MA 485-12 (Probability), Chernov — Midterm test #2

Show your work. — Tue, Feb 15

1. (5 pts) A continuous random variable X has the density function $f(x) = x^2/3$ for $-1 \le x \le 2$ and 0 elsewhere. Compute the following:

(a) $EX =$ Answer: $\int_{-1}^{2} x \cdot x^2/3\, dx = 5/4$

(b) $EX^2 =$ Answer: $\int_{-1}^{2} x^2 \cdot x^2/3\, dx = 11/4$

(c) $\operatorname{Var} X =$ Answer: $11/5 - (5/4)^2 \approx 0.6375$

(d) $\sigma_X =$ Answer: $\sqrt{\operatorname{Var} X} \approx 0.798$

(e) Find the moments of all order $k \ge 1$. Answer: $\int_{-1}^{2} x^k \cdot x^2/3\, dx = \frac{2^{k+3}-(-1)^{k+3}}{3(k+3)}$

2. (5 pts) By using the table for $\Phi(x)$ on page 266, find the following probabilities for two normal random variables, $Z = N(0,1)$ and $X = N(-2,4)$.

(a) $P(Z > 2.42) =$ Answer: $1 - \Phi(2.42) = .0058$

(b) $P(-1.26 < Z < 1.03) =$ Answer: $\Phi(1.03) - \Phi(-1.26) = 0.7447$

(c) $P(|Z| \leq 1.84) =$ Answer: $\Phi(1.84) - \Phi(-1.84) = 0.9342$

(d) $P(1.62 < Z \leq 3.14) =$ Answer: $\Phi(3.14) - \Phi(1.62) = 1 - 0.9474 = 0.0526$

(e) $P(-4.64 < X < 1.32) =$ Answer: $F_X(1.32) - F_X(-4.64) = \Phi((1.32+2)/2) - \Phi((-4.64+2)/2) = \Phi(1.66) - \Phi(-1.32) = 0.8581$

(f) What is the type (and parameters) of the random variable $Y = 5 - 6Z$?

Answer: $\mu = 5$, $\sigma^2 = (-6)^2 = 36$. So, $Y = N(5, 36)$

(g) [Bonus] What is the type (and parameters) of the random variable $W = 8 + 4X$?

Answer: $X = -2 + 2Z$, so $W = 8 + 4(-2 + 2Z) = 8Z$, hence $W = N(0, 64)$.

3. (4 pts) The lifetime of a radioactive atom is an exponential random variable X with half-life $\bar{t}_{1/2} = 15$ (years).

(a) Find the parameter $\lambda =$ Answer: $\lambda = \ln 2/15 = 0.0462$

(b) Write down the distribution function $F(x) =$ Answer: $= 1 - e^{-0.0462\,x}$ for $x > 0$

(c) Write down the density function $f(x) =$ Answer: $= 0.0462\,e^{-0.0462\,x}$ for $x > 0$

(d) Give values for EX and $\mathrm{Var}\,X$. Answer: $EX = 1/\lambda = 21.65$, $\mathrm{Var}\,X = 1/\lambda^2 = 468.5$

(e) Compute $P(X > 30) =$ Answer: $1 - F(30) = e^{-0.0462 \cdot 30} = 0.25$

(f) Find the conditional probability $P(X > 35/X > 5) =$ Answer: $= P(X > 30) = 0.25$ (no memory)

4. (6 pts) Let X be a uniform random variable $X = U(0,1)$.

(a) Find the distribution and density functions for the random variable $Y = 4 - \sqrt{2X}$.

Solution: $0 < X < 1$, hence $0 < \sqrt{2X} < \sqrt{2}$, so $4 - \sqrt{2} < Y < 4$.

$F_Y(y) = P(Y \le y) = P(4 - \sqrt{2X} \le y) = P(\sqrt{2X} \ge 4 - y)$
$= P(2X \ge (4-y)^2) = P(X \ge (4-y)^2/2) = 1 - F_X((4-y)^2/2) = 1 - (4-y)^2/2$

$f_Y(y) = F'_Y(y) = 4 - y$ for $4 - \sqrt{2} < Y < 4$.

(b) Find the distribution and density functions for the random variable $W = 1 - 3/X$.

Solution: $0 < X < 1$, so $1/X > 1$, and $1 - 3/X < -2$, hence $Y < -2$.

$F_Y(y) = P(Y \le y) = P(1 - 3/X \le y) = P(3/X \ge 1 - y)$
$= P(X \le 3/(1-y)) = F_X(3/(1-y)) = 3/(1-y)$

$f_Y(y) = F'_Y(y) = 3/(1-y)^2$ for $y < -2$

[Bonus] Find the distribution function of the random variable $V = 1 - (1 - 2X)^2$.
Note: do not forget to find the range for the new random variables.

5. (5 pts) Suppose X and Y are two independent random variables such that $EX = 5$, $\text{Var}X = 16$, $EY = -1$ and $\text{Var}Y = 9$. Let $Z = 3X - 4Y - 15$. Compute the following:

(a) $EZ =$ Answer: $3 \cdot 5 - 4 \cdot (-1) - 15 = 4$

(b) $\sigma_Z =$ Answer: $\sqrt{288} \approx 16.97$

Since $\text{Var}\, Z = 3^2 \cdot \text{Var}\, X + (-4)^2 \cdot \text{Var}\, Y = 9 \cdot 16 + 16 \cdot 9 = 288$

(c) $EX^2 =$ Answer: $\text{Var}\, X + (EX)^2 = 16 + 5^2 = 41$

(d) $EY^2 =$ Answer: $\text{Var}\, Y + (EY)^2 = 9 + (-1)^2 = 10$

(e) $E(2X^2 - 3XY - 4Y^2) =$ Answer: $2 \cdot 41 - 3 \cdot 5 \cdot (-1) - 4 \cdot 10 = 57$

(f) [Bonus] $E[XY(X + Y)] =$

Answer: $E(X^2Y + XY^2) = EX^2 \cdot EY + EX \cdot EY^2 = 41 \cdot (-1) + 5 \cdot 10 = 9$

6. (5 pts) Random variables X and Y are uniform on the interval $(-1, 1)$ and independent.

(a) Find the probability that $X^2 + Y^2 < 1$.

Solution: $f_{X,Y}(x, y) = 1/4$ for $0 < x, y < 1$.

Answer: $\pi/4$

(b) Find the probability that $|X - Y| < 1$.

Answer: $3/4$

[Bonus] Find the probability that $|X| + |Y| < 0.2$.

Answer: 0.02

7. (5 pts) An engine has 4 components. The lifetime (time to failure) of each component is a uniform random variable on the interval $(0, 5)$ (in years), and their lifetimes are independent. Find the distribution function $F_E(x)$ and the density function $f_E(x)$ of the lifetime of the engine in the following cases:

(a) The engine fails when one of its components fails.

Answer: $F_E(x) = 1 - (1 - F(x))^4 = 1 - \left(1 - \frac{x}{5}\right)^4$.

$f_E(x) = F_E'(x) = \frac{4}{5}\left(1 - \frac{x}{5}\right)^3$ for $0 < x < 5$

(b) The engine only fails when all of its components fail.

Answer: $F_E(x) = [F(x)]^4 = \left(\frac{x}{5}\right)^4$.

$f_E(x) = F_E'(x) = \frac{4}{5}\left(\frac{x}{5}\right)^3$ for $0 < x < 5$

(c) [Bonus] The engine fails when two of its components fail.

Answer: $F_E(x) = 1 - \left(1 - \frac{x}{5}\right)^4 - 4\left(1 - \frac{x}{5}\right)^3\left(\frac{x}{5}\right)$.

$f_E(x) = F_E'(x) = \frac{12}{25}x\left(1 - \frac{x}{5}\right)^2$ for $0 < x < 5$

8. (5 pts) A random variable X takes the following values with the corresponding probabilities:

X	-1	0	1	2
P	0.3	0.1	0.4	0.2

Let $Y = X^2$. Compute the following:

(a) $P(X + Y < 2) =$ Answer: $0.3 + 0.1 = 0.4$

(b) $EX =$ Answer: 0.5

(c) $EY =$ Answer: 1.5

(d) $E(XY) =$ Answer: 1.7

(e) [Bonus] $\text{Var}(X + Y) =$ Answer: 4.8