MA 485-25 (Probability), Dr Chernov — Midterm Test II

Show your work. Each problem is 4 pts. — Thu, Feb 15, 2001

1. A continuous random variable $X$ has density function $f(x) = 1 - cx$ for $1 < x \le 3$ (and zero elsewhere).

(a) Find the the value of $c$.

Answer:

$$\int_1^3 (1 - cx)\, dx = 2 - 4c = 1$$

so $c = 1/4$

(b) Find the distribution function $F(x)$ of $X$

Answer:

$$F(x) = \int_1^x \left(1 - \frac{t}{4}\right)\, dt = x - x^2/8 - 7/8$$

for $1 < x \le 3$.

(c) Find $P(1.5 < X < 4)$ Answer: $F(4) - F(1.5) = 1 - F(1.5) = 21/32 = 0.65625$

(d) Find $P(X = 3)$ Answer: 0

(e) Find $P(|X| < 2)$ Answer: $F(2) - F(-2) = F(2) - 0 = 0.625$

(f) Find $F(3.5)$ Answer: 1

2. By using the table for $\Phi(x)$ on page 488, find the following probabilities for two normal random variables, $Z = N(0, 1)$ and $X = N(-5, 9)$.

(a) $P(Z > 0.54) =$ Answer: 0.2946

(b) $P(-2.6 < Z < 1.8) =$ Answer: 0.9594

(c) $P(|Z| \geq 2) =$ Answer: 0.0456

(d) $P(2 < Z \leq 5) =$ Answer: 0.0228

(e) $P(-6.2 < X < 1) =$ Answer: 0.6326

(f) What is the type (and parameters) of the random variable $Y = 3 - 2X$?

Answer: since $X = -5+3Z$, then $Y = 3-2(-5+3Z) = 13-6Z$, hence $Y = N(13, 36)$.

3. A continuous random variable $X$ has distribution function

$$F(x) = \frac{x^3 + 8}{9}$$

for $-2 \leq x \leq 1$. Compute the following:

(a) $E(X) =$

Solution: First, we find the density: $f(x) = x^2/3$ for $-2 < x < 1$.

Then

$$E(X) = \int_{-2}^{1} \frac{x^3}{3}\, dx = -\frac{5}{4} = -1.25$$

(b) $E(X^2) =$

Answer:

$$E(X^2) = \int_{-2}^{1} \frac{x^4}{3}\, dx = \frac{11}{5}$$

(c) $\mathrm{Var}(X) =$ Answer: $11/5 - (-5/4)^2 = 0.6375$

(d) $\sigma_X =$ Answer: $\sqrt{0.6375} \approx 0.7984$

(e) Find the moments of $X$ of all order $k \geq 1$.

Answer:

$$E(X^k) = \int_{-2}^{1} \frac{x^{k+2}}{3}\, dx = \frac{1 - (-2)^{k+3}}{3(k+3)}$$

4. Let $X$ be an exponential random variable with parameter $\lambda = 5$.

(a) Find the half-life $\bar{t}_{1/2}$. Is it less than 0.15?

Answer: $\bar{t}_{1/2} = (\ln 2)/5 \approx 0.139$

(b) Write down the distribution function $F(x)$ and density function $f(x)$.

Answer:

$$F(x) = 1 - e^{-5x}$$
$$f(x) = 5e^{-5x}$$

(c) Give values of $E(X)$ and $\mathrm{Var}(X)$. Answer: $1/5$ and $1/25$.

(d) Compute $P(X > E(X) + 3\sigma_X) =$

Answer:

$$P(X > 1/\lambda + 3/\lambda) = P(X > 4/\lambda) = 1 - F(4/\lambda) = e^{-4}$$

(e) Find the conditional probability

$$P(X > E(X) + 4\sigma_X / X > E(X))$$

Compare to the answer in (d). Explain.

Answer: $e^{-4}$. Same, by the memoryless property and because $E(X) = \sigma_X$.

5. Random variables $X$ and $Y$ have joint density function $f(x, y) = 2$ in the region defined by the inequalities $x > 0$, $y > 0$, and $x + y < 1$ (and zero elsewhere). Sketch that region.

(a) Find the probability $P(X^2 + Y^2 < 0.09)$.

Answer: $2\pi(0.3)^2/4 = 9\pi/200$ (it is a quarter of a disk of radius 0.3)

(b) Find the probability $P(|X - Y| < 0.2)$.

Answer: 0.36 (it is a central strip around the line $y = x$)

[Bonus] Find the probability $P((X - 1)^2 + Y^2 < 0.25)$

Answer: $2\pi(0.5)^2/8 = \pi/16$ (it is one-eight of a disk of radius 0.5).

6. Suppose $X$ and $Y$ are two independent random variables such that $E(X) = -10$, $\mathrm{Var}(X) = 4$, $E(Y) = 5$ and $\sigma_Y = 3$. Let $V = 8 - 2X + 3Y$. Compute the following:

(a) $EV =$ Answer: $8 - 2 \cdot (-10) + 3 \cdot 5 = 43$

(b) $\sigma_V =$

Answer:

$$\mathrm{Var}(V) = (-2)^2 \cdot 4 + 3^2 \cdot 3^2 = 97$$
$$\sigma_Y = \sqrt{97}$$

(c) $EX^2 =$ Answer: $(-10)^2 + 4 = 104$

(d) $EY^2 =$ Answer: $5^2 + 3^2 = 34$

(e) $E(3Y^2 - X^2 - 2XY) =$ Answer: 98

7. Let $X$ be a uniform random variable $X = U(-1, 1)$.

(a) Find the distribution and density functions for the random variable $Y = 2 + \frac{4}{X+1}$.

Answer: range is $Y > 4$,

$$F_Y(y) = P(Y \le y) = P\left(2 + \frac{4}{X+1} \le y\right)$$

$$= P\left(X \ge \frac{4}{y-2}\right) = 1 - F_X\left(\frac{4}{y-2}\right) = 1 - \frac{2}{y-2}$$

$$f_Y(y) = F_Y'(y) = \frac{2}{(y-2)^2}$$

[Bonus] Find $E(Y)$

Answer:

$$E(Y) = \int_4^\infty \frac{2y}{(y-2)^2}\, dy = \infty$$

[Bonus] Find the distribution function of the random variable $V = e^{X^2}$.

Answer: range is $1 < V < e$,

$$F_V(v) = P(V \le v) = P(e^{X^2} \le v) = P(X^2 \le \ln v)$$

$$= P(-\sqrt{\ln v} \le X \le \sqrt{\ln v}) = F_X(\sqrt{\ln v}) - F_X(-\sqrt{\ln v}) = \sqrt{\ln v}$$

$$f_V(v) = F_V'(v) = \frac{1}{2v\sqrt{\ln v}}$$

8. A system has 6 components. The lifetime (time to failure) of each component is an exponential random variable with parameter $\lambda = 1$, and their lifetimes are independent. Find the distribution function $F_S(x)$ and the density function $f_S(x)$ of the lifetime of the system in the following cases:

(a) It is a parallel system, i.e. it functions as long as at least one component works.

Answer:

$$F(x) = (1 - e^{-x})^6$$
$$f(x) = 6e^{-x}(1 - e^{-x})^5$$

(b) It is a series system, i.e. it functions as long as all components work.

Answer:

$$F(x) = 1 - e^{-6x}$$
$$f(x) = 6e^{-6x}$$

(c) [Bonus] It is a "4-out-of-6 system", i.e. it works as long at least 4 components work.

9. Two discrete random variables, $X$ and $Y$, have joint probability function $p(x, y) = \frac{1}{6}x^2y$ for $x = -1, 1$ and $y = 1, 2$ (and zero elsewhere).

(a) Mark the actual values $X, Y$ of this pair of random variables on the $x, y$ plane and indicate the probability of each point.

(b) Check that the sum of all probabilities is one, i.e.

$$\sum_{x,y} p(x, y) = 1$$

Answer: $1/6 + 1/6 + 1/3 + 1/3 = 1$

(c) Find $P(X = Y)$ Answer: $= p(1, 1) = 1/6$

(d) Find $P(X + Y < 2)$ Answer: $= p(-1, 1) + p(-1, 2) = 1/6 + 1/3 = 1/2$

(e) Find $E(X)$ Answer: 0

(f) Find $\mathrm{Var}(X)$ Answer: 1

(g) [Bonus] Are $X$ and $Y$ independent random variables? Explain. Answer: Yes.

(h) [Bonus] Find $E(XY)$ Answer: 0