Spatial structure of Sinai-Ruelle-Bowen measures

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June 23, 2014

Abstract

Sinai-Ruelle-Bowen measures are the only physically observable invariant measures for billiard dynamical systems under small perturbations. These measures are singular, but as it was noted in [1], marginal distributions of spatial and angular coordinates are absolutely continuous. We generalize these facts and provide full mathematical proofs.

1 Introduction

This work is motivated by our earlier studies [1] of physically observable properties of Sinai-Ruelle-Bowen (SRB) measures for 2D periodic Lorentz gases (Sinai billiards) with finite horizon, under small perturbations.

Sinai billiards and their perturbations have some rough singularities, but on the other hand they are strongly hyperbolic [3]. Precisely, their Lyapunov exponents are non-zero and unstable vectors grow uniformly, at a rate $\geq C\Lambda^n$, where n is the collision counter and $\Lambda > 1$ the so called hyperbolicity constant. While Sinai billiards are equilibrium systems preserving a smooth (Liouville) measure, their perturbations are non-equilibrium systems whose natural invariant measures (steady states) are usually singular – they are SRB measures.

An SRB measure is an ergodic invariant probability measure with absolutely continuous conditional distributions on unstable manifolds. SRB measures are the only physically observable measures because their basins of attraction have positive Lebesgue volume; see [14] and [11, Sect. 5.9]. Perturbations of Sinai billiards have unique SRB measures, which are mixing and Bernoulli [3, 4].

In [1] we studied particularly interesting perturbations of Sinai billiards where the particle moved under a small constant external field $\mathbf{E}$ subject to a Gaussian thermostat that kept its speed constant. For that model, the SRB measure was constructed long ago [7, 8], and it was proved that the (global) current $\mathbf{J}$ of the particle satisfied $\mathbf{J} = \sigma\mathbf{E} + o(|\mathbf{E}|)$, with the conductivity σ given by a standard Green-Kubo formula.

In that model, as well as in many other perturbations of Sinai billiards, the SRB measure is singular with respect to the Lebesgue measure (the Hausdorff dimension of the SRB measure is lower than that of the phase space [7]). It is also argued in statistical mechanics that many multiparticle systems (gases and fluids) under external forces develop nonequilibrium steady states

that behave as SRB measures, in particular they are singular with respect to the phase volume (this is known as "chaotic hypothesis" or "Axiom C" [10]).

However one rarely observes the a steady state on the entire phase space to see its singularity. Usually one observes selected variables, such as positions or velocities of certain moving particle(s). And computer simulations show that those selected variables have surprisingly continuous distributions. In [1] we showed that for the Lorentz gas model three selected variables – the local particle density, the local current, and the angular velocity distribution have continuous densities. We derived Green-Kubo type formulas for those densities.

In [1] we only sketched the arguments and presented numerical evidence supporting our conclusions. In this paper we provide full mathematical proofs and generalize our conclusions to wider classes of perturbations and selected variables.

2   Model

Our work is an extension of papers [3] and [4], and for consistency we follow their definitions and notations whenever possible. We also refer the reader to these papers for more details on the model.

Let $\mathcal{D} = \mathbb{T}^2 \setminus \cup_{i=1}^{k} \mathcal{B}_i$ be a 2D torus without a finite union of disjoint open convex domains $\mathcal{B}_i$ whose boundary is C^3 smooth and has non-vanishing curvature. Two particular tables of that sort are shown in Figure 1.
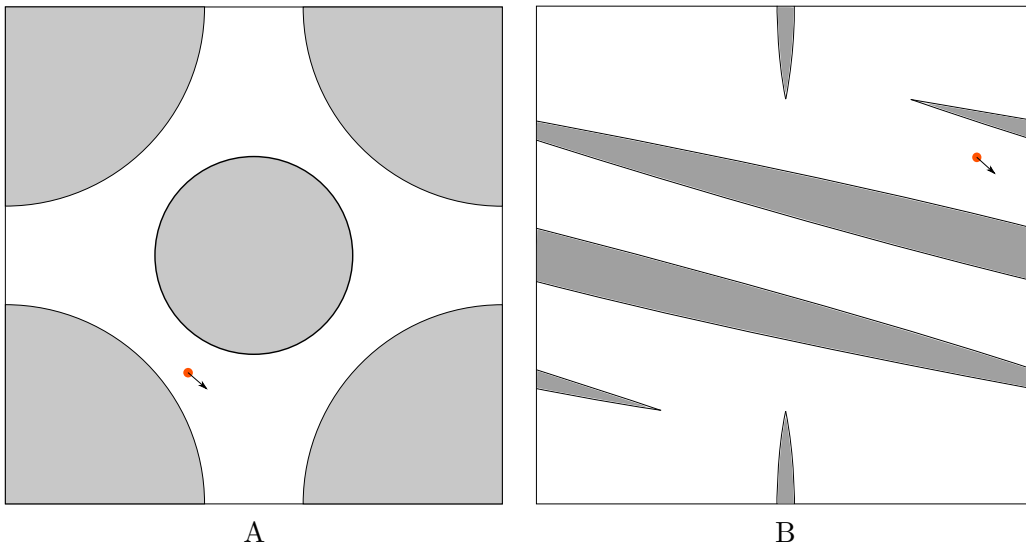

Figure 1: Tables A and B used in our numerical experiments with Gaussian Thermostat.

A particle moves in $\mathcal{D}$ according to equations

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = \mathbf{F}(\mathbf{p}, \mathbf{q}) \end{cases} \tag{1}$$

where $\mathbf{F}(\mathbf{p}, \mathbf{q}) = (F_1, F_2)$ is a stationary (independent of time) force. Collisions with obstacles are elastic (the speed is preserved) and specular (the angle of reflection is equal to the angle of

incidence). We are studying this model under three assumptions on the geometry of the table and the force:

Assumption A (additional integral). A smooth function $\mathcal{E}(\mathbf{q}, \mathbf{p})$ is preserved by the dynamics, $\Omega := \{\mathcal{E}(\mathbf{p}, \mathbf{q}) = \text{const}\}$ is a compact 3D manifold, and for every $\mathbf{q}$ and $\mathbf{p} \neq 0$ the ray $\{(\mathbf{q}, s\mathbf{p}), s > 0\}$ intersects Ω in exactly one point.

Under this assumption, Ω can be parameterized by (x, y, θ) where $(x, y) \in \mathcal{D}$ is the position on the torus and θ is the angle of motion measured between $\mathbf{p}$ and the positive x-axis. Equations of motion can now be given as

$$\dot{x} = p \cos \theta, \quad \dot{y} = p \sin \theta, \quad \dot{\theta} = ph$$

where

$$p = \|\mathbf{p}\| > 0 \quad \text{and} \quad h = (-F_1 \sin \theta + F_2 \cos \theta)/p^2.$$

Note that p does not have to be constant, but it is bounded away from 0 and infinity: $0 < p_{\min} \leq p \leq p_{\max} < \infty$ (due to the compactness of Ω).

For a function f on Ω let f_x, f_y, f_θ denote its partial derivatives, and $\|f\|_{C^2}$ the maximum of f and its first and second partial derivatives over Ω. Let $B_0 = \max\left(p_{\min}^{-1}, \|p\|_{C^2}, \|h\|_{C^2}\right)$.

Assumption B (smallness of the force). The force $\mathbf{F}$ and its first derivatives are small:

$$\max(|h|, |h_x|, |h_y|, |h_\theta|) \leq \delta_0$$

More precisely, we require that for any given $B_* > 0$ there is a small $\delta_* = \delta_*(\mathcal{D}, B_*)$ such that all our results hold whenever $B_0 < B_*$ and $\delta_0 < \delta_*$.

Assumption C (finite horizon). There is an $L > 0$ so that every straight line on the torus of length L crosses the interior of at least one obstacle. (Both tables A and B in Figure 1 have finite horizon.)

A particular example of the force is *Gaussian thermostat*, which was the subject of paper [1], being a physically interesting model of electrical conductance. There $\mathbf{F} \cdot \mathbf{p} = 0$, thus $\mathcal{E}(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\|\mathbf{p}\|^2$ is preserved by the dynamics. For more examples see [3, Section 2].

3 Standard notation and facts

Flow Φ^t acts on the *phase space* Ω, which is a 3D manifold.

Collision space $\mathcal{M} \subset \Omega$ is a set of points where the particle undergoes a collision with $\partial \mathcal{D}$. Now $\mathcal{M}$ can be parameterized by (r, φ) where r is an arclength parameter along $\partial \mathcal{D}$ and φ is an angle between the particle's outgoing velocity and the inward normal to $\partial \mathcal{D}$. Note that $-\pi/2 \leq \varphi \leq \pi/2$ (see [3]), thus $\mathcal{M}$ can be identified with a finite union of cylinders $\cup_{i=1}^k \partial \mathcal{B}_i \times [-\pi/2, \pi/2]$, hence the collision space is independent of the force $\mathbf{F}$.

Collision map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is the natural first return map on $\mathcal{M}$. It preserves a unique SRB measure ν; see [3]. We denote the time between collisions by $\tau : \mathcal{M} \to \mathbb{R}$. Now Φ^t can be represented as a suspension flow with base $\mathcal{M}$ and the ceiling function τ. The flow Φ^t preserves a unique SRB measure μ; see [4]. The map $\mathcal{F}$ and the flow Φ^t are ergodic, mixing, and Bernoulli, they enjoy strong statistical properties [4].

We will use subscript "0" in ν_0, $\mathcal{F}_0$, μ_0, Φ_0^t etc. to refer to the unperturbed (billiard) dynamics on $\mathcal{D}$, i.e., to the case $\mathbf{F} = 0$.

There is a simple relation between μ and ν: if $F : \Omega \to \mathbb{R}$ is a bounded function such that $f(X) = \int_0^{\tau(X)} F(\Phi^t(X)) \, dt$, then

$$\mu(F) = \frac{\nu(f)}{\nu(\tau)}. \tag{2}$$

In addition to natural singularities of $\mathcal{F}$ (the preimages of grazing collisions characterized by $\varphi = \pm\pi/2$) we need to cut $\mathcal{M}$ into countably many *homogeneity strips* along the lines $\{\varphi = \pm(\pi/2 - k^{-2})\}$ for all $k \geq k_0$, forcing $\mathcal{F}$ to be discontinuous on the preimages of these lines as well; see [3].

Collision space $\mathcal{M}$ has a measurable partition into *homogeneous unstable manifolds* (or h-fibers) that are increasing curves in the (r, φ) coordinates with slopes uniformly bounded away from 0 and ∞ and uniformly bounded curvature. H-fibers end on *singularity curves* that are images of the lines $\{\varphi = \pm\pi/2\}$ and boundaries of the homogeneity strips. It is important for us that the singularity curves are nondecreasing in (r, φ) coordinates and there are countably many of them. For almost every point $X \in \mathcal{M}$ (with respect to both the Lebesgue measure on $\mathcal{M}$ and the SRB measure ν) there exists an h-fiber $\gamma = \gamma(X)$ that contains X. The SRB measure ν on $\mathcal{M}$ may be singular with respect to the Lebesgue measure, but its conditional distributions on h-fibers are absolutely continuous with respect to the arclength measure.

For $X, Y \in \mathcal{M}$ we define the future separation time $\mathbf{s}_+(X, Y)$ as the first $n \geq 0$ for which $\mathcal{F}^n(X)$ and $\mathcal{F}^n(Y)$ belong to different connected components of $\mathcal{M}$. Similarly, $\mathbf{s}_-(X, Y)$ is the first $n \geq 0$ for which $\mathcal{F}^{-n}(X)$ and $\mathcal{F}^{-n}(Y)$ belong to different connected components of $\mathcal{M}$.

A function $f : \mathcal{M} \to \mathbb{R}$ is *dynamically Hölder continuous* if there are $0 < \theta_f < 1$ and $C_f > 0$ such that for any X and Y lying on one unstable curve[1]

$$|f(X) - f(Y)| \leq C_f \theta_f^{\mathbf{s}_+(X,Y)}$$

and for any X and Y lying on the same stable curve

$$|f(X) - f(Y)| \leq C_f \theta_f^{\mathbf{s}_-(X,Y)}.$$

Dynamical Hölder continuity implies boundedness of f. The class of dynamically Hölder continuous functions is large, for example it includes all piecewise Hölder-continuous functions whose discontinuities coincide with those of $\mathcal{F}^{\pm m}$ for some $m \geq 0$.

We will say that a function ρ is *regular* on an unstable curve γ if

$$|\ln \rho(X) - \ln \rho(Y)| \leq C_r \theta_r^{\mathbf{s}_+(X,Y)} \tag{3}$$

where $\theta_r = \Lambda^{-1/6} < 1$ and C_r is a sufficiently large constant that is determined by the geometry of the table and can be chosen arbitrarily high.

A *standard pair* is (γ, ν_γ) is an unstable curve γ with a probability measure ν_γ on it which has a regular density with respect to the arclength measure. More generally, a *standard family* is an arbitrary collection $\mathcal{G} = (\gamma_\alpha, \nu_\alpha)$, $\alpha \in \mathfrak{A}$, of standard pairs with a probability factor measure $\lambda_\mathcal{G}$ on the index set $\mathfrak{A}$ (one can naturally define a metric on the space of all standard pairs, see

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[1]A curve is *unstable* if its tangent vectors belong to unstable cones [9, Sect. 4.5], i.e. $C_1 < d\varphi/dr < C_2$ for some positive constants $C_1 < C_2$. Note that unstable curves are defined on $\mathcal{M}$ before it is cut into connected components, i.e., they can cross singularity lines and borders of the homogeneity strips.

[5, Proposition 8.1], then $\mathfrak{A}$ becomes a metric space with the respective Borel σ-algebra). Every standard family $\mathcal{G}$ naturally induces a measure on $\mathcal{M}$ by

$$\nu_{\mathcal{G}}(A) = \int_{\mathfrak{A}} \nu_{\alpha}(A \cap \gamma_{\alpha}) \, d\lambda_{\mathcal{G}}(\alpha).$$

Every point X on the unstable curve $\gamma \in \mathcal{G}$ breaks it into two pieces. Denote by $r_{\mathcal{G}}(X)$ the length of the shorter one. Let

$$\mathcal{Z}_{\mathcal{G}} = \sup_{\varepsilon > 0} \frac{\nu_{\mathcal{G}}(\{r_{\mathcal{G}}(X) < \varepsilon\})}{\varepsilon}. \tag{4}$$

A standard family $\mathcal{G}$ is **proper** if $\mathcal{Z}_{\mathcal{G}} < C_p$ where C_p is a large but fixed constant. A standard family consisting of all h-fibers together with the conditional measures induced by ν is proper [4, p. 96].

4 Regularity of projections

We will show that despite the singularity of the SRB measure with respect to the Lebesgue measure, its projections that are *transverse* to the unstable manifolds have continuous densities.

The collision space $\mathcal{M}$ admits a measurable partition Γ into h-fibers $\gamma \subset \mathcal{M}$. The SRB measure ν induces conditional probability measures ν_{γ} on h-fibers $\gamma \in \Gamma$ and a factor measure λ on Γ with a standard σ-algebra (see, e.g., [9, p. 287]). The measures ν_{γ} are absolutely continuous with respect to the arclength on γ. Moreover, the corresponding density functions ρ_{γ} are C^1 smooth (see, e.g., [9, Sect. 5.2]) and regular as defined above.

The length of h-fibers, as a function

$$\begin{aligned} L: \quad & \mathcal{M} \to \mathbb{R} \\ & X \mapsto |\gamma| \quad \text{for } \gamma \ni X \end{aligned}$$

is measurable by the dominated convergence theorem: for almost every $X \in \mathcal{M}$ let $B_n(X)$ be a connected component of the domain of $\mathcal{F}^{-n}$ which contains X. Let $L_n(X) = \sup\{|\gamma| : \gamma \in \Gamma, \gamma \subset B_n(X)\}$ be a supremum of lengths of h-fibers in $B_n(X)$. Then $L(X) = \lim_{n \to \infty} L_n(X)$ (see the beginning of Chapter 5 in [9] for more information on the structure of Γ.) Hence the length of h-fibers is also a measurable function on Γ, by a straightforward verification.

Since Γ is a proper standard family, $\mathcal{Z}_{\Gamma} < \infty$, and therefore [9, Sect. 7.4]

$$\int_{\Gamma} \frac{d\lambda(\gamma)}{|\gamma|} < \infty. \tag{5}$$

This allows us to renormalize the conditional measures and the factor measure replacing $d\nu_{\gamma}$ and $d\lambda(\gamma)$ as follows:

$$d\hat{\nu}_{\gamma} = |\gamma| d\nu_{\gamma} \qquad \text{and} \qquad d\hat{\lambda}(\gamma) = \frac{d\lambda(\gamma)}{|\gamma|}.$$

The new factor measure $\hat{\lambda}$ is still finite, according to (5).

Since ν_{γ} is a probability measure for each γ, and ρ_{γ} is C^1 smooth and regular, ρ_{γ} is bounded by $|\gamma|^{-1} e^{-C_r}$ from below and by $|\gamma|^{-1} e^{C_r}$ from above, and thus the density $\hat{\rho}_{\gamma}$ of the new measure $\hat{\nu}_{\gamma}$ is also C^1 smooth and bounded *uniformly* by e^{-C_r} and e^{C_r}.

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Lemma 4.1. *The ν-measure of every h-fiber $\gamma \in \Gamma$ is zero, i.e. $\lambda(\gamma) = \hat{\lambda}(\gamma) = 0$.*

Proof. Assume that an h-fiber γ has positive measure. Recall that ρ_γ is bounded away from 0 by $|\gamma|^{-1}e^{-C_r}$. Note that the collision map $\mathcal{F}$ is piecewise continuous and bijective. For every $n > 0$ the curve $\mathcal{F}^{-n}(\gamma)$ is a piece of some h-fiber; it carries the same measure as γ, but its length is $O(\Lambda^{-n}|\gamma|)$. By the Poincaré recurrence theorem $\mathcal{F}^{-n}(\gamma)$ must overlap with γ infinitely many times, which implies that the measure of a relatively long piece of γ is equal to the measure of an arbitrarily short piece, hence density ρ_γ cannot be bounded. $\square$

In the following theorem $\mathcal{X}$ denotes an abstract manifold, but Γ is still the partition of our collision space $\mathcal{M}$ and $\hat{\lambda}$ is still the factor measure defined above.

Theorem 4.1. *Let $\mathcal{X}$ be a compact Riemannian manifold equipped with Lebesgue measure dx. Assume that for each $\gamma \in \Gamma$ there is a function $p_\gamma \colon \mathcal{X} \to \mathbb{R}$, which is bounded uniformly in γ. Define a (possibly signed) measure ξ on $\mathcal{X}$ by*

$$\xi(A) = \int_\Gamma \zeta_\gamma(A)\, d\hat{\lambda}(\gamma), \qquad \zeta_\gamma(A) = \int_A p_\gamma(x)\, dx$$

where we assume that $\zeta_\gamma(A)$ is measurable, as a function of γ, for every measurable $A \subset \mathcal{X}$. Assume that for every point $x \in \mathcal{X}$

$$\hat{\lambda}\{\gamma \in \Gamma \colon p_\gamma \text{ is discontinuous at } x\} = 0. \tag{6}$$

Then the measure ξ has a continuous density on $\mathcal{X}$ with respect to dx. If, in addition, for every $x \in \mathcal{X}$

$$\hat{\lambda}\{\gamma \in \Gamma \colon p_\gamma(x) > 0\} > 0 \tag{7}$$

then the density of ξ is strictly positive and bounded away from zero.

Proof. For $x \in \mathcal{X}$ and $r > 0$ let $B_r(x) \subset \mathcal{X}$ denote the ball of radius r centered at x and $|B_r(x)|$ its Lebesgue volume. Then for $\hat{\lambda}$-almost every $\gamma \in \Gamma$

$$\lim_{r \to 0} \frac{\zeta_\gamma(B_r(x))}{|B_r(x)|} = p_\gamma(x).$$

By the bounded convergence theorem $p_\gamma(x)$ is a measurable function on Γ and

$$\lim_{r \to 0} \frac{\xi(B_r(x))}{|B_r(x)|} = p(x) := \int_\Gamma p_\gamma(x)\, d\hat{\lambda}(\gamma).$$

By the Lebesgue differentiation theorem $p(x)$ is almost everywhere on $\mathcal{X}$ equal to the density of ξ. The continuity and positivity of $p(x)$ under our assumptions follows directly from the bounded convergence theorem. $\square$

Next we show how Theorem 4.1 implies the continuity of various projections of the SRB measure ν. In all our cases, every $x \in \mathcal{X}$ will be a discontinuity point for at most countably many functions $p_\gamma(x)$. This, along with Lemma 4.1, will guarantee the assumption (6).

4.1 Angular distribution for the collision map

Let $\mathcal{X} = [-\pi/2, \pi/2]$ and $P\colon \mathcal{M} \to \mathcal{X}$ be the projection onto the φ-coordinate. Let ξ be the corresponding pushforward of ν.

Theorem 4.2. *The measure ξ is absolutely continuous on $\mathcal{X} = [-\pi/2, \pi/2]$ with a positive continuous density.*

Proof. For $A \subset \mathcal{X}$ we have $\xi(A) = \nu(P^{-1}(A))$ and

$$\xi(A) = \int_\Gamma \zeta_\gamma(A)\, d\hat{\lambda}(\gamma), \qquad \zeta_\gamma(A) = \hat{\nu}_\gamma\left(\gamma \cap P^{-1}(A)\right)$$

Since the h-fibers are increasing curves in the (r, φ) coordinates with slopes uniformly bounded away from 0 and ∞ and the densities $\hat{\rho}_\gamma$ of the respective measures $\hat{\nu}_\gamma$ are uniformly bounded, all the measures ζ_γ have uniformly bounded piecewise continuous densities on $\mathcal{X}$ (the densities of ζ_γ correspond to p_γ in Theorem 4.1).

Recall that the h-fibers terminate on singularity curves, and every line $\varphi = \mathrm{const}$ intersects at most countably many singularities. Thus at most countably many unstable manifolds terminate on that line, hence at most countably many projected densities have discontinuities at any given $\varphi \in \mathcal{X}$. This guarantees (6), and (7) follows from [4, Lemma 3.3] because the preimage of each line $\varphi = \mathrm{const}$ is a finite union of stable curves. For every measurable $A \subset \mathcal{X}$, the measurability of the function $\gamma \mapsto \zeta_\gamma(A)$ follows from the measurability of the partition Γ. Now the result follows from Theorem 4.1. $\square$

Exactly the same argument shows that projection onto the r coordinate has a positive continuous density.

4.2 Projection on $\mathcal{D}$

We leave the collision space $\mathcal{M}$ and project the SRB measure μ for the flow Φ^t onto the table $\mathcal{D}$. Let $P\colon \Omega \to \mathcal{D}$ be the projection of the phase space onto the configuration space, and ξ be the pushforward of μ.

Theorem 4.3. *The measure ξ is absolutely continuous on $\mathcal{D}$ with a positive continuous density.*

Proof. According to formula (2), for any set $A \subset \mathcal{D}$

$$\xi(A) = \mu\left(P^{-1}(A)\right) = \frac{\nu(f_A)}{\nu(\tau)} \quad \text{with}$$

$$f_A(X) = \int_0^{\tau(X)} \mathbf{1}_A(P(\Phi^t(X)))\, dt.$$

Therefore

$$\xi(A) = \frac{1}{\nu(\tau)} \int_\Gamma \hat{\nu}_\gamma(f_A)\, d\hat{\lambda}(\gamma).$$

We observe that the map $A \mapsto \hat{\nu}_\gamma(f_A)$ defines a measure on $\mathcal{D}$ with a piecewise continuous density. This measure is supported on the *trace* of h-fiber γ (see Figure 2), on which the density is positive
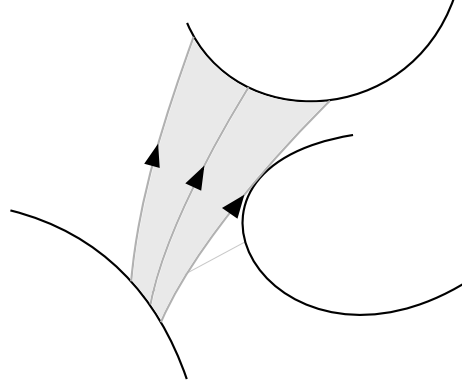
Figure 2: $\hat{\nu}_\gamma(f_A)$ is supported on trajectories that start from an h-fiber γ

and continuous. This density corresponds to p_γ in Theorem 4.1. The densities p_γ on $\mathcal{D}$ are bounded uniformly in γ because the density of $\hat{\nu}_\gamma$ on each γ is uniformly bounded.

Furthermore, for any $x \in \mathcal{D}$ define the set

$$E_x := \left\{ X \in \mathcal{M} : \{\Phi^t(X)\}_{t=0}^{\tau(X)} \cap P^{-1}(x) \text{ is not empty} \right\}$$

of points in the collision space. The trajectories starting from E_x pass through x and therefore correspond to a focusing wave front [9, Sect. 3.7]. In other words, E_x consists of a finite number of decreasing curves in the (r, φ)-coordinates, which have countably many intersections with singularity curves on which h-fibers terminate. Recall that h-fibers also terminate on the preimages of lines $\{\varphi = \pm\pi/2\}$. It is clear that if $x \in \mathcal{D}$ is a point of discontinuity for the density p_γ of some h-fiber γ, then γ must terminate on E_x. Therefore x can be a point of discontinuity for at most countably many densities p_γ. This guarantees (6), and (7) again follows from [4, Lemma 3.3] because E_x is a finite union of stable curves. Now the result follows from Theorem 4.1. $\qquad\square$

Note that the density of the measure ξ on $\mathcal{D}$ is given by

$$p(x) := \frac{1}{\nu(\tau)} \int_\Gamma p_\gamma(x) \, d\hat{\lambda}(\gamma).$$

Next we show that the velocity field is continuous in the following sense. For every point $\mathbf{q} \in \mathcal{D}$ let $\bar{\mathbf{p}}(\mathbf{q}) = \mu_{\mathbf{q}}(\mathbf{p})$ denote the average velocity vector, where $\mu_{\mathbf{q}}$ denotes the conditional measure induced by μ on the section of the phase space Ω corresponding to the fixed footpoint $\mathbf{q}$.

Theorem 4.4. *The velocity vector field $\bar{\mathbf{p}}(\mathbf{q})$ is continuous on $\mathcal{D}$.*

Proof. In place of the SRB measure for the flow μ we use a signed measure μ_1, defined by $\mu_1(F) = \mu(v_1 F)$ for any function F on Ω, where $v_1(X)$ is the horizontal component of the velocity of the particle at $X \in \Omega$. Then the projection of μ_1 onto $\mathcal{D}$ is a signed measure ξ_1 defined by

$$\xi_1(A) = \mu_1(P^{-1}(A)) = \mu\left(v_1 \mathbf{1}_{P^{-1}(A)}\right) = \frac{\nu(f_{1,A})}{\nu(\tau)}, \text{ where}$$

$$f_{1,A}(X) = \int_0^{\tau(X)} \left(v_1 \mathbf{1}_{P^{-1}(A)}\right)\left(\Phi^t(X)\right) dt,$$

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and

$$\xi_1(A) = \frac{1}{\nu(\tau)} \int_\Gamma \hat{\nu}(f_{1,A}) \, d\hat{\lambda}(\gamma).$$

Once again, the map $A \mapsto \hat{\nu}(f_{1,A})$ defines a (signed) measure on $\mathcal{D}$ with density $p_{1,\gamma}$ that has the same properties as p_γ above, so that Theorem 4.1 applies to prove that ξ_1 has continuous a density on $\mathcal{D}$ given by

$$p_1(\mathbf{q}) := \frac{1}{\nu(\tau)} \int_\Gamma p_{1,\gamma}(\mathbf{q}) \, d\hat{\lambda}(\gamma)$$

Note that the average horizontal velocity on the set $A \subset \mathcal{D}$ is given by $\mu\left(v_1 \mathbf{1}_{P^{-1}(A)}\right) / \mu\left(\mathbf{1}_{P^{-1}(A)}\right) = \xi_1(A)/\xi(A)$. Taking into account that ξ and ξ_1 have continuous densities p and p_1, and $p > 0$ everywhere, the average horizontal velocity at every point $X \in \mathcal{D}$ is well defined as $p_1(\mathbf{q})/p(\mathbf{q})$ and is continuous. The same argument works for the vertical component of the velocity. $\qquad\square$

Figure 3 shows the computed velocity field in a system with a Gaussian thermostated force $\mathbf{F}$ directed horizontally to the right.
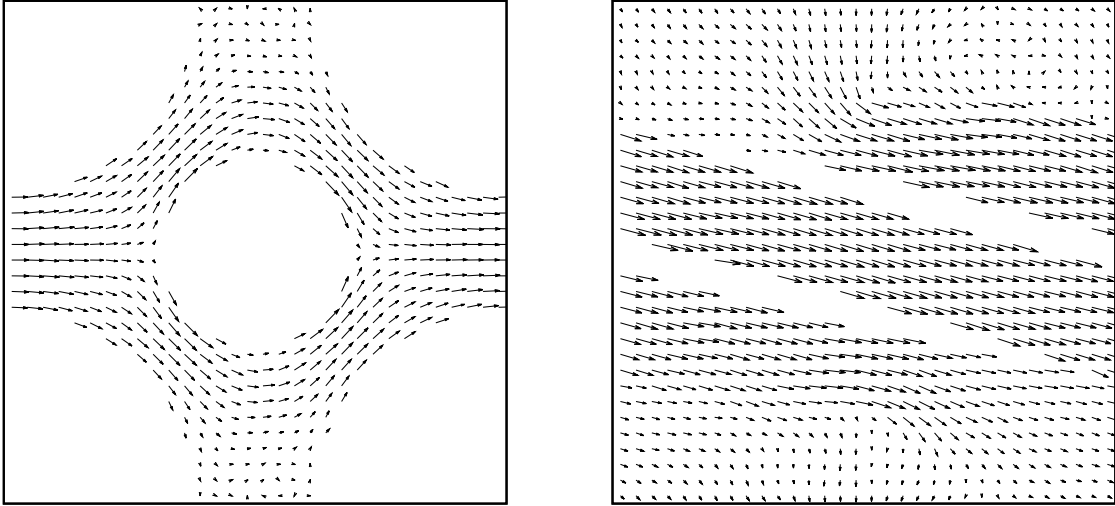

Figure 3: Velocity fields on tables A and B (see Figure 1).

4.3   Angular distribution for the flow with Gaussian thermostat

This section is restricted to a specific model — a constant external field with Gaussian thermostat. The force is given by $\mathbf{F} = \mathbf{E} - \frac{\mathbf{E} \cdot \mathbf{p}}{|\mathbf{p}|^2}$, where $\mathbf{E}$ is a small nonzero constant vector. This model was the subject of our paper [1].

We can choose the coordinate system so that $\mathbf{E}$ points in the positive x direction, then $\theta \in [-\pi, \pi]$ (see Assumption A) measures the angle between the particle velocity and the field $\mathbf{E}$. It is a direct verification that the only straight trajectories (where $\dot{\theta} \neq 0$) are those parallel to the field $\mathbf{E}$, i.e., $\theta \in \{0, \pm\pi\}$. This fact is used in the construction below.

Let $\mathcal{X} = [-\pi, \pi]$, denote by $P: \Omega \to \mathcal{X}$ the projection of the phase space onto the θ coordinate, and by ξ the pushforward of μ.

Theorem 4.5. *The measure ξ is absolutely continuous on $\mathcal{X}$ with a positive continuous density.*

Proof. For every set $A \subset \Omega$ we have:

$$\mu(A) = \frac{1}{\nu(\tau)} \int_\Gamma \hat{\nu}_\gamma(f_A) \, d\hat{\lambda}(\gamma), \quad \text{where}$$

$$f_A(X) = \int_0^{\tau(X)} \mathbf{1}_A(\Phi^t(X)) \, dt.$$

Denote $\mu_\gamma(A) := \hat{\nu}_\gamma(f_A)$ and let $\hat{\mu}_\gamma$ be the projection of μ_γ on $\mathcal{X}$. Then for any set $B \subset \mathcal{X}$

$$\xi(B) = \frac{1}{\nu(\tau)} \int_\Gamma \hat{\mu}_\gamma(B) \, d\hat{\lambda}(\gamma).$$

Let $S_\gamma := \{\Phi^t(X): X \in \gamma \text{ and } 0 \leq t \leq \tau(X)\}$ be again the *trace* of γ. Then $\mu_\gamma(A)$ is supported on S_γ and has continuous and uniformly (in γ) bounded density on it.

Since S_γ is a compact smooth 2-dimensional manifold in the phase space Ω and h-fibers correspond to *strongly divergent families of trajectories*[2], the angle between S_γ and the θ-axis is bounded above by a global constant that is less than $\pi/2$. Therefore, $\hat{\mu}_\gamma$ has a density on $\mathcal{X}$. Moreover, the area and the size (diameter) of S_γ are uniformly (in γ) bounded above and below by positive constants. Thus the density of $\hat{\mu}_\gamma$ is bounded above uniformly in γ.

The density of $\hat{\mu}_\gamma$ may have a discontinuity at $\theta_0 \in \mathcal{X}$ only if S_γ has a curve on its boundary where $\theta \equiv \theta_0$. The boundary of S_γ consists of four parts:

$$S_\gamma^1 := \{\Phi^t(X): X \in \gamma \text{ and } t = 0\},$$
$$S_\gamma^2 := \{\Phi^t(X): X \in \gamma \text{ and } t = \tau(X)\},$$
$$S_\gamma^3 := \{\Phi^t(X): X = X_1 \text{ and } 0 \leq t \leq \tau(X)\},$$
$$S_\gamma^4 := \{\Phi^t(X): X = X_2 \text{ and } 0 \leq t \leq \tau(X)\},$$

where X_1 and X_2 are the endpoints of γ.

The angle θ cannot be constant on any sub-curve of S_γ^1, because γ is an increasing curve in the (r, φ) coordinates. On S_γ^3 or S_γ^4, it can be a constant only if the whole trajectory is parallel to $\mathbf{E}$, i.e., $\theta \in \{0, \pm\pi\}$. The trajectories where $\theta \equiv 0$ or $\theta \equiv \pm\pi$ make a finite union of flat wave fronts, with at most countably many intersections with singularity curves and preimages of $\{\varphi = \pm\pi/2\}$, on which h-fibers terminate. Hence there are at most countably many γ's for which S_γ^3 or S_γ^4 has a sub-curve where $\theta \equiv 0$ or $\theta \equiv \pm\pi$.

It may happen that S_γ^2 contains a subset of positive length on which $\theta \equiv \text{const}$. Note, however, that for every $\theta_0 \in \mathcal{X}$ the set

$$H_{\theta_0}: \ = \{X \in \mathcal{M}: P(\Phi^{0^-}(X)) = \theta_0\}$$

---

[2]Let κ denote the curvature of a cross-section of S_γ orthogonal to the flow; then *strong divergence* means $0 < \kappa_{\min} < \kappa < \kappa_{\max} < \infty$ for some global constants $\kappa_{\min}$ and $\kappa_{\max}$, see [3, Secion 3].

of reflection points where the "incoming" velocity vector makes angle θ_0 with the field $\mathbf{E}$, is a finite union of smooth curves, so its preimage $\mathcal{F}^{-1}(H_{\theta_0})$ is a finite union of smooth curves, too. It is now clear that there could be at most countably many γ's that partially coincide with $\mathcal{F}^{-1}(H_{\theta_0})$.

Overall, for every $\theta_0 \in \mathcal{X}$ there are at most countably many γ's for which the boundary of S_γ contains a curve on which $\theta \equiv \theta_0$. This guarantees (6).

To satisfy (7), for every $\theta_0 \in \mathcal{X}$ consider a set

$$E_{\theta_0}: \ = \{X \in \mathcal{M} \colon P(\Phi^{0+}(X)) = \theta_0\}$$

of reflection points where the "outgoing" velocity vector makes angle θ_0 with the field $\mathbf{E}$. It consists of a finite number of decreasing curves, and its preimage $\mathcal{F}^{-1}(E_{\theta_0})$ consists of a finite number of stable curves. Consider two sets

$$\Gamma_{\theta_0}: \ = \{\gamma \colon \gamma \text{ intersects } E_{\theta_0}\} \quad \text{and} \quad \Gamma'_{\theta_0}: \ = \{\gamma \colon \gamma \text{ terminates on } E_{\theta_0}\}.$$

It follows from [4, Lemma 3.3] that Γ_{θ_0} has positive $\hat{\lambda}$-measure in Γ, and Γ'_{θ_0} is at most countable, because it has at most countably many intersection points with singularity curves and preimages of $\{\varphi = \pm\pi/2\}$, on which h-fibers terminate. Therefore, $\hat{\lambda}(\Gamma_{\theta_0} \setminus \Gamma'_{\theta_0}) > 0$. Observe that for every $\gamma \in \Gamma_{\theta_0} \setminus \Gamma'_{\theta_0}$ the density of the projected measure $\hat{\mu}_\gamma$ is positive at θ_0. This implies (7).

Now the result follows from Theorem 4.1. $\qquad\square$

Figure 4 shows the density of ξ constructed via computer simulation for a system under a small external force directed horizontally to the right, with Gaussian thermostat.
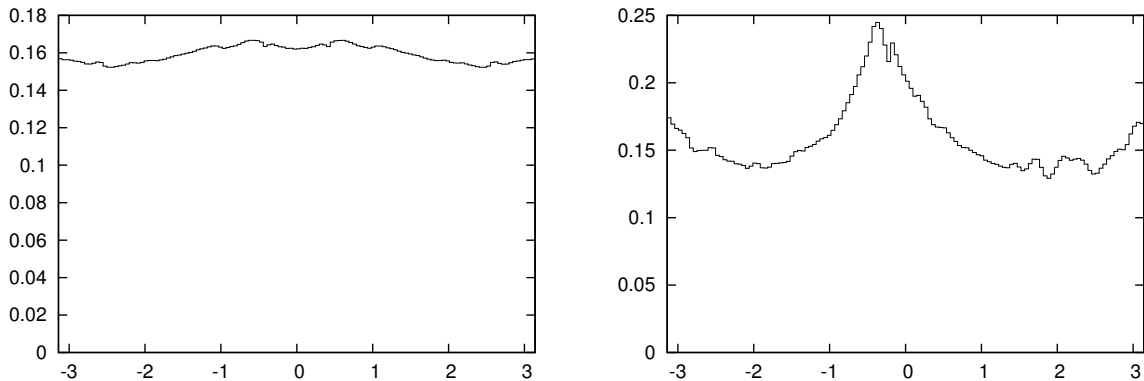

Figure 4: Densities of angular distributions for a Gaussian thermostated force with the same small field on the tables A and B (see Figure 1).

5 Linear response

Suppose that the force $\mathbf{F} = \mathbf{F}_\varepsilon$ is parameterized by a parameter $\varepsilon \in [0, \bar{\varepsilon}]$. More precisely, Assumption B in section 2 now takes form

$$\max(|h|, |h_x|, |h_y|, |h_\theta|) \le C_0 \varepsilon$$

with some $C_0 > 0$ and $B_0 > 0$ independent of ε. We will add the subscript ε to our symbols to emphasize the dependence of the dynamics on ε. Let $g_\varepsilon = d\mathcal{F}_\varepsilon^{-1}\nu_0/d\nu_0$ be the Jacobian of $\mathcal{F}_\varepsilon$ with

respect to the unperturbed billiard invariant measure ν_0. Denote $\Delta_\varepsilon := (1 - g_\varepsilon)/\varepsilon$ for $\varepsilon > 0$ and assume that $\Delta_0 := \lim_{\varepsilon \to 0}(1 - g_\varepsilon)/\varepsilon$ exists almost everywhere with respect to ν_0.

We will make a rather technical, but not too restrictive assumption on functions Δ_ε, namely that each h-fiber of the map $\mathcal{F}_0$ can be divided into no more than N_Δ pieces (for some constant N_Δ), on which they are Hölder continuous with a constant $C_\Delta > 0$ and exponent $1/6$, i.e. $|\Delta_\varepsilon(X) - \Delta_\varepsilon(Y)| \leq C_\Delta |X - Y|^{1/6}$, and that $|\Delta_\varepsilon| < C_\Delta$. We need exponent $1/6$ to connect Δ_ε to a proper standard family, as it will be shown further.

For example, for the Gaussian thermostat in Section 4.3, the function Δ_ε satisfies our assumptions with Hölder exponent $1/2$, and constant C_Δ determined by the maximum and minimum curvature of the obstacles; see [6, Sect. 8].

Assume that we are observing a function f_ε on $\mathcal{M}$ that may also change with ε. Let f_ε be bounded uniformly in ε and dynamically Hölder continuous with respect to $\mathcal{F}_\varepsilon$, with constants $C' = C_{f_\varepsilon}$ and $\theta' = \theta_{f_\varepsilon}$ independent from ε. Suppose the limit $f_0 := \lim_{\varepsilon \to 0} f_\varepsilon$ exists almost everywhere with respect to ν_0.

Theorem 5.1.
$$\nu_\varepsilon(f_\varepsilon) - \nu_0(f_\varepsilon) = \varepsilon \sum_{k=1}^\infty \nu_0\left((f_0 \circ \mathcal{F}_0^k)\Delta_0\right) + o(\varepsilon). \tag{8}$$

Remark 5.1. It may not be true that $\nu_0(f_\varepsilon) = \nu_0(f_0) + C\varepsilon + o(\varepsilon)$, this part of response is determined by the character of f_ε.

Proof. We start with a Kawasaki-type formula [4, Eq. (2.15)]:

$$\nu_\varepsilon(f_\varepsilon) = \nu_0(f_\varepsilon) + \sum_{k=1}^\infty \varepsilon\, \nu_0\left[\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k\right)\Delta_\varepsilon\right].$$

Here terms of the series decay exponentially fast and uniformly in ε. We are going to factor ε out of the series and prove that we still have a series with terms converging to zero exponentially and uniformly in ε.

Note that the integrands $\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k\right)\Delta_\varepsilon$ are bounded and pointwise converge to $\left(f_0 \circ \mathcal{F}_0^k\right)\Delta_0$, as $\varepsilon \to 0$. Therefore it is enough to show that there exist constants $C > 0$ and $0 < \theta < 1$ so that $\left|\nu_0\left[\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k\right)\Delta_\varepsilon\right]\right| \leq C\theta^k$ for every k.

We decompose Δ_ε into $\Delta_\varepsilon^+ - \Delta_\varepsilon^-$ where $\Delta_\varepsilon^+ = 1 + \Delta_\varepsilon \mathbf{1}_{\Delta_\varepsilon > 0}$ and $\Delta_\varepsilon^- = 1 - \Delta_\varepsilon \mathbf{1}_{\Delta_\varepsilon < 0}$. Then $\nu_0(\Delta_\varepsilon^+) = \nu_0(\Delta_\varepsilon^-)$ since $\nu_0(\Delta_\varepsilon) = 0$.

Let γ be a piece of an h-fiber corresponding to the unperturbed dynamics, on which Δ_ε is Hölder continuous with constant C_Δ and exponent $1/6$, and $\gamma(X, Y)$ be its subcurve that terminates at points X and Y. Then

$$\begin{aligned}
\left|\ln \frac{\Delta_\varepsilon^\pm(X)}{\Delta_\varepsilon^\pm(Y)}\right| &= \left|\ln\left(1 + \frac{\Delta_\varepsilon^\pm(X) - \Delta_\varepsilon^\pm(Y)}{\Delta_\varepsilon^\pm(Y)}\right)\right| \\
&\leq \left|\Delta_\varepsilon^\pm(X) - \Delta_\varepsilon^\pm(Y)\right| \leq C_\Delta |\gamma(X,Y)|^{1/6} \\
&\leq C_\Delta C_1 \Lambda^{-\frac{1}{6}s_+(X,Y)} = C_\Delta C_1 \theta_r^{s_+(X,Y)}
\end{aligned}$$

where constant C_1 is determined by the billiard geometry, see [9, Formula (5.32)], and θ_r comes from (3).

If ρ_γ is the density of the conditional measure induced by ν_0 on γ, then

$$\left| \ln \frac{\rho_\gamma(X)\Delta_\varepsilon^\pm(X)}{\rho_\gamma(Y)\Delta_\varepsilon^\pm(Y)} \right| \leq (C_r + C_\Delta C_1)\theta_r^{s+(X,Y)},$$

where constant C_r also comes from (3). We are free to choose C_r arbitrarily large, and replacing C_r with $C_r + C_\Delta C_1$ we make $\rho_\gamma\Delta^\pm$ a regular function on γ.

Thus the densities $\rho_\gamma\Delta_\varepsilon^\pm$ on the pieces of h-fibers, where Δ_ε is continuous, specify standard families $\mathcal{G}_\varepsilon^\pm$ such that $\nu_{\mathcal{G}_\varepsilon^\pm}(h) = \nu_0(\Delta_\varepsilon^\pm h)/\nu_0(\Delta_\varepsilon^\pm)$ for every function h.

The standard family $\mathcal{G}$ consisting of all h-fibers with their conditional measures corresponding to the unperturbed dynamics, is proper, i.e., $\mathcal{Z}_\mathcal{G} < C_p$; see equation (4). We want to show that $\mathcal{G}_\varepsilon^\pm$ is also proper.

To each standard pair $(\gamma, \nu_\gamma) \in \mathcal{G}$ there correspond at most N_Δ standard pairs $(\gamma_i, \nu_{\gamma_i})$ in $\mathcal{G}_\varepsilon^\pm$ with regular densities. Then, in the notations of equation (4),

$$\frac{\sum_i \nu_{\gamma_i}\left\{ X : r_{\mathcal{G}_\varepsilon^\pm}(X) < \varepsilon \right\}}{\varepsilon} \leq N_\Delta e^{C_r} \frac{\nu_\gamma\left\{ X : r_\mathcal{G}(X) < \varepsilon \right\}}{\varepsilon}.$$

Therefore $\mathcal{Z}_{\mathcal{G}_\varepsilon^\pm} \leq N_\Delta e^{C_r} \mathcal{Z}_\mathcal{G} \leq N_\Delta e^{C_r} C_p$. We are now free to choose C_p large enough to make the standard families $\mathcal{G}_\varepsilon^\pm$ proper.

By the equidistribution property [4, Proposition 2.2]

$$\left| \nu_{\mathcal{G}_\varepsilon^\pm}\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k \right) - \nu_\varepsilon\left(f_\varepsilon \right) \right| \leq C_8 \theta_8^k,$$

where $C_8 > 0$ and $0 < \theta_8 < 1$ are independent from ε. Coupled with

$$\nu_0\left[\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k \right) \Delta_\varepsilon \right] = \nu_0\left(\Delta_\varepsilon^+ \right)\left(\nu_{\mathcal{G}_\varepsilon^+}\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k \right) - \nu_{\mathcal{G}_\varepsilon^-}\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k \right) \right)$$

this gives

$$\nu_0\left[\left(f_\varepsilon \circ \mathcal{F}_\varepsilon^k \right) \Delta_\varepsilon \right] \leq 2\nu_0\left(\Delta_\varepsilon^+ \right) C_8 \theta_8^k.$$

We note that $\nu_0\left(\Delta_\varepsilon^+ \right) \leq 1 + C_\Delta < \infty$, which completes the proof. $\qquad\square$

We proved the linear response formula for the map $\mathcal{F}_\varepsilon$ but not for the flow, because no estimates on correlations for perturbed billiard flows are available.

Acknowledgement

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