MA 485-4A (Probability), Dr. Chernov — Midterm test #2

Show your work. Each problem is 20 points. — Thu, Oct 28, 2004

1. Two random variables X and Y have joint density function $f(x, y) = c$ on the rectangle $\{0 < x < 5,\ 0 < y < 1\}$ (and zero elsewhere).

(a) Find $c =$. Answer: 1/5

(b) Find $P(X > Y) =$. Answer: 9/10.

(c) Find $P(X^2 + Y^2 < 1) =$. Answer: $\pi/20$.

(d) Find $P(X = Y) =$. Answer: 0.

(e) For $0 < x < 5$, find $P(X < x) =$. Answer: $x/5$.

(f) For $0 < y < 1$, find $P(Y < y) =$. Answer: y.

(Bonus) Find the density functions of X and Y. Answer: 1/5 and 1.

(Bonus) Are X and Y independent? Answer: yes.

2. By using the table for $\Phi(x)$ on page 266, find the following probabilities for two normal random variables: $Z = N(0,1)$ and $X = N(-10, 25)$.

(a) $P(-16 < X < 3.5) =$. Answer: 0.8814.

(b) $P(X^2 > 64) =$. Answer: 0.6554.

(c) $P(Z < -2.77) =$. Answer: 0.0028.

(d) What is the type (and parameters) of the random variable $Y = 3(2-X)+4$? Answer: normal, $Y = N(40, 225)$.

3 Let $X = U(0,1)$. Find the distribution function and the density function for the random variable $Y = (1-\sqrt{X})^2$. For extra credit, find $E(Y)$.

Answers: range for Y is $0 < Y < 1$,

$$F_Y(y) = P((1-\sqrt{X})^2 \leq y) = P(X \geq (1-\sqrt{y})^2) = 1 - (1-\sqrt{y})^2$$

and

$$f_Y(y) = \frac{1}{\sqrt{y}} - 1$$

for $0 < y < 1$.

Bonus: $E(Y) = 1/6$.

4. A continuous random variable X has density function $f(x) = c/x$ for $1 \leq x \leq 10$.

(a) Find $c =$. Answer: $c = 1/\ln 10$.

(b) Find the distribution function $F(x)$. Answer: $F(x) = \ln x/\ln 10$.

(c) $E(X) =$. Answer: $E(X) = 9/\ln 10$.

(d) $E(X^2) =$. Answer: $E(X^2) = 99/(2\ln 10)$.

(e) $\mathrm{Var}(X) =$. Answer: 6.22.

(f) $\sigma_X =$. Answer: 2.494.

5a. Let X be an exponential random variable with $E(X) = 5$. Find λ and $t_{1/2}$. Is $t_{1/2}$ greater than 4? Find the conditional probability $P(X > 2004/X > 1999)$.

Answer: $\lambda = 1/5$, $t_{1/2} = 5 \ln 2 \approx 3.46$ (less than 4);

$$P(X > 2004/X > 1999) = P(X > 5) = e^{-1}$$

5b. Suppose X and Y are two independent random variables such that $E(X) = 9$, $E(Y) = 3$, $\sigma_X = 2$ and $\sigma_Y = 4$. Let $W = 15 - 3X + 2(Y + 5)$. Compute $E(W)$ and σ_W.

Answer: $E(W) = 4$, $\text{Var}(W) = 100$, $\sigma_W = 10$.

[Bonus problem] A system has 15 components. The lifetime (time to failure) of each component is a random variable with distribution function $F(x) = \frac{x^2}{x^2+1}$ for $x > 0$, and their lifetimes are independent. The system works as long as m components are working, $1 \leq m \leq 15$. Find the distribution function $F_S(x)$ and the density function $f_S(x)$ of the lifetime of the system.

Answer:

$$\begin{aligned} F_S(x) &= \sum_{i=0}^{m-1} \binom{15}{i} \left(1 - \frac{x^2}{x^2+1}\right)^i \left(\frac{x^2}{x^2+1}\right)^{15-i} \\ &= \frac{1}{(x^2+1)^{15}} \sum_{i=0}^{m-1} \binom{15}{i} x^{30-2i} \end{aligned}$$

and

$$f_S(x) = 15 \binom{14}{m-1} \frac{2x^{31-2m}}{(x^2+1)^{16}}$$