MA 485-12 (Probability), Dr. Chernov — Midterm test #1
Show your work. — Tue, Jan 25 2000

1. (4 pts) A friend rolls two dies and tells you that the numbers shown differ by less than 4 (i.e. the dies show numbers X and Y such that $|X - Y| \leq 3$). Given this, find the probability that the sum of the two numbers is at least 11.

Answers:

$P = \frac{3/36}{30/36} = \frac{1}{10}$

2. (5 pts) In a class of 25 students, the math teacher selects three students at random for a test in math. The next day, the physics teacher selects four students randomly for a test in physics. The teachers made their choices independently. What is the chance that no students are selected by both teachers? What is the chance that exactly one (unlucky) student is selected by both teachers?

Answers:

Total number of choices: $N = C_{25,3} \cdot C_{25,4}$

Number of choices without overlap: $N_0 = C_{25,3} \cdot C_{22,4}$

Number of choices with exactly one common person: $N_1 = C_{25,3} \cdot 3 \cdot C_{22,3}$

Chance of no overlaps: $N_0/N \approx 0.578$

Chance of one overlap: $N_1/N \approx 0.365$

3. (4 pts) You have 30 dimes and 30 pennies in two boxes. The first box contains 20 dimes and 3 pennies, the second box contains 10 dimes and 27 pennies. You chose a box at random and then pick a coin out of that box at random. What is the probability that it is a dime?

Answers:

$P = \frac{10}{10+27} \cdot \frac{1}{2} + \frac{20}{20+3} \cdot \frac{1}{2} \approx 0.57$

4. (5 pts) A doctor's office receives 4 calls per hour, on the average. The receptionist takes a 12 minutes coffee break. What is the chance that somebody calls during her coffee break? What is the chance that at least two people will call?

Answers:

Average (typical) number of calls per 12 minutes is $4 \cdot 12/60 = 0.8$

This is λ for the Poisson formula.

$P(\geq 1 \text{ calls}) = 1 - P(0 \text{ calls}) = 1 - e^{-\lambda} = 1 - e^{-0.8} \approx 0.55$

$P(\geq 2 \text{ calls}) = 1 - P(0 \text{ calls}) - P(1 \text{ call}) = 1 - e^{-\lambda} - \lambda e^{-\lambda} \approx 0.19$

5. (5 pts) You toss three coins: a dime, a nickel, and a penny. Let A = "the dime shows Heads", B = "the number of Heads shown by all the three coins is even", and C = "the nickel and the penny show the same sides". Compute the probabilities $P(A)$, $P(B)$, $P(C)$. Are A and B independent? Are A and C independent? Are B and C independent? Are the three events A, B, and C jointly independent?

Answers:

$P(A) = 1/2$, $P(B) = 1/2$, $P(C) = 1/2$.

$P(A \cap B) = 1/4$, $P(A \cap C) = 1/4$, $P(B \cap C) = 1/4$.

So, they are pairwise independent.

$P(A \cap B \cap C) = 0$. They are not independent.

6. (4 pts) In an engine, four identical components work in parallel, so that as long as one component works the engine is running. The reliability of the components are 90%, 85%, 75% and 70%, respectively. They work or fail independently from each other. What is the probability that the engine will run? Bonus question: find the probability that exactly one component fails.

Answers:

$P(\text{engine works}) = 1 - P(\text{engine fails}) = 1 - 0.1 \cdot 0.15 \cdot 0.25 \cdot 0.3 \approx 0.9988$

$P(\text{one component fails}) \approx 0.42$

7. (4 pts) A discrete random variable X takes the following values with the corresponding probabilities:

X	-3	-1	0	1	2	3
P	0.1	0.2	0.1	0.2	0.15	0.25

Compute the following:

(a) $P\{X = 2\}$ = Answer: 0.15

(b) $P\{X = -2\}$ = Answer: 0

(c) $P\{X \leq 0\}$ = Answer: 0.4

(d) $P\{X$ is negative$\}$ = Answer: 0.3

(e) $P\{|X| \leq 2\}$ = Answer: 0.65

(f) (conditional probability) $P\{|X| \leq 1 \,/\, X \leq 0\}$ = Answer: 0.75

(d) Sketch the graph of the distribution function of X.

8. (5 pts) A continuous random variable X has the following density function:

$$f(x) = \begin{cases} x^2 & \text{for } -1 < x \leq 1 \\ 1/3 & \text{for } 1 < x \leq 2 \end{cases}$$

(a) Find the distribution function of X

Answer:

$$F(x) = \begin{cases} \int_{-1}^{x} x^2\,dx = \frac{x^3}{3} + \frac{1}{3} & \text{for } -1 \leq x \leq 1 \\ \int_{-1}^{x} f(x)\,dx = \frac{2}{3} + \frac{x-1}{3} & \text{for } 1 \leq x \leq 2 \end{cases}$$

(b) Sketch the graph of the distribution function $F(x)$. Show its values at least on the interval $-3 < x < 3$. Make sure the function is continuous.

(c) $P(-0.3 < X < 2.3) =$ Answer: ≈ 0.68

(d) $P(X = 1) =$ Answer: 0

(e) $P(0.3 < X < 1.5) =$ Answer: 0.491

(f) $F(3) =$ Answer: 1

(g) $F(-3) =$ Answer: 0

(h) $f(3) =$ Answer: 0

9. (4 pts) Suppose X has a density function $f(x) = c/x^2$ when $x > 1$ and 0 otherwise. What value of c makes this a density function? Find the distribution function of X.

Answers:

$c = 1$

$F(x) = 1 - \frac{1}{x}$ for $x > 1$

10. [Bonus] Let X be a geometric random variable with parameter p, i.e. $X =$ geometric(p). For two arbitrary positive integers $k > m \geq 1$, find the probability

$$P(m \leq X \leq k)$$

Answers:

$P(m \leq X \leq k) = q^{m-1} - q^k$