MA 485-1E (Probability), Dr Chernov | Midterm Test II
Show your work. Each problem is 25 pts. Full credit is 100 pts. | October 31, 2003

1. By using the table for $\Phi(x)$ on page 488, find the following probabilities for two normal random variables: $Z = N(0,1)$ and $X = N(-3,25)$.

(a) $P(|X| > 4) =$

Answer: $P(X > 4) + P(X < -4) = (1 - \Phi(1.4)) + \Phi(-0.2) = 0.5015$

(b) $P(|Z - 1| < 0.32) =$

Answer: $P(0.68 < Z < 1.32) = \Phi(1.32) - \Phi(0.68) = 0.1544$

(c) Conditional probability $P(Z < 2/Z > 1) =$

Answer: $P(1 < Z < 2)/P(Z > 1) = 0.1359/0.1587 = 0.8563$

(d) What is the type (and parameters) of the random variable $Y = 6 + X/5$?

Answer: $Y = 6 + (-3 + 5Z)/5 = 6 - 0.6 + Z = 5.4 + Z$, so $Y = N(5.4, 1)$.

2. A continuous random variable X has density function $f(x) = c + 2x$ for $1 \leq x \leq 2$.
(a) Find the value of c

Answer: $c = -2$.

(b) Find the distribution function $F(x)$

Answer: $F(x) = x^2 - 2x + 1$ for $1 \leq x \leq 2$.

(c) $E(X) =$

Answer: $E(X) = 5/3$.

(d) $E(X^2) =$

Answer: $E(X^2) = 17/6$.

(e) $\mathrm{Var}(X) =$

Answer: $\mathrm{Var}(X) = 1/18$.

(f) $\sigma_X =$

Answer: $\sigma_X = \sqrt{2}/6$.

3. Let X =exponential(λ).

(a) Find the median m and the mean $E(X)$. Which one is larger? For an extra credit, find the quartiles $q_1 = \pi_{1/4}$ and $q_3 = \pi_{3/4}$.

Answer: $m = t_{1/2} = \ln 2/\lambda$ and $E(X) = 1/\lambda$. Since $\ln 2 < 1$, then $m < E(X)$.

(b) Find the distribution function and the density function for the random variable $Y = 1 - \ln X$.

Answer: the range for Y is $-\infty < Y < \infty$,

$$F_Y(y) = P\left(X \geq e^{1-y}\right) = e^{-\lambda e^{1-y}}$$

$$f_Y(y) = \lambda e^{1-y} e^{-\lambda e^{1-y}}$$

4. Random variables X and Y have a joint density function $f_{X,Y}(x, y) = c$ on the domain defined by $0 < x < 4$, $y > 0$, and $x > 4y$ (and $f_{X,Y}(x, y) = 0$ elsewhere).

(a) Sketch this domain.

Answer: a triangle with vertices $(0, 0)$, $(4, 0)$, and $(4, 1)$.

(b) Find the value of c.

Answer: $c = 1/$(Area of the triangle)$= 1/2$.

(c) Find the probability $P(X < 1)$.

Answer: $P(X < 1) = 1/16$.

(d) Find the probability $P(X^2 + Y^2 = 1)$.

Answer: 0, because the area of a circular arc is zero.

[Bonus] Find the marginal distribution functions $F_X(x)$ and $F_Y(y)$ and the marginal density functions $f_X(x)$ and $f_Y(y)$. (Give an argument.)