MA 485-1E (Probability), Dr Chernov — Midterm Test II

Show your work. Full credit is 20 pts. — Fri, Nov 1, 2002

1. (6 pts) A continuous random variable X has density function

$$f(x) = 1 - cx$$

for $2 \le x \le 4$. Do the following:

(a) Find the value of c. Solution:

$$1 = \int_2^4 1 - cx\,dx = -6c + 2$$

hence $c = 1/6$.

(b) Find the distribution function $F(x)$. Solution:

$$F(x) = \int_2^x 1 - x/6\,dx = x - x^2/12 - 5/3$$

for $2 < x < 4$.

(c) $E(X) =$

$$\int_2^4 x(1 - x/6)\,dx = 26/9$$

(d) $E(X^2) =$

$$\int_2^4 x^2(1 - x/6)\,dx = 26/3$$

(e) $\mathrm{Var}(X) =$

$$\frac{26}{3} - \left(\frac{26}{9}\right)^2 \approx 0.321$$

(f) $\sigma_X =$

$$\sqrt{0.321} \approx 0.567$$

(Bonus) Find the moments of X of all orders $k \ge 1$.

2. (4 pts) Let X be an exponential random variable with $\lambda = 7$.

(a) Find the half-life $t_{1/2}$. Is it less than 0.1?

Solution: $t_{1/2} = \ln 2/\lambda \approx 0.099$. Yes, it is less than 0.1.

(b) Find $E(X)$.

Solution: $E(X) = 1/\lambda = 1/7$.

(c) Write down the density function $f(x)$. [Bonus question: What is the maximal value of $f(x)$ and where is it attained?]

Solution: $f(x) = \lambda e^{-\lambda x} = 7e^{-7x}$.

Maximum is 7, attained at $x = 0$.

(d) Compute $P(X > 10)$.

Solution: $P(X > 10) = 1 - F(10) = e^{-10\lambda} = e^{-70}$.

(e) Find the conditional probability $P(X > 99/X > 89)$.

Solution: by lack of memory, it is $= P(X > 10) = e^{-70}$.

3. (4 pts) By using the table for $\Phi(x)$ on page 488, find the following probabilities for two normal random variables, $Z = N(0, 1)$ and $X = N(-1, 4)$.

(a) $P(-2.98 < Z < 0.15) =$

Solution: $\Phi(0.15) - \Phi(-2.98) = 0.5598 - (1 - 0.9986) = 0.5584$.

(b) $P(|X| > 0.8) =$

Solution:

$$\begin{aligned} P(|X| > 0.8) = P(X > 0.8) + P(X < -0.8) = 1 - F(0.8) + F(-0.8) \\ = 1 - \Phi\left(\frac{0.8+1}{2}\right) + \Phi\left(\frac{-0.8+1}{2}\right) \\ = 1 - 0.8159 + 0.5398 = 0.7239 \end{aligned}$$

(c) $P(X \leq 7.3) =$

Solution:

$$\Phi\left(\frac{7.3+1}{2}\right) \approx 1$$

(d) What is the type (and parameters) of the random variable $Y = 6 + 5X$?

Solution: $Y = 6 + 5(-1 + 2Z) = 1 + 10Z$, hence $Y = N(1, 100)$.

4. (4 pts) Let X be a uniform random variable $X = U(0, 1)$.

Find the distribution function and the density function for the random variable

$$Y = \sqrt{2 - X}$$

Solution: Range of Y is $1 \le Y \le \sqrt{2}$.

Note that $F_X(x) = x$ for $0 < x < 1$. Then

$$F_Y(y) = P(Y \le y) = P(\sqrt{2 - X} \le y) = P(X \ge 2 - y^2) = 1 - (2 - y^2) = y^2 - 1$$

and

$$f_Y(y) = F'_Y(y) = 2y$$

for $1 \le y \le \sqrt{2}$.

[Bonus] Find $E(Y)$

Answer:

$$E(Y) \approx 1.22$$

5. (4 pts) Random variables X and Y have a joint density function $f_{X,Y}(x, y) = 4$ on the domain defined by $x > 0$, $y > 0$ and $2x + y < 1$ (and $f_{X,Y}(x, y) = 0$ elsewhere).

(a) Sketch this domain.

It is a triangle with vertices $(0, 0)$, $(0, 1)$ and $(0.5, 0)$.

(b) Find the probability $P(Y < X)$.

Answer: $P(Y < X) = 1/3$.

(c) Find the probability $P(X = Y)$.

Answer: $P(X = Y) = 0$.

[Bonus] Are X and Y independent?

Answer: no, they are not independent.