MA 485-4A (Probability), Dr. Chernov — Midterm test #1

Show your work. — Tue, Sep 28, 2004

1 (12 pts). The letters $s, s, s, t, t, t, i, i, a, c$ are arranged in a random order. What is the probability that they will spell the word "statistics"?

Answer:

$$\frac{3!\,3!\,2!}{10!} = \frac{1}{50400} \approx 0.00198\%$$

2 (18 pts). A discrete random variable X has probability function

$$p(x) = c(x^2 + 2), \qquad \text{for} \quad x = 0, \pm 1, \pm 2$$

and zero elsewhere.

(a) determine the value of c. Answer: $c = 1/20$.

(b) $P\{-2 < X < 1\} =$ Answer: 0.25.

(c) $P\{X \text{ is positive}\} =$ Answer: 0.45.

(d) $P\{X \text{ is even }\} =$ Answer: 0.7.

(e) find conditional probability $P(|X| \leq 1 \,/\, X \geq 0)$. Answer: 5/11.

(Bonus) Plot the probability function of $Y = X^2$

3 (16 pts). A two-stage experiment is performed: first, a number X is picked at random from the set $\{1, 2, 3, 4\}$, and, second, a fair coin is tossed X times, yielding Y heads and $X - Y$ tails.

(a) Find the probability that $Y = 1$.

Answer: by the law of total probability

$$P(Y = 1) = \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4}\right) = \frac{13}{32}$$

(b) Suppose that $Y = 1$. Use the Bayes formula to find the probability that $X = 3$.

Answer:

$$P(X = 3/Y = 1) = \frac{\frac{1}{4} \times \frac{3}{8}}{\frac{13}{32}} = \frac{3}{13}$$

4 (12 pts). A fair coin is tossed until both heads and tails appear at least once. Let X be the number of tosses. Determine all possible values of X and their probabilities (that is, find a general formula for $P(X = k)$ for all possible k). Find $P(X \geq 20)$.

Answers: Possible values: $k = 2, 3, 4, \ldots$. Probabilities

$$P(X = k) = \frac{1}{2^{k-1}}$$

$$P(X \geq 20) = \sum_{n=19}^{\infty} \frac{1}{2^n} = \frac{1}{2^{18}}$$

[Bonus] Find the probability that X is even.

Answer:

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{2} \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}$$

5 (14 pts). Suppose A, B, and C are independent events such that $P(A) = 1/4$, $P(B) = 1/3$ and $P(C) = 1/2$. Determine the probabilities that exactly 0, 1, 2, and 3 of them occur.

Answers:

$$P(0) = \frac{1}{4}, \quad P(1) = \frac{11}{24}, \quad P(2) = \frac{1}{4}, \quad P(3) = \frac{1}{24}$$

6 (12 pts). The probability of triplets in human birth is approximately 0.001.

(a) What is the probability that there will be at least one set of triplets among 700 births in a large hospital?

Answer (by Poisson approximation with $\lambda = 0.7$:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.7} \approx 0.504$$

(b) What is the probability that there will be at most one set of triplets among 700 births in a large hospital?

Answer (by Poisson approximation with $\lambda = 0.7$:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-0.7} + 0.7\, e^{-0.7} \approx 0.844$$

7 (16 pts). Let X be a Poisson random variable with parameter $\lambda = 21.7$. Show that the probability function $p(k) = P(X = k)$ increases until $k = 21$ and then decreases, so that it takes its maximum at $k = 21$.

Solution: the probability function is

$$p(k) = \frac{21.7^k}{k!}\, e^{-21.7}$$

Observe that

$$\frac{p(k)}{p(k-1)} = \frac{21.7}{k}$$

so that

$$\frac{p(k)}{p(k-1)} > 1 \qquad \text{for} \quad k \leq 21$$

and

$$\frac{p(k)}{p(k-1)} < 1 \qquad \text{for} \quad k \geq 22$$

(Bonus) Investigate the probability function of a Poisson random variable with parameter $\lambda = 21$. Where does it increase? decrease? take maximum?

Similar solution: the probability function is

$$p(k) = \frac{21^k}{k!}\, e^{-21}$$

Observe that

$$\frac{p(k)}{p(k-1)} = \frac{21}{k}$$

so that

$$\frac{p(k)}{p(k-1)} > 1 \qquad \text{for} \quad k \leq 20$$

and

$$\frac{p(k)}{p(k-1)} < 1 \qquad \text{for} \quad k \geq 22$$

but

$$p(21) = p(20),$$

which is the maximum value of $p(k)$.