
\mathrm{Vol}(T^{-n}\mathcal{U}_\varepsilon(\mathcal{S})) &= 2\int \nu_U^f(T^{-n}\mathcal{U}_\varepsilon(\mathcal{S}) \cap U)\, d\mu^f(U) \\
&\le 2\int [\nu_U(U)]^{-1}\nu_U(x \in U :\, r_{U_{n+1},n+1}(x) < C'\varepsilon\Lambda_{\max})\, d\mu^f(U) \\
&\le 2\int \Big[(\alpha\Lambda_{\min})^{n+1}\, \nu_U(x \in U :\, r_{U,0}(x) < C'\varepsilon\Lambda_{\max}/\Lambda_{\min}^{n+1})/\nu_U(U) \\
&+ C'\varepsilon\Lambda_{\max}\beta\delta_0^{-1}(1-\alpha)^{-1}\Big]\, d\mu^f(U) \\
&\le 4(\alpha\Lambda_{\min})^{n+1}\int \nu_U^f(r_{U,0}(x) < C'\varepsilon\Lambda_{\max}/\Lambda_{\min}^{n+1})\, d\mu^f(U) \\
&+ 2C'\varepsilon\Lambda_{\max}\beta\delta_0^{-1}(1-\alpha)^{-1}\mathrm{Vol}(M)
\end{aligned}$$

where we used (3.1) and (2.13). Clearly, the last integral is $\le$const$\cdot\varepsilon\Lambda_{\min}^{-n-1}$. This proves (H5).

We now prove (H6) and (H7). Let U be an admissible u-manifold and $\{U_n\}$ its admissible u-filtration. Based on (3.1) we have

$$\nu_U(U \cap T^{-n}\mathcal{U}_\varepsilon(\mathcal{S})) \le \nu_U(r_{U_{n+1},n+1} < C'\varepsilon\Lambda_{\max}) \le C'\varepsilon\Lambda_{\max}Z_{n+1} \cdot \nu_U(U) \tag{3.2}$$

The assumption (H7) now follows from Theorem 2.1 with $q = 1$ and $B_U = C'\Lambda_{\max}\max\{Z_0, \bar\beta\delta_0^{-1}\}$, $n_U = a\ln Z_0 + b$, $C = C'\Lambda_{\max}\bar\beta\delta_0^{-1}$. The assumption (H6) follows in the same way, provided we can prove the sheer existence of unstable manifolds, which we do next.

Let $\Lambda > 1$ and $\varepsilon > 0$. Define

$$M^{\pm}_{\Lambda,\varepsilon} = \{x \in M^{\pm} : \rho(T^{\pm n}x, \mathcal{S}) > \varepsilon\Lambda^{-n} \quad \forall n \geq 0\}$$

and

$$M^{\pm}_{\Lambda} = \cup_{\varepsilon>0} M^{\pm}_{\Lambda,\varepsilon} \qquad M^0_{\Lambda} = M^+_{\Lambda} \cap M^-_{\Lambda}$$

The following is standard [9, 13]:

Fact. Let $1 < \Lambda < \Lambda_{\min}$ and $\varepsilon > 0$. Then $\forall x \in M^-_{\Lambda,\varepsilon}$ there is a LUM $W^u(x)$ such that $\rho(x, \partial W^u(x)) \geq \varepsilon$. Similarly, $\forall x \in M^+_{\Lambda,\varepsilon}$ there is an LSM $W^s(x)$ such that $\rho(x, \partial W^s(x)) \geq \varepsilon$.

Therefore, stable manifolds exist everywhere on M^-_{Λ}, and unstable ones everywhere on M^+_{Λ}. For $x \in M^-_{\Lambda,\varepsilon}$ we denote by $W^u_{\varepsilon}(x)$ the ε-ball in $W^u(x)$ centered at x, in the $\rho_{W^u(x)}$ metric. It is, indeed, a ball, since $\rho_{W^u(x)}(x, \partial W^u(x)) \geq \varepsilon$. Similarly, $W^s_{\varepsilon}(x)$ is defined $\forall x \in M^+_{\Lambda}$. We will call $W^s_{\varepsilon}(x)$ and $W^u_{\varepsilon}(x)$ stable and unstable *disks* of radius ε through x, respectively.

Lemma 3.2 $\forall \Lambda > 1$ *we have* $M^0_{\Lambda} \neq \emptyset$.

Pesin [9] proved this lemma under the assumption (H5), that we already proved, for a larger class of hyperbolic systems than we study here. On the other hand, Young [13] provided a direct argument for 2-D case, which we will extend below to our systems.

Let U be an admissible u-manifold, and let

$$\bar{\nu}_N = \frac{1}{N} \sum_{i=0}^{N-1} T^i_* \nu_U$$

This is a pre-compact sequence of Borel measures on $\bar{M}$. Any limit point $\hat{\mu}$ of this sequence, normalized, is a T-invariant probability measure concentrated on M^0. Theorem 2.1, see also the above proof of (H7), ensures that $\exists C > 0$ such that $\hat{\mu}(\mathcal{U}_{\varepsilon}(\mathcal{S})) \leq C\varepsilon$, $\forall \varepsilon > 0$. Then the standard application of Borel-Cantelli lemma [13] proves Lemma 3.2. $\square$

Proposition 3.1 is proved.$\square$

This concludes the proof of parts (a) and (b) of Theorem 1.1. $\square$

Corollary 3.3 ([9, 13]) *For any SRB measure* μ *and any* $\Lambda > 1$ *we have* $\mu(M^0_{\Lambda}) = 1$, *i.e., LUM's and LSM's exist a.e. with respect to any SRB measure.*

Remark. Applying Borel-Cantelli lemma to ν_U rather than $\hat{\mu}$ yields that for any u-manifold U we have $\nu_U(U \setminus M^+_{\Lambda}) = 0$, i.e. an LSM W^s_x exists for ν_U-a.e. point $x \in U$.

Having proved the first two parts of Theorem 1.1, we conclude that all the SRB measures satisfying the assumptions of the part (c) are actually mixing and Bernoulli.

4 Refined filtration of u-manifolds

The techniques of Section 2 are not enough to obtain the statistical properties of T. We will be dealing with rectangles defined later in Sect. 5. Those are made of points whose both stable and unstable manifolds are large enough. The results of Sect. 2 allow us only to control the sizes of unstable manifolds and their iterates. In order to locate points on a given unstable manifold with large enough stable manifolds, we have to, according to the Fact given in the previous section, discard the points whose orbits come too close to the singularity manifold $\mathcal{S}$.

Technically, we again consider the iterations of an admissible u-manifold W under T^n, $n \geq 0$. The admissible u-filtration $\{W_n\}$ constructed in Sect. 2 will be refined here, so that points in W whose images come too close to the singularity manifolds will be set apart and no longer iterated under T. This will create countably many gaps in W in which stable manifolds fail to be long enough.

We start with a technical construction around the singularity manifold $\mathcal{S}$. For every $\delta' \ll \delta_0$ we define two parallel hypersurfaces at distance δ' from every singularity manifold $\mathcal{S}_j$ (located on both sides of $\mathcal{S}_j$). They are obtained by moving every point $x \in \mathcal{S}_j$ the distance δ' from $\mathcal{S}_j$ along the normal vectors to $\mathcal{S}_j$ at x in both directions from $\mathcal{S}_j$. Since δ_0 is less than the minimum radius of curvature of $\mathcal{S}_j$, the resulting hypersurfaces will be smooth $\forall \delta' < \delta_0$. We also make sure that those hypersurfaces terminate on the same components of $\mathcal{S}$ as the original manifold $\mathcal{S}_i$. We denote by $\hat{\mathcal{S}}_j^{\delta'}$ the *union* of these two hypersurfaces, $\forall j = 1, \ldots, r$.

Now, fix a $\Lambda \in (1, \Lambda_{\min})$ and let $\delta_2 \ll \delta_1 = \delta_0/(2\bar{\beta})$. The two parameters Λ and δ_2 will govern all our further constructions in this section.

Let W be an admissible u-manifold and $\{W_n\}$ its admissible u-filtration defined in Sect. 2. For any $n \geq 0$ we define an open subset $W_n' \subset W_n$ by

$$W_n' = \left\{ x \in W_n : \, 0 < \rho(T^n x, \mathcal{S}) < \delta_2 \Lambda^{-n} \right\} \tag{4.1}$$

This is a set of points whose n-th iterates come too close to the singularity manifolds.

Observe that the sets W_n', $n \geq 0$, may overlap. So, we define $W_n^0 = W_n' \setminus (\overline{W_0'} \cup \cdots \cup \overline{W_{n-1}'})$. This set consists of points whose trajectories come too close to $\mathcal{S}$ at time n, not earlier. Hence, W_n^0 is a gap in W created at the n-th iteration. We then set $W_0^1 = W$ and $W_n^1 = W \setminus (\overline{W_0'} \cup \cdots \cup \overline{W_{n-1}'})$ for $n \geq 1$. Thus, W_n^1 is the part of W that survives n iteration of T without coming too close to $\mathcal{S}$.

All W_n^0 and W_n^1 are n-admissible open subsets of W. We call the two collections $\{W_n^1\}$ and $\{W_n^0\}$ the *refinement of the u-filtration* $\{W_n\}$, or a refined u-filtration. We denote it by $(\{W_n\}, \{W_n^1\}, \{W_n^0\})$.

We put $W_\infty^1 = \cap_{n \geq 0} W_n^1$. Observe that $W_\infty^1 \subset M_{\Lambda,\delta_2}^+$, and so a stable disk $W_{\delta_2}^s(x)$ of radius δ_2 exists at every point $x \in W_\infty^1$.

Next, we characterize the 'sizes' of the u-manifolds $T^n W_n^1$ and $T^n W_n^0$, $n \geq 0$, in the manner similar to that of Sect. 2. Define $\forall n \geq 0$

$$Z_n^1 = Z[W, W_n^1, n] \qquad \text{and} \qquad Z_n^0 = Z[W, W_n^0, n]$$

based on (2.4). In the case $W_n^1 = \emptyset$ we have, of course, $Z_n^1 = 0$, but this will never actually happen in our further constructions. Observe that $Z_0^1 = Z_0$, where Z_0 was defined in Theorem 2.1. Put also

$$w_n^1 = \nu_W(W_n^1)/\nu_W(W) \qquad \text{and} \qquad w_n^0 = \nu_W(W_n^0)/\nu_W(W)$$

Observe that $w_n^1 = 1 - w_0^0 - \cdots - w_{n-1}^0$ and $w_n^1 \searrow w_\infty^1 \stackrel{\text{def}}{=} \nu_W(W_\infty^1)/\nu_W(W)$ as $n \to \infty$.

Theorem 4.1 *Let W be an admissible u-manifold and $\{W_n\}$ its admissible u-filtration. Let $1 < \Lambda < \Lambda_{\min}$ and $\delta_2 \ll \delta_1$. Then the refinement $(\{W_n\}, \{W_n^1\}, \{W_n^0\})$ of the u-filtration $\{W_n\}$ satisfies the following bounds:*
(i) *we have*

$$Z_1^1 \leq \alpha Z_0^1 + \beta\delta_0^{-1} \tag{4.2}$$

with the same $\alpha \in (0,1)$ and $\beta > 0$ as in Theorem 2.1, and for any $n \geq 2$

$$Z_n^1 \leq \alpha^n Z_0^1 + \beta\delta_0^{-1}(1 + \alpha + \cdots + \alpha^{n-1}) \tag{4.3}$$

(ii) *for any $n \geq 0$ we have $Z_n^0 \leq (3K_0 + 1)Z_n^1$;*
(iii) *for any $n \geq 0$ we have $w_n^0 \leq Z_n^0 C'\delta_2\Lambda^{-n}$ with some constant $C' = C'(T) > 0$.*

Remark. This theorem is a refinement of Theorem 2.1. Part (i) essentially states that even after the removal of the parts of T^nW that come too close to singularities, the remaining components grow exponentially fast, on the average. The part (iii) asserts that the total measure of the gaps created at the n-th iteration is exponentially small in n (quite natural). The part (ii) ensures that the boundaries of the gaps are 'not too ugly': this is necessary to control further images of the gaps (coming into play later, in Sect. 6), we will prove that they, too, grow fast enough, on the average.

Proof. To prove (4.2), we only need to modify Step 3 of the proof of Theorem 2.1. According to (4.1), the sets W_0^0 and W_1^1 are made by cutting W with the hypersurfaces $\mathcal{S}_{j_i}$ and $\hat{\mathcal{S}}_{j_i}^{\delta_2}$, $1 \leq i \leq k$. Thus, in addition to $k \leq K_0$ singularity hyperplanes Σ_j, $1 \leq j \leq k$, in the notations used in the proof of Theorem 2.1, we now have k pairs of their parallel copies, which we shall call Σ_j' and Σ_j'', $1 \leq j \leq k$.

Lemma 2.2 admits the following easy modification:

Lemma 4.2 *Let W be a domain in $\mathbb{R}^{d_u} \subset \mathbb{R}^d$ with piecewise smooth boundary, and Σ', Σ'' two parallel hyperplanes in $\mathbb{R}^d$. Denote by B the layer in $\mathbb{R}^d$ between Σ' and Σ''. For any $\varepsilon > 0$ put $\mathcal{U}_\varepsilon^0 = \{x \in W : \rho_W(x, \partial W) < \varepsilon\}$ and $\mathcal{U}_{\varepsilon,1}'' = \{x \in W \setminus B : \rho_W(x, \Sigma' \cup \Sigma'') < \varepsilon\}$. Then $\nu_W(\mathcal{U}_{\varepsilon,1}'' \setminus \mathcal{U}_\varepsilon^0) \leq \nu_W(\mathcal{U}_\varepsilon^0)$.*

We apply Lemma 4.2 to each pair of hyperplanes Σ_j', Σ_j'', $1 \leq j \leq k$ and then sum over $j = 1, \ldots, k$ as we did in Corollary 2.3. This proves (4.2).

The bound (4.3) follows from (4.2) by induction on n, as in Step 5 of the proof of Theorem 2.1. We simply apply the bound (4.2) to every connected component of $T^nW_n^1$, which is an admissible u-manifold.

To prove (ii) for $n = 0$, we apply Corollary 2.3 to the entire collection of $3k$ hyperplanes $\Sigma_j, \Sigma'_j, \Sigma''_j$, $1 \leq j \leq k$, and then get

$$\nu_W(x \in W_0^0 : \rho(x, \partial W_0^0) < \varepsilon) \leq (3K_0 + 1) Z_0^1 \, \varepsilon \nu_W(W)$$

To prove (ii) for $n \geq 1$, we apply the above argument to every connected component of $T^n W_n^1$.

To prove (iii), observe that, in the notations of (4.1), $\forall x \in W_n^0$ we have

$$\rho_{T^n W_n^0(x)} \left(T^n x, \partial T^n W_n^0(x) \right) \leq \rho_{T^n W_n^0(x)} \left(T^n x, \mathcal{S} \right) \leq C' \delta_2 \Lambda^{-n}$$

where $C' > 0$ depends on the minimum angle between $\mathcal{S}$ and the unstable cone family. We apply (2.3) and (2.4) with $U = W$, $V = W_n^0$, $\varepsilon = C'\delta_2\Lambda^{-n}$ and observe that then

$$\nu_W(x \in W_n^0 : r_{W_n^0, n}(x) < C'\delta_2\Lambda^{-n}) = \nu_W(W_n^0) = w_n^0 \cdot \nu_W(W)$$

This gives (iii). Theorem 4.1 is proved. $\square$.

Corollary 4.3 *Let* $\bar{\beta} = 2\beta/(1-\alpha)$, $\delta_1 = \delta_0/(2\bar{\beta})$, $a = -(\ln \alpha)^{-1}$ *and* $b = \max\{0, a \ln[\delta_0(1-\alpha)/\beta]\}$ *as in Section 2. Let* $\bar{Z}_0 = \max\{Z_0, \bar{\beta}/\delta_0\}$. *Then*
(i) $Z_n^1 \leq \bar{Z}_0$ *and* $Z_n^0 \leq (3K_0 + 1)\bar{Z}_0$ *for all* $n \geq 0$*;*
(ii) $Z_n^1 \leq \bar{\beta}/\delta_0 = (2\delta_1)^{-1}$ *for all* $n \geq a \ln Z_0 + b$*;*
(iii) $w_n^0 \leq C''\bar{Z}_0\delta_2\Lambda^{-n}$ *for all* $n \geq 0$, *where* $C'' = (3K_0 + 1)C'$*;*
(iv) $w_n^1 \geq 1 - C''\bar{Z}_0\delta_2/(1 - \Lambda^{-1})$ *for all* $n \geq 1$*;*
(v) $\nu_W(W_\infty^1) \geq \nu_W(W) \cdot \left[1 - C''\bar{Z}_0\delta_2/(1 - \Lambda^{-1})\right]$

The values Z_n^1 and Z_n^0 do not characterize the average size of the components of $T^n W_n^1$ or $T^n W_n^0$, respectively, in the sense of Section 2, since W_n^1 and W_n^0 are not subsets of full measure in W. To characterize the average sizes of the components of any n-admissible open subset $V \subset W$ we will use the quantity

$$Z[V, n] := \sup_{\varepsilon > 0} \frac{\nu_W(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot \nu_W(V)} = Z[W, V, n] \times \frac{\nu_W(W)}{\nu_W(V)} \tag{4.4}$$

This value depends on V but not on W. It characterizes the average size of the components of $T^n V$ just like $Z[U, V, n]$ did in Section 2 for subsets $V \subset U$ of full measure. Accordingly, the values of

$$Z[W_n^1, n] = Z_n^1/w_n^1 \qquad \text{and} \qquad Z[W_n^0, n] = Z_n^0/w_n^0$$

characterize the average size of the components of $T^n W_n^1$ or $T^n W_n^0$, respectively.

In our further constructions, the set W_∞^1 will be often very dense in W, so that $w_\infty^1 > 0.9$. We call this a special case, and Corollary 4.3 then implies:

Special case. If $w_\infty^1 > 0.9$, then for all $n \geq a \ln Z_0 + b$ we have $Z[W_n^1, n] \leq 0.6/\delta_1$. We will say then that the components of $T^n W_n^1$ are large enough, on the average.

Remark. The values of $Z[U, V, n]$ in (2.4) and the values of $Z_n^{1,0}$, $w_n^{1,0}$ in this section will certainly not change if we replace the Lebesgue measures, ν_U in (2.4) and ν_W here, by any measures proportional to those. It is also straightforward that all the results of Sections 2 and 4 extend to countable disjoint unions of admissible u-manifolds with finite measures that are linear combinations of the Lebesgue measures on individual components. Precisely, let $U = \cup_k U^{(k)}$ be a countable union of pairwise disjoint admissible u-manifolds and $\hat{\nu}_U = \sum_k u_k \nu_{U^{(k)}}$, with some $u_k > 0$, a finite measure on U. Then $Z[U, V, n]$ is still defined by (2.4), with ν_U replaced by $\hat{\nu}_U$, for any set $V = \cup_k V^{(k)}$, where $V^{(k)}$ are some n-admissible open subsets of $U^{(k)}$. The definition of u-filtration and the proof of Theorem 2.1 go through with only minor obvious changes. Likewise, the definitions and results of this section apply to any countable union $W = \cup W^{(k)}$ of admissible u-manifolds with any finite measure $\hat{\nu}_W = \sum_k u_k \nu_{W^{(k)}}$, provided we use the same parameters Λ and δ_2 for all $W^{(k)}$.

Final Remark. Let W' be an admissible u-manifold, $k \geq 1$, and $V' \subset W'$ a k-admissible open subset. Then $W = T^k V'$ is a finite or countable union of admissible u-manifolds. The measure $\tilde{\nu}_W := T_*^k \nu_{W'} | W$ on W is almost uniform (proportional to the Lebesgue measure ν_W) on each component of W, according to Key Remark of Sect. 2. All the results of Sections 2 and 4 will then apply to $(W, \tilde{\nu}_W)$, instead of (W, ν_W), but the slight nonuniformity of the measure $\tilde{\nu}_W$ with respect to ν_W might slightly affect the values of the constants, such as α, β, a, b. The smaller δ_0, the smaller changes in the constants will be inflicted. In all that follows we assume that the constants are adjusted accordingly, so that the results of Sections 2 and 4 apply to pairs $(W, \tilde{\nu}_W)$ as above.

Lastly, we generalize the above special case:

Proposition 4.4 *Let $(\{W_n\}, \{W_n^1\}, \{W_n^0\})$ be a refined u-filtration of an admissible u-manifold W, such that $w_\infty^1 = p > 0$. Then for all $n \geq a_1(\ln Z_0 - \ln p) + b_1$ we have $\nu_W(W_\infty^1)/\nu_W(W_n^1) \geq 0.9$ and $Z[W_n^1, n] \leq 0.6/\delta_1$, i.e. the components of $T^n W_n^1$ will be large enough, on the average. Here $a_1 = a + (\ln \Lambda)^{-1}$ and b_1 is another constant determined by $\alpha, \beta, \delta_0, \Lambda, C''$.*

Remark. Loosely speaking, it takes const$\cdot(\ln Z_0 - \ln p)$ iterations to grow the components of $T^n W_n^1$ large, on the average. Recall that it takes const$\cdot \ln Z_0$ iterations to grow the components of $T^n W$, where Z_0 characterizes the initial size of W. Now, the manifolds $T^n W$ lose, along the way, some of the mass in gaps, and only a fraction, p, of the initial mass survives. Therefore, additional const$\cdot \ln p$ iterations are required to recover the losses due to gaps.

Proof. Due to the part (ii) of Corollary 4.3, we have $Z_n^1 \leq (2\delta_1)^{-1}$, and hence $Z[W_n^1, n] \leq (2\delta_1 p)^{-1}$, for all $n \geq n' := a \ln Z_0 + b$. Due to the part (iii) of the same corollary, we have $\sum_{i=n}^{\infty} w_i^0 \le p/20$ for all $n \ge n'' := \log_\Lambda[20\, C'' \bar{Z}_0 p^{-1}/(1-\Lambda^{-1})]$. Observe that $n', n'' \ge 0$ and let $k = n' + n''$. The set $\tilde{W} := T^k W_k^1$ is a finite or countable union of admissible u-manifolds. It carries the measure $\tilde{\nu}_{\tilde{W}} := T_*^k \nu_W | \tilde{W}$, so that the results of this section apply to $(\tilde{W}, \tilde{\nu}_{\tilde{W}})$, according to Final Remark. The subsets $T^k W_m^1 \subset \tilde{W}$, $m \ge k$, correspond to a refined u-filtration $(\{\tilde{W}_n\}, \{\tilde{W}_n^1\}, \{\tilde{W}_n^0\})$ of $\tilde{W}$ with δ_2 replaced by $\delta_2 \Lambda^{-k}$, so that $T^k W_m^1 = \tilde{W}_{m-k}^1$, $\forall m \ge k$. Since $k \ge n'$, we have $Z[\tilde{W}, \tilde{W}, 0] = Z[W_k^1, k] \le (2\delta_1 p)^{-1}$. Since $k \ge n''$, we have

$$\tilde{w}_\infty^1 = \tilde{\nu}_{\tilde{W}}(\tilde{W}_\infty^1)/\tilde{\nu}_{\tilde{W}}(\tilde{W}) = \nu_W(W_\infty^1)/\nu_W(W_k^1) \ge 0.9$$

Thus, the refined u-filtration $(\{\tilde{W}_n\}, \{\tilde{W}_n^1\}, \{\tilde{W}_n^0\})$ of $\tilde{W}$ falls in the above special case. Hence, $Z[\tilde{W}_n^1, n] \le 0.6/\delta_1$ for all $n \ge n''' := -a \ln(2\delta_1 p) + b$. Therefore, for the original refined u-filtration of W, we have $Z[W_n^1, n] \le 0.6/\delta_1$ for all $n \ge n' + n'' + n'''$. It is then an easy calculation that $n' + n'' + n''' \le a_1(\ln Z_0 - \ln p)$+const. $\square$.

Final Remark (Part 2). The above proposition also applies to any pair $(W, \tilde{\nu}_W)$ described in Final Remark before the proposition. Likewise, some further results stated and proved for admissible u-manifolds, W, with Lebesgue measures ν_W, will also apply to measures $\tilde{\nu}_W = T_*^k \nu_{T^{-k}W}$ on W for any $k \ge 1$ such that T^{-k} is defined on W. We will assume this without any more reminders.

5 Rectangles

The key instrument in Young's proofs [13] of statistical properties of hyperbolic dynamical systems is a set with hyperbolic product structure. Its full definition is quite long, but for uniformly hyperbolic maps studied here such a set is just a rectangle or parallelogram in Sinai-Bowen sense, cf. [11, 2].

Definition. A subset $R \subset M^0$ is called a *rectangle* if $\exists \varepsilon > 0$ such that for any $x, y \in R$ there is an LSM $W^s(x)$ and an LUM $W^u(y)$, both of diameter $\le \varepsilon$, that meet in exactly one point, which also belongs in R. We denote that point by $[x, y] = W^s(x) \cap W^u(y)$.

In all our rectangles, we will have $\varepsilon < \delta_0$.

A subrectangle $R' \subset R$ is called a u-subrectangle if $W^u(x) \cap R = W^u(x) \cap R'$ for all $x \in R'$. Similarly, s-subrectangles are defined. We say that a rectangle R' u-crosses another rectangle R if $R' \cap R$ is a u-subrectangle in R and an s-subrectangle in R'.

We introduce some more notation. Let $x \in M$ and $r \in (0, \delta_0)$. We denote by $S_r(x)$ any s-manifold that is a ball of radius r centered at x in its own metric, $\rho_{S_r(x)}$. By that we mean $\rho_{S_r(x)}(x, y) = r$, $\forall y \in \partial S_r(x)$. We call such $S_r(x)$ an *s-disk*. In order to define s-disks also around points close to ∂M we extend the cone families C^u and C^s continuously beyond the boundaries of M into the δ_0-neighborhood of M. Then s-disks $S_r(x)$ exist $\forall x \in M, \forall r \in (0, \delta_0)$. Note that $S_r(x)$ is by no means uniquely determined by x and r.

Let U be a u-manifold of diameter $< \delta_0$, and $x \in M$. Clearly, any s-disk $S_{\delta_0}(x)$ can meet U in at most one point (as we always require δ_0 be small enough). We call

$$H_x(U) = \{y \in U : y = S_{\delta_0}(x) \cap U \quad \text{for some} \quad S_{\delta_0}(x)\}$$

the *s-shadow* of x on U.

We say that a point $x \in M$ is overshadowed by a u-manifold U if $\forall S_{\delta_0}(x)$ we have $S_{\delta_0}(x) \cap U \neq \emptyset$. Note that in this case, of course, $\rho(x, U) \leq \delta_0$. We call

$$\rho^s(x, U) = \sup_{S_{\delta_0}(x)} \rho_{S_{\delta_0}(x)}(x, S_{\delta_0}(x) \cap U)$$

the *s-distance* from x to U (this one is also $\leq \delta_0$ whenever defined).

Let U, U' be two u-manifolds of diameters $< \delta_0$. We call

$$H_U(U') = \cup_{x \in U} H_x(U')$$

the s-shadow of U on U'. We say that U' overshadows U if it overshadows every point $x \in U$. In this case we define

$$\rho^s(U, U') = \sup_{x \in U} \rho^s(x, U')$$

the s-distance from U to U'. It is not symmetric, since no two u-manifolds can simultaneously overshadow each other: geometrically, U' overshadows U if U is close to U' and U' stretches all the way along U and a little beyond it.

Let $\Lambda \in (1, \Lambda_{\min})$ be the one fixed in Sect. 4. We assume that δ_0, and hence δ_1, are small enough, so that $M^-_{\Lambda,\delta_1} \neq \emptyset$. Therefore,

$$A_{\delta_1} \stackrel{\text{def}}{=} \{x \in M : \text{the unstable disk} \quad W^u_{\delta_1}(x) \quad \text{exists}\} \neq \emptyset$$

(recall, cf. Sect. 3, that $W^u_\varepsilon(x)$ is the ball of radius ϵ centered at x in the local unstable manifold $W^u(x)$).

Let $z \in A_{\delta_1}$. Consider $W(z) := W^u_{\delta_1/3}(z)$, the 'central part' of the existing unstable disk $W^u_{\delta_1}(z)$. It is an admissible u-manifold, and a perfect ball in its own metric. It is an easy exercise that for a perfect ball W of radius δ in $\mathbb{R}^{d_u}$ one has $Z[W, W, 0] = d_u/\delta$. Since the manifolds $W(z)$, $z \in A_{\delta_1}$, actually have some (bounded) sectional curvature, $Z[W(z), W(z), 0]$ might be larger than $3d_u/\delta_1$, but if δ_1 is small enough, that difference is not big, and we will have

$$Z[W(z), W(z), 0] \leq 4d_u/\delta_1 \tag{5.1}$$

for all $z \in A_{\delta_1}$.

Let $\delta_2 \ll \delta_1$ to be specified below, and $(\{W_n(z)\}, \{W^1_n(z)\}, \{W^0_n(z)\})$ the refined u-filtration of $W(z)$ defined in Sect. 4 and governed by the two parameters Λ and δ_2.

Lemma 5.1 *If δ_2/δ_1 is small enough, then $\forall z \in A_{\delta_1}$ we have $\nu_{W(z)}(W^1_\infty(z)) \geq 0.9 \cdot \nu_{W(z)}(W(z))$.*

This follows from (5.1) and the part (v) of Corollary 4.3, provided

$$\frac{\delta_2}{\delta_1} \leq \frac{1-\Lambda^{-1}}{40\, C'' d_u} \tag{5.2}$$

Convention. We will treat our small parameters δ_i, $i \geq 0$, in the following way. On the one hand, all of them are assumed to be small, and on the other hand, the ratios δ_{i+1}/δ_i, $i \geq 0$, are also small. Moreover, we will fix their ratios δ_{i+1}/δ_i, $i \geq 1$, at some points below, but still allow them to vary altogether with their ratios fixed.

In fact, the ratio $\delta_1/\delta_0 = (2\bar{\beta})^{-1}$ is already fixed in Section 2. We now fix δ_2/δ_1 that satisfies (5.2). Recall that $\forall x \in W^1_\infty(z)$ a stable disk $W^s_{\delta_2}(x)$ exists, cf. Sect. 4.

Lemma 5.2 *Let $z \in A_{\delta_1}$, and consider a refined u-filtration $(\{W_n(z)\}, \{W^1_n(z)\}, \{W^0_n(z)\})$ of the unstable disk $W(z) = W^u_{\delta_1/3}(z)$. Then $\forall n \geq n_0' := a\ln(16d_u) + \max\{1, a\ln[\beta\delta_0^{-1}/(1-\alpha)]\}$ we have*
(i) $Z^1_n < (2\delta_1)^{-1}$ and $Z[W^1_n(z), n] < 0.6/\delta_1$;
(ii) $\nu_{W(z)}(x \in W^1_n(z) : r_{W^1_n(z),n}(x) > \delta_1) > 0.4 \cdot \nu_{W(z)}(W^1_n(z)) > 0.4 \cdot \nu_{W(z)}(W^1_\infty(z))$.
In other words, (ii) means that at least 40% of the points in $T^nW^1_n(z)$ (with respect to the measure induced by $\nu_{W(z)}$) lie a distance $\geq \delta_1$ away from the boundaries of $T^nW^1_n(z)$.

Proof. This follows from the part (ii) of Corollary 4.3, Lemma 5.1 and (5.1), recall also a similar bound (2.14). □

Remark. Let $z \in A_{\delta_1}$. For a moment, let $W(z) = W^u_\varepsilon(z)$ be the stable disk of any radius $\varepsilon \in (\delta_1/3, \delta_1)$. That disk $W(z)$ is larger than $W^u_{\delta_1/3}(z)$, and so (5.1) still holds. Therefore, the statements (i) and (ii) of the above lemma hold as well. Furthermore, if, again for a moment, we decrease δ_2 thus making the ratio δ_2/δ_1 smaller than the one fixed above, then Lemma 5.1 will still hold, and then so will (i) and (ii) of Lemma 5.2.

In the next proposition, we consider the iterations of two nearby unstable manifolds and prove that the ρ^s-distance between them decreases exponentially. Let $\delta_3 \ll \delta_2$, to be specified later.

Proposition 5.3 *Let W be an admissible u-manifold, and W' another u-manifold that overshadows W and $\rho^s(W,W') \leq \delta_3$. Let $(\{W_n\}, \{W^1_n\}, \{W^0_n\})$ be a refined u-filtration of W. Then $\forall n \geq 1$ and any connected component V of W^1_n there is a connected domain $V' \subset W' \setminus \mathcal{S}^{(n)}$ such that the u-manifold T^nV' overshadows the admissible u-manifold T^nV, and $\rho^s(T^nV, T^nV') \leq \delta_3\Lambda_{\min}^{-n}$.*

Proof. The proof easily goes by induction on n, so that it suffices to prove the proposition for $n = 1$. Put $n = 1$, and let V be a connected component of W_0^1. Then $\forall x \in V$ we have $\rho(x, \mathcal{S}) > \delta_2$ by (4.1), so that any s-disk $S_{2\delta_3}(x)$ will cross W' but not $\mathcal{S}$, provided δ_3/δ_2 is small enough. Hence, the s-shadow $H(V, W')$ belongs in one connected component of $W' \setminus \mathcal{S}$. It is then easy to see by direct inspection that its image under T overshadows TV and the s-distance from TV to that image is $\leq \delta_3 \Lambda_{\min}^{-1}$. $\square$

For any $z \in A_{\delta_1}$ we define a 'canonical' rectangle $R(z)$ as follows: $y \in R(z)$ iff $y = W^s_{\delta_2}(x) \cap W^u$ for some $x \in W^1_\infty(z)$ and for some LUM W^u that overshadows $W(z) = W^u_{\delta_1/3}(z)$, and such that $\rho^s(W(z), W^u) \leq \delta_3$. Observe that if $\delta_3/\delta_2 < c'$, where $c' > 0$ is determined by the minimum angle between the stable and unstable cone families, then every W^u that overshadows $W(z)$ and is δ_3-close to it in the above sense will meet all stable disks $W^s_{\delta_2}(x)$, $x \in W^1_\infty(z)$. In that case $R(z)$ will be a rectangle, indeed. We fix the ratio δ_3/δ_2 now as follows:

$$\delta_3/\delta_2 = \min\{c', 1 - \Lambda^{-1}, 1/3\} \tag{5.3}$$

For any connected subdomain $V \subset W(z)$ the set $R_V(z) := \{y \in R(z) : W^s(y) \cap V \neq \emptyset\}$ is an s-subrectangle in $R(z)$ "with base V". Let $n \geq 1$. The partition of $W^1_n(z)$ into connected components, $\{V\}$, induces a partition of $R(z)$ into s-subrectangles $\{R_V(z)\}$ that those components as bases. Let $R_V(z)$ be one of those s-subrectangles. It follows from Proposition 5.3 that $T^n R_V(z)$ is a rectangle. We call every rectangle $T^n R_V(z)$ a component of the set $T^n R(z)$, note that the entire set $T^n R_V(z)$ does not have to be a rectangle itself. We next consider the intersections of $T^n R_V(z)$ with $R(z')$ for $z' \in A_{\delta_1}$.

First, we prove a technical lemma. It says that if two unstable manifolds come close to each other at some points that are in their middle parts, then they must be close enough to each other 'all the way'.

Lemma 5.4 *There is a $c_1 > 0$ such that $\forall z, z' \in A_{\delta_1}$ such that $\rho(z, z') < c_1\delta_3$, the LUM $W^u_{\delta_1/2}(z')$ overshadows the LUM $W(z) = W^u_{\delta_1/3}(z)$, and $\rho^s(W(z), W^u_{\delta_1/2}(z')) \leq \delta_3/2$. Likewise, the LUM $W^u_{\delta_1}(z)$ overshadows the LUM $W^u_{\delta_1/2}(z')$, and $\rho^s(W^u_{\delta_1/2}(z'), W^u_{\delta_1}(z)) \leq \delta_3/2$.*

Proof. We will only prove the first statement, the second one is completely similar. We need to prove that $\forall x \in W(z)$ we have $\rho^s(x, W^u_{\delta_1/2}(z')) \leq \delta_3/2$. Assume that it is not the case, i.e.

$$\exists x \in W(z) : \qquad \rho^s(x, W^u_{\delta_1/2}(z')) > \delta_3/2 \tag{5.4}$$

Observe that $\forall m \geq 0$, the map T^{-m} is defined and smooth on both $W^u_{\delta_1/2}(z)$ and $W^u_{\delta_1/2}(z')$. The distance between the inverse images $W_m := T^{-m}(W^u_{\delta_1/2}(z))$ and $W'_m := T^{-m}(W^u_{\delta_1/2}(z'))$ grows with m, and eventually these images may be separated by a singularity manifold. Let $m \geq 1$ be the largest integer that satisfies two conditions:
(i) T^{-m} is smooth on a connected domain in M that contains both $W^u_{\delta_1}(z)$ and $W^u_{\delta_1}(z')$;
(ii) $\rho(T^{-m}z, W'_m) \leq \delta_1$.

Observe that $\rho^s(z, W^u_{\delta_1/2}(z')) \le C' c_1 \delta_3$ for some C' determined by the minimum angle between the stable and unstable cone families. It is easy to see that $\rho^s(T^{-m}z, W'_m) \le C'' c_1 \delta_3 \Lambda^m_{\max}$, where C'' is another constant determined by the minimum angle between the stable and unstable cone families. On the other hand, we have the following lower bound on the inner radius of the manifold W_m: $\rho_{W_m}(T^{-m}z, \partial W_m) \ge \delta_1 \Lambda^{-m}_{\max}/2$. It follows from (H3) that the LUM's W_m and W'_m cannot be separated by singularities as long as the distance between them is $\ll$ the inner radius of W_m. Therefore, $2c_1\delta_3\delta_1^{-1}\Lambda^{2m}_{\max} \ge c''/C''$, where c'' is some constant determined by the minimum angle between the unstable cone family and $\mathcal{S}$. Recall that δ_3/δ_2 and δ_2/δ_1 are fixed, so that $\delta_3 = \tilde{c}\delta_1$ with some $\tilde{c}$ =const> 0. Hence, $m \ge \ln[(2\tilde{c}c_1)^{-1}c''/C'']/\ln \Lambda^2_{\max}$, so that m can be made arbitrarily large by choosing c_1 in the lemma very small. On the other hand, assuming (5.4) we arrive at

$$\rho^s(T^{-m}x, W'_m) \ge \delta_3 \Lambda^m_{\min}/2 = \tilde{c}\delta_1 \Lambda^m_{\min}/2$$

When m is large enough, the right hand side will be $\gg \delta_1$, which contradicts the requirement (ii) above (note that both manifolds W_m and W'_m will be tiny, of diameter $\ll \delta_1$). This proves the lemma. $\square$

We now get back to the intersections of $T^n R_V(z)$ with $R(z')$ for $z' \in A_{\delta_1}$. We set $\delta_4 = c_1\delta_3$. We also fix $n''_0 = \min\{n \ge 1 : \Lambda^n_{\min} > 2\}$

Proposition 5.5 *Let $z \in A_{\delta_1}$ and $n \ge n''_0$. Let V be a connected component of $W^1_n(z)$ and $x \in V$ such that $r_{V,n}(x) > \delta_1$ and $\rho(T^n x, z') < \delta_4$ for some $z' \in A_{\delta_1}$. Then the rectangle $T^n R_V(z)$ u-crosses the rectangle $R(z')$, i.e. $T^n R_V(z) \cap R(z')$ is* (i) *a u-subrectangle in $R(z')$ and* (ii) *an s-subrectangle in $T^n R_V(z)$.*

Proof. According to Lemma 5.4, the LUM $T^n V$ overshadows $W(z')$, and $\rho^s(W(z'), T^n V) \le \delta_3/2$. According to our choice of n''_0 and Proposition 5.3, the LUM $W^u(y)$ for every $y \in T^n R_V(z)$ overshadows $W(z')$, and $\rho^s(W(z'), W^u(y)) \le \delta_3$. This implies (ii). To prove (i), we need to show that $\forall x' \in W^1_\infty(z')$ the point $y' = W^s_{\delta_3}(x') \cap T^n V$ belongs in $T^n R_V(z)$. It is enough to show that the point $y = T^{-n}y' \in W^1_\infty(z)$. Firstly, $y \in V \subset W^1_n(z)$, so we only need to prove that $y \in W^1_{n+1+k}(z)$, $\forall k \ge 0$. Observe that $\rho(T^k x', \mathcal{S}) > \delta_2 \Lambda^{-k}$, $\forall k \ge 0$, since $x' \in W^1_\infty(z')$. Next, observe that $\rho(T^k x', T^k y') < \delta_3 \Lambda^{-k}$, $\forall k \ge 0$, by virtue of Proposition 5.3. Therefore, $\rho(T^k y', \mathcal{S}) > (\delta_2 - \delta_3)\Lambda^{-k} > \delta_2 \Lambda^{-n-k}$, the last inequality follows from (5.3). Thus, $y \in W^1_{n+1+k}(z)$, $\forall k \ge 0$. $\square$.

6 Rectangular structure and return times

The scheme of our proof of the part (c) of Theorem 1.1 is the following. Let μ be an SRB measure such that (T^n, μ) is ergodic $\forall n \ge 1$. According to the last remark of Sect. 3, μ is also mixing and Bernoulli. Clearly, there is a $\delta_0 > 0$ and a $z_1 \in A_{\delta_1}$ (remember that $\delta_1 = \delta_0/(2\bar{\beta})$) such that $\mu(R(z_1)) > 0$. Note also that for any other ergodic SRB measure

$\mu' \neq \mu$ we have $\mu'(R) = 0$. We fix such a δ_0 and one such $z_1 \in A_{\delta_1}$. We then denote, for brevity, $R = R(z_1)$, $W = W(z_1)$, $W^1_\infty = W^1_\infty(z_1)$, etc.

Let $\mathcal{Z} = \{z_1, z_2, \dots, z_p\}$ be a finite δ_4-dense subset of A_{δ_1} containing the above point z_1. We call $\mathcal{R} = \cup_i R(z_i)$ the rectangular structure. It is a finite union of rectangles that most likely overlap and do not cover M or even the support of μ.

We will partition the set W^1_∞ into a countable collection of subsets $W^1_{\infty,k}$, $k \geq 0$, such that for every $k \geq 1$ there is an integer $r_k \geq 1$ such that for the s-subrectangle $R_k \subset R$ with base $W^1_{\infty,k}$ the set $T^{r_k}(R_k)$ will be a u-subrectangle in some $R(z_i)$, $z_i \in \mathcal{Z}$. By the s-subrectangle $R_k \subset R$ with base $W^1_{\infty,k}$ we mean the set $R_k = \{x \in R : W^s(x) \cap W^1_\infty \in W^1_{\infty,k}\}$. Topologically, $W^1_{\infty,k}$, $k \geq 1$, are d_u-dimensional Cantor sets for systems with singularities. We will call them *gaskets*. We consider the fact that $T^{r_k}(R_k)$ is a u-subrectangle in some $R(z_i)$ as a *proper return* (of R_k into $\mathcal{R}$). We define a function $r(x)$ on W^1_∞ by $r(x) = r_k$ for $x \in W^1_{\infty,k}$, $k \geq 1$, and $r(x) = \infty$ for $x \in W^1_{\infty,0}$. We call $r(x)$ the return time.

L.-S. Young proved [13] the following:

Theorem 6.1 *If $\int_{W^1_\infty} r(x)\, d\nu_W < \infty$, then there is an SRB measure μ_R concentrated on $\cup_{n\geq 0} T^n R$. That measure is ergodic, thus unique.*

Theorem 6.2 *If $\nu_W\{r(x) > n\} \leq C\theta^n$, $\forall n \geq 1$, for some $C > 0$, $\theta \in (0,1)$, then the system (T, μ_R) enjoys an exponential decay of correlations and a central limit theorem.*

We state these theorems here in a slightly wider version than Young did in [13]. One gets her original theorems if one sets $p = 1$, i.e. when the rectangular structure contains just one rectangle. However, Young worked with finite unions of (overlapping) rectangles in Section 7 of [13], and showed that it was equivalent to working with one rectangle.

Alternatively, one can define the returns of R to itself rather than to $\cup_i R(z_i)$ and then apply the original Young's theorems (with $p = 1$) directly. This can be done by using the mixing property of the measure μ, along the lines of [3, 4], but this is not necessary in view of the above.

The uniqueness of μ_R in Theorem 6.1 implies $\mu_R = \mu$. Note also that if $T^{r_k}(R_k) \subset R(z_i)$, then $\mu(R(z_i)) > 0$, so there are no possible returns to rectangles $R(z_i) \subset \mathcal{R}$ of zero μ-measure, i.e. they can be simply ignored. In summary, it remains to define the function $r(x)$ and prove an exponential tail bound:

$$\nu_W\{r(x) > n\} \leq C\theta^n \tag{6.1}$$

for some $C > 0$, $\theta \in (0,1)$, and all $n \geq 1$.

In the rest of this section, we define the partition $W^1_\infty = \cup_k W^1_{\infty,k}$ and the return time $r(x)$. Our definition consists in several steps.

Initial growth. First, we take $n_1 = \max\{n_0', n_0''\}$. According to Lemma 5.2, we have (a) $Z^1_{n_1} < (2\delta_1)^{-1}$ and $Z[W^1_{n_1}, n_1] < 0.6/\delta_1$, i.e. the components of $T^{n_1} W^1_{n_1}$ are large enough, on the average, and

(b) $\nu_W\{x \in W^1_{n_1} : r_{W^1_n,n}(x) \geq \delta_1\} \geq 0.4\, \nu_W(W^1_{n_1})$, i.e. at least 40% of the points in $T^{n_1}W^1_{n_1}$ (with respect to the measure induced by ν_W) lie a distance $\geq \delta_1$ away from $\partial T^{n_1}W^1_{n_1}$.
(Recall that (b) actually follows from (a), cf. Lemma 5.2.) Let $W^g = T^{n_1}W^1_{n_1}$, and $\tilde{\nu}_{W^g} = T^{n_1}_* \nu_W | W^g$ the induced measure on W^g. For every connected component $V \subset W^g$ such that $\exists x_V \in V : \rho_V(x_V, \partial V) \geq \delta_1$ we arbitrarily fix one such point x_V. Then $x_V \in A_{\delta_1}$, and $\exists z_V \in \mathcal{Z}$ such that $\rho(x_V, z_V) < \delta_4$. We fix one such z_V, too. Then we label the set $T^{-n_1}(V \cap R(z_V))$ as one of our gaskets $W^1_{\infty,k}$, and we define $r_k = n_1$ on it. According to Proposition 5.5, $T^{r_k}(R_k)$ is a u-subrectangle in $R(z_V)$, indeed. Note that we are defining at most one gasket in each component V of W^g. We will sometimes slightly abuse the terminology and call the set $V \cap R(z_V)$ a gasket, too (strictly speaking, it is the image of a gasket).

Lemma 6.3 *There is a $q = q(T) > 0$ such that, independently of the choice of the points x_V and z_V in the components $V \subset W^g$, the just defined gaskets $W^1_{\infty,k}$ satisfy*

$$\nu_W\left(\cup W^1_{\infty,k}\right) \geq q\, \nu_W(W^1_{n_1})$$

Proof. The lemma follows from Lemmas 5.1 and 5.2, along with the absolute continuity (2.2). □

Thus, a certain fraction ($\geq q$) of W^g returns at the n_1-th iteration. This is the earliest return in our construction. Further returns are harder to define, and we first explain why. Let $n > n_1$ and $\exists x \in V : \rho_V(x, \partial V) > \delta_1$ for some connected component V of $T^n W^1_n$. If we arbitrarily pick some points x_V and z_V as before, then the set $T^{-n}(V \cap R(z_V))$ may overlap with some previously defined gaskets $W^1_{\infty,k}$, so we cannot label it as another gasket. To avoid possible overlaps, we perform the following construction.

Capture. Every connected component V of W^g where a point x_V has been picked is now subdivided into two connected sets: $V^c := W^u_{\delta_1/2}(x_V)$ and $V^f := V \setminus V^c$. Observe that V^c overshadows $W(z_V)$, according to Lemma 5.4, and so the gasket $V \cap R(z_V)$ defined above lies wholly in V^c. We say that V^c is 'captured' at the n_1-th iteration. The rest of V, i.e. the set V^f, is 'free to move'. The captured parts of W^g are taken out of circulation, for the moment, and the rest of W^g, let us call it W^f, is mapped further under T, it contains no points of the previously defined gaskets. Denote $W^f_n = W^f \cap T^{n_1}W^1_{n_1+n}$ for $n \geq 0$. Observe that the manifolds W^f_n, $n \geq 0$, correspond to a refined u-filtration $\{W^f_n, W^{f,1}_n, W^{f,0}_n\}$ of the u-manifold W^f in the sense of Sect. 4 with δ_2 replaced by $\delta_2\Lambda^{-n_1}$, so that $W^f_n = W^{f,1}_n$, $\forall n \geq 0$.

We would like to see, first of all, that the components of $T^n W^f_n$ for some $n \geq 0$ are large, on the average, precisely that $Z[W^f_n, n] < 0.6/\delta_1$. This may not be the case for $n = 0$, for the following reasons. The removal of the captured parts from W^g will create more boundary in the remaining part, W^f, and also reduce its measure. As a result,

$Z[W^f, W^f, 0] > Z[W^g, W^g, 0] = Z[W^1_{n_1}, n_1]$. However, the ε-neighborhood of the boundary increases at most twice $\forall\varepsilon > 0$, cf. the remark after Lemma 2.2. It is also clear that $\tilde{\nu}_{W^g}(W^f) > \tilde{\nu}_{W^g}(W^g)/2$. Therefore, $Z[W^f, W^f, 0] < 4 \cdot Z[W^1_{n_1}, n_1] < 2.4/\delta_1$. Applying then the part (ii) of Corollary 4.3 to the manifold W^f, which is possible according to Final Remark of Section 4, with δ_2 replaced by $\delta_2\Lambda^{-n_1}$ yields $Z[W^f, W^f_n, n] < (2\delta_1)^{-1}$ for all $n \geq n_2$, where $n_2 := [-\ln 9.6/\ln\alpha]+1$. Also, the part (iv) of the same corollary, along with (5.2), yields $\tilde{\nu}_{W^g}(W^f_n) > (1 - 0.06\,\Lambda^{-n_1}/d_u)\,\tilde{\nu}_{W^g}(W^f) > 0.9\,\tilde{\nu}_{W^g}(W^f)$ for all $n \geq 0$. Therefore, due to (4.4), we have $Z[W^f_n, n] = Z[W^f, W^f_n, n]\,\tilde{\nu}_{W^g}(W^f)/\tilde{\nu}_{W^g}(W^f_n) < 0.6/\delta_1$ for all $n \geq n_2$, as desired. In other words, it takes a fixed number of iterations, n_2, to recover the lost average size of the freely moving manifold, $T^n W^f_n$, $n \geq 0$, after the removal of the captured parts from W^g. As soon as this is done, i.e. at the iteration $n = n_2$, at least 40% of the image $T^n W^f_n$, will lie a distance $\geq \delta_1$ from its boundary, just as in the claim (b) above.

Now we inductively repeat the above procedure of picking points x_V, z_V in the large components V of the freely moving manifold, defining new gaskets $V \cap R(z_V)$, capturing disks covering the newly defined gaskets, moving the remaining manifold another n_2 iterations under T until its components grow large enough, on the average, etc. According to Lemma 6.3, the points of the freely moving manifold are being captured at an exponential rate: at least a fraction $q > 0$ of them is captured every n_2 iterations of T. Let $t_0(x)$, $x \in W^1_\infty$, be the number of iterations it takes to capture the image of the point x. Observe that $t_0(x) = n_1 + kn_2$ for some $k = 0, 1, \ldots$. Lemma 6.3 implies that

$$\nu_W(t_0(x) > n)/\nu_W(W^1_\infty) \leq C_0\theta_0^n \tag{6.2}$$

with $\theta_0 = q^{1/n_2} < 1$ and some $C_0 > 0$. In particular, $t_0(x) < \infty$ for a.e. $x \in W^1_\infty$.

Release. Next, we take care of the captured parts of the manifolds $T^n W^1_n$, $n \geq 1$. They are all very similar. Let $B^c \subset T^{n_c} W^1_{n_c}$ be a connected part captured at the n_c-th iteration of T, $n_c \geq n_1$. Then B^c is a perfect ball of radius $\delta_1/2$ in some connected component of $T^{n_c} W^1_{n_c}$. It carries the measure $\tilde{\nu}_{B^c} = T^{n_c}_* \nu_W | B^c$. The center x_c of the disk B^c belongs in A_{δ_1}, and there is a point $z_c \in \mathcal{Z}$ such that $\rho(x_c, z_c) < \delta_4$ and such that the set $B^c_R := B^c \cap R(z_c)$ makes a new gasket at the moment of capture. The points of the gasket successfully return to $R(z_c)$, i.e. $r(x) = n_c$ for $x \in T^{-n_c} B^c_R$. Denote also $B^c_\infty = B^c \cap T^{n_c} W^1_\infty$. We now have to take care of the set $B^c_\infty \setminus B^c_R$.

Denote $B^c_n = B^c \cap T^{n_c} W^1_{n_c+n}$ for $n \geq 0$. According to the remark after Lemma 5.2, we have

$$Z[B^c, B^c, 0] \leq 4d_u/\delta_1 \qquad \text{and} \qquad Z[B^c_n, n] < 0.6/\delta_1,\ \forall n \geq n'_0 \tag{6.3}$$

In other words, it takes n'_0 iterations of T to make the components of $T^n B^c_n$ large enough, on the average.

In order to define a new gasket in any large component V of $T^n B^c_n$ and avoid possible overlaps with the image $T^n B^c_R$ of the old gasket B^c_R, we will make sure that V contains no points of $T^n B^c_R$. To get a control of that, we define a 'release time' (we will call it also 'point release time'), $f(x)$, for points $x \in B^c_\infty \setminus B^c_R$. Loosely speaking, a point x

will be 'released' if $T^{f(x)}(x)$ is sufficiently far from $T^{f(x)}B^c_R$, so that for all $n \geq f(x)$ the component of $T^n B^c_n$ containing $T^n x$ will contain no points of $T^n B^c_R$.

The definition of the release time requires a classification of points $x \in B^c_\infty \setminus B^c_R$.

Type I points are such that there is an LSM $W^s(x)$ meeting the manifold $W^u_{\delta_1}(z_c)$ in one point, call it $h(x)$. Then $h(x) \notin W^1_\infty(z_c)$, otherwise x would have belonged in B^c_R. Hence, either $h(x) \in W^u_{\delta_1}(z_c) \setminus W^u_{\delta_1/3}(z_c)$ or $h(x) \in W^0_m(z_c)$ for some $m = m(x) \geq 0$. In the former case, we set $m(x) = 0$ and $\varepsilon(x) = \rho(h(x), W^u_{\delta_1/3}(z_c))$. In the latter case we set $\varepsilon(x) = \rho(T^m h(x), \partial T^m W^0_m(z_c))$. We now define the release time to be $f(x) = m(x) + \log_\Lambda(\delta_0/\varepsilon(x))$, one formula for both cases.

Type II points have no local stable manifolds that extend to $W^u_{\delta_1}(z_c)$. Let $x \in B^c_\infty$ be such a point. According to the second statement in Lemma 5.4, $\rho^s(x, W^u_{\delta_1}(z_c)) \leq \delta_3/2$. Hence, no local stable manifold $W^s(x)$ contains a stable disk of radius $\delta_3/2$ around x. Therefore, $x \notin M^+_{\Lambda,\delta_3/2}$, in virtue of the Fact of Section 3. Let then $m = m(x) = \min\{m' > 0 : \rho(T^{m'}x, \mathcal{S}) \leq \delta_3\Lambda^{-m'}/2\}$. We claim that, on the component of $T^m B^c_m$ containing $T^m x$, there are no points of $T^m B^c_R$ in the $(\delta_2\Lambda^{-m}/2)$-neighborhood of $T^m x$. Indeed, if some point $y \in T^m B^c_R$ were there, its local stable manifold $W^s(y)$ would contain a point $y' \in T^m W^1_\infty(z_c)$, which is at distance $\leq \delta_3\Lambda^{-m}$ from y. Then $\rho(y', \mathcal{S}) \leq \delta_2\Lambda^{-m}$, since $\delta_3 \leq \delta_2/3$, cf. (5.3). This, however, contradicts the definition of $W^1_\infty(z_c)$, cf. (4.1). We now define the release time to be $f(x) = 2m(x) + \log_\Lambda(2\delta_0/\delta_2)$.

It is clear that for any point $x \in B^c_\infty \setminus B^c_R$ of either type I or II and any $n \geq f(x)$ the point $T^n x$ should be at least the distance δ_0 from $T^n B^c_R$ (measured along $T^n B^c_n$), so that, in fact, the component of $T^n B^c_n$ containing $T^n x$ does not intersect $T^n B^c_R$ at all.

Therefore, we are free to define new gaskets and capture new disks on any component $V \subset T^n B^c_n$ that contains at least one released point, i.e. such that $\exists x \in T^{-n}V : f(x) \leq n$. We can only define a gasket, however, if $\exists x \in V : \rho_V(x, \partial V) \geq \delta_1$, i.e. if V is large enough. Hence the next step in our construction.

Growth. To get a control on the size of the components of $T^n B^c_n$, we gather, for every $n \geq 0$, the components $V \subset T^n B^c_n$ released at the n-th iteration. We say that V is released at the n-th iteration if at least one point of V is released at this iteration, and none of the points of the component of $T^i B^c_i$ that contains $T^{-(n-i)}V$ is released at the i-th iteration for any $i = 0, \ldots, n-1$. In that case we define another function, $s(x) = n$, on $B^c_\infty \cap T^{-n}V$. We call $s(x)$ the 'component release time' (as opposed to the point release time $f(x)$ defined earlier). Observe that $s(x)$ is defined for each $x \in B^c_\infty \setminus B^c_R$ and $s(x) \leq f(x)$.

Fix an $s \geq 0$ (the 'component release time') and let

$$\tilde{W} = \tilde{W}(s) = \cup\{V \subset T^s B^c_s : s(x) = s \ \ \forall x \in B^c_\infty \cap T^{-s}V\} \tag{6.4}$$

be the union of the components of $T^s B^c_s$ released exactly at the s-th iteration. The manifold $\tilde{W}$ carries the measure $\tilde{\nu}_{\tilde{W}} = T^s_* \tilde{\nu}_{B^c} | \tilde{W}$. Observe that the sets $\tilde{W} \cap T^s B^c_{s+n}$,

$n \geq 0$, correspond to a refined u-filtration $\{\tilde{W}_n, \tilde{W}_n^1, \tilde{W}_n^0\}$ of $\tilde{W}$ in the sense of Sect. 4 with δ_2 replaced by $\delta_2 \Lambda^{-n_c - s}$, so that $\tilde{W} \cap T^s B_{s+n}^c = \tilde{W}_n^1$. Denote

$$p(s) = \tilde{\nu}_{\tilde{W}}(\tilde{W}_\infty^1)/\tilde{\nu}_{\tilde{W}}(\tilde{W}) = \tilde{\nu}_{\tilde{W}}(\tilde{W} \cap T^s B_\infty^c)/\tilde{\nu}_{\tilde{W}}(\tilde{W}) \tag{6.5}$$

In a trivial case, when $p(s) = 0$, there is nothing in $\tilde{W}$ to worry about, and we disregard such a release time s. If $p(s) > 0$, then Proposition 4.4 applies to $(\tilde{W}, \tilde{\nu}_{\tilde{W}})$, according to Final Remark (Part 2). Hence, $\exists n \geq 1$ such that $Z[\tilde{W}_n^1, n] \leq 0.6/\delta_1$, i.e. the components of $T^n \tilde{W}_n^1$ are large enough, on the average. Let g be the minimum of such n's. We call g the 'growth time' and define another function, $g(x) = g$ on $B_\infty^c \cap T^{-s}\tilde{W}$ (note that $g(x)$ is a constant function on $B_\infty^c \cap T^{-s}\tilde{W}$, and it only depends on s, so we will also write it as $g(s)$).

Consider now the manifold $\hat{W} = T^g \tilde{W}_g^1$ and the measure $\tilde{\nu}_{\hat{W}} = T_*^g \tilde{\nu}_{\tilde{W}} | \hat{W}$ on it. Denote $\hat{W}_\infty^1 = T^g(\tilde{W}_\infty^1) = T^g(\tilde{W} \cap T^s B_\infty^c)$ the subset of $\hat{W}$ we will keep track of. According to Proposition 4.4, we have
(c) $\tilde{\nu}_{\hat{W}}(\hat{W}_\infty^1) > 0.9\, \tilde{\nu}_{\hat{W}}(\hat{W})$, and
(d) $Z[\hat{W}, \hat{W}, 0] \leq 0.6/\delta_1$, so that at least 40% of the points in $\hat{W}$ (with respect to the measure $\tilde{\nu}_{\hat{W}}$) lie a distance $\geq \delta_1$ away from $\partial \hat{W}$.
Next, we define new gaskets and capture disks covering them on the large components of $\hat{W}$, as we did to W^g early in this section. Then we move the remaining parts of $\hat{W}$ under T^{n_2}, again define new gaskets and capture new disks, etc., exactly repeating the procedure applied to W^g during the period of initial growth. Let $t(x)$ be the 'capture time' for $x \in \hat{W}_\infty^1$, i.e. the minimum of $t \geq 0$ such that $T^t x$ belongs in a captured disk. Note that $T^t x$ might be luckily covered by a gasket, then it returns to $\mathcal{R}$, or else it has to be iterated further under T.

Lemma 6.4 *We have $\tilde{\nu}_{\hat{W}}(t(x) > n)/\tilde{\nu}_{\hat{W}}(\hat{W}_\infty^1) \leq C_0 \theta_0^n$ with the same constants as in (6.2).*

Proof. The lemma follows from the properties (c) and (d) of the manifold $\hat{W}$ just like Lemma 6.3 and (6.2) followed from the similar properties of the manifold W^g. □

Summary. We summarize the ideas of our constructions. For every release time $s \geq 0$ we take the union $\tilde{W}$ of the components of $T^s B_s^c$ released exactly at the s-th iteration, iterate them further g times without capturing or defining gaskets, then they become large enough, on the average. Then our construction repeats inductively. We define new gaskets and capture new disks on the components of $T^t \tilde{W}$, $t \geq g$, the gaskets make successful return to $\mathcal{R}$ at the time they are defined, the captured points around gaskets are iterated further and eventually released, the released components grow in size until they become large enough, on the average, then new gaskets are defined, etc. For a.e. point $x \in W_\infty^1$, the cycle 'growth→capture→release→growth…' repeats until the point returns to $\mathcal{R}$ at a moment of capture. If it never returns, however, we put it into $W_{\infty,0}^1$ and set $r(x) = \infty$. This concludes our definition of the partition $W_\infty^1 = \cup_k W_{\infty,k}^1$ and the return time $r(x)$.

7 Exponential tail bound

In this section we prove the exponential tail bound (6.1). First, we show that the points of any captured disk B^c are being released at an exponential rate.

Lemma 7.1 *There are $C_1 > 0$ and $\theta_1 \in (0,1)$ such that for every captured disk B^c we have $\tilde{\nu}_{B^c}(f(x) > n)/\tilde{\nu}_{B^c}(B^c) < C_1\theta_1^n$, $\forall n \geq 0$.*

Proof. Recall that we have defined the point release time $f(x)$ separately for the captured points of types I and II. First, we take care of points of type I. Recall that for every point x of type I we have defined a point $h(x) \in W^u_{\delta_1}(z_c)$ and two numbers, $m(x) \geq 0$ and $\varepsilon(x) > 0$. In view of the absolute continuity (2.2), it is enough to estimate the measure $\nu_{W^u_{\delta_1}(z_c)}\{h(x) : f(x) > n\}$. The measure of the set $\{h(x) : m(x) > n/2\}$ is exponentially small in n due to the part (iii) of Corollary 4.3 and (5.1). Next, for every $0 \leq m \leq n/2$, the measure of the set $\{h(x) : m(x) = m \ \& \ \varepsilon(x) < \delta_2\Lambda^{-n/2}\}$ is exponentially small in n, uniformly in m, in view of the definition of Z^0_m and the part (i) of Corollary 4.3 and (5.1). Thus, the points of type I obey our claim.

For any point x of type II with $m(x) = m$, observe that $\rho_V(T^m x, \partial V \cup \mathcal{S}) < C'\delta_3\Lambda^{-m}/2$, where V is the component of $T^m B^c$ containing $T^m x$, and $C' > 0$ is a constant depending on the minimum angle between unstable cones and $\mathcal{S}$. Hence, $\rho_{V'}(T^{m+1}x, \partial V') \leq C'\Lambda_{\max}\delta_3\Lambda^{-m}/2$, where V' is the component of $T^{m+1}B^c$ containing $T^{m+1}x$. The measure of the set of such points is then exponentially small in m in view of the definition of Z_{m+1} in Sect. 2 as applied to the u-manifold $U = B^c$, combined with the part (ii) of Theorem 2.1 and (6.3). □.

For brevity, we normalize the measure $\tilde{\nu}_{B^c}$, so that $\tilde{\nu}_{B^c}(B^c) = 1$. The next lemma shows that the released components in the images of any captured disk B^c grow at an exponential rate:

Lemma 7.2 *There are $C_2 > 0$ and $\theta_2 \in (0,1)$ such that for every captured disk B^c we have $\tilde{\nu}_{B^c}(s(x) + g(x) > n) < C_2\theta_2^n$, $\forall n \geq 0$.*

Proof. Let $s \geq 0$ be fixed. We will use the notations $\tilde{W}(s)$ and $p(s)$ introduced by (6.4) and (6.5). On the set $B^c_\infty \cap T^{-s}\tilde{W}(s)$, the functions $s(x) = s$ and $g(x) = g(s) = g$ have constant values. Proposition 4.4 implies that $g \leq a_1(\ln Z[\tilde{W}(s), \tilde{W}(s), 0] - \ln p(s)) + b_1$. Let $q(s) := \tilde{\nu}_{B^c}(T^{-s}\tilde{W}(s))$, then $Z[\tilde{W}(s), \tilde{W}(s), 0] \leq 4d_u(\delta_1 q(s))^{-1}$, according to (6.3). Therefore, $g \leq -a_1 \ln[p(s)q(s)] + \text{const}$, so that

$$p(s)q(s) \leq \text{const} \cdot \exp\left(-g/a_1\right) \tag{7.1}$$

Observe that

$$p(s)q(s) = \tilde{\nu}_{B^c}(B^c_\infty \cap T^{-s}\tilde{W}(s)) \leq C_1\theta_1^s \tag{7.2}$$

due to Lemma 7.1, since $s(x) \leq f(x)$. Now let $\theta_2^2 = \max\{\theta_1, e^{-1/a_1}\}$. Then (7.1) and (7.2) imply that, for all $s \geq 0$,

$$\tilde{\nu}_{B^c}(B^c_\infty \cap T^{-s}\tilde{W}(s)) = p(s)q(s) \leq \text{const} \cdot \theta_2^{s+g}$$

The lemma now follows immediately. □.

We are now in a position to prove the tail bound (6.1). Let $x \in W^1_\infty$. The point x first goes through the period of initial growth, which takes $t_0(x)$ iterations. Then it is captured and either returns or goes through one or more cycles of 'release→growth→capture' before it makes return. Let $N(x) \geq 0$ be the number of cycles the point x goes through before it returns, and $s_i(x), g_i(x), t_i(x)$ be the lengths of the 'release', 'growth', and 'capture' periods, respectively, in the i-th cycle. Then

$$r(x) = t_0(x) + \sum_{i=1}^{N(x)} [s_i(x) + g_i(x) + t_i(x)]$$

We have already proved exponential tail bounds for $t_0(x)$, cf. (6.2), for $t_i(x)$, cf. Lemma 6.4, and for the sum $s_i(x) + g_i(x)$, cf. Lemma 7.2. Furthermore, since a certain fraction ($\geq q$) of every captured disk makes return (due to Lemma 5.1 and the absolute continuity (2.2), in the same way as in the proof of Lemma 6.3), we get an exponential tail bound on $N(x)$: $\nu_W(N(x) > n) \leq (1-q)^n$ for all $n \geq 0$.

Now an exponential tail bound on $r(x)$ can be obtained by a standard argument developed in [4] (pp. 129–130) and used in [13] (Sublemma 6 in Section 7).

Instead of repeating that argument, we present a different one here, of a completely probabilistic nature. Its relevance to our previous discussion will be quite clear. Let ξ_n, $n \geq 1$, be a sequence of independent identically distributed random variables taking positive integral values and satisfying an exponential tail bound $P(\xi_i = n) \leq c_1 \lambda_1^n$ for some $c_1 > 0$, $\lambda_1 \in (0,1)$. Let also N be a random variable independent from all ξ_i's, taking positive integral values, and satisfying an exponential tail bound $P(N = n) \leq c_2 \lambda_2^n$ for some $c_2 > 0$ and $\lambda_2 \in (0,1)$. Let $S_N = \sum_{i=1}^N \xi_i$.

Proposition 7.3 *The random variable S_N satisfies an exponential tail bound $P(S_N = n) \leq c\lambda^n$ with some $c > 0$ and $\lambda \in (0,1)$.*

Proof. The generating function

$$G_\xi(z) = \sum_{n=1}^\infty P(\xi_i = n)\, z^n$$

is analytic in the open disk $|z| < \lambda_1^{-1}$. The generating function of S_N is

$$G_{S_N}(z) = \sum_{n=1}^\infty P(N = n) G_\xi^n(z)$$

Since $|G_\xi(z)| \leq 1$ on the closed unit disk $|z| \leq 1$, then for any $1 < A < \min\{\lambda_1^{-1}, \lambda_2^{-1}\}$ we have $|G_\xi(z)| \leq A$ on some larger open disk $|z| < 1 + \varepsilon_A$, $\varepsilon_A > 0$. Then $G_{S_N}(z)$ is an analytic function in the open disk $|z| < 1+\varepsilon_A$. This implies $P(S_N = n) \leq \text{const}\cdot(1+\varepsilon')^{-n}$ for $\varepsilon' < \varepsilon_A$. □

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Proposed running head: Statistical properties